Constructive Control by Adding Candidates

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY ADDING AN UNLIMITED NUMBER OF CANDIDATES ($\mathcal{E}$-CCAUC).

Given:

- Disjoint sets C and D of candidates,
- a list V of votes over $C \cup D$, and
- a distinguished candidate $p \in C$.

Question: Is there a subset D' of D such that p is the unique winner of the $\mathcal{E}$ election $(C \cup D', V)$?

Constructive Control by Adding Candidates

Definition (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY ADDING (A LIMITED NUMBER OF) CANDIDATES ($\mathcal{E}$-CCAC).

Given:
- Disjoint sets C and D of candidates,
- a list V of votes over $C \cup D$,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is there a subset D' of D such that $\|D'\| \leq k$ and p is the unique winner of the $\mathcal{E}$ election $(C \cup D', V)$?

Constructive Control by Deleting Candidates

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY DELETING CANDIDATES ($\mathcal{E}$-CCDC).

Given:
- A set C of candidates,
- a list V of votes over C,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is it possible to delete up to k candidates from C such that p is the unique winner of the resulting $\mathcal{E}$ election?

Constructive Control by Partition of Candidates

Definition (Bartholdi, Tovey, and Trick (1992) &
Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY PARTITION OF CANDIDATES ($\mathcal{E}$-CCPC).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition C into C_1 and C_2 such that p is the unique winner (w.r.t. V) of the final stage of the two-stage election in which

- the winners of (C_1, V) surviving the tie-handling rule
- run against all candidates in C_2?

- "Ties eliminate" (TE): Only unique winners proceed to final stage.
- "Ties promote" (TP): All winners proceed to final stage.

Constructive Control by Run-Off Partition of Candidates

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY RUN-OFF PARTITION OF CANDIDATES ($\mathcal{E}$-CCRPC).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition C into C_1 and C_2 such that p is the unique winner (w.r.t. V) of the final stage of the two-stage election in which the run-off is between

- the winners of (C_1, V) surviving the tie-handling rule and
- the winners of (C_2, V) surviving the tie-handling rule?

- “Ties eliminate” (TE): Only unique winners proceed to final stage.
- “Ties promote” (TP): All winners proceed to final stage.

Constructive Control by Adding Voters

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY ADDING VOTERS ($\mathcal{E}$-CCAV).

Given:

- A set C of candidates,
- a list V of registered votes over C and an additional list W of as yet unregistered votes over C,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is there a subset W' of W such that $\|W'\| \leq k$ and p is the unique winner of the $\mathcal{E}$ election $(C, V \cup W')$?

Constructive Control by Deleting Voters

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY DELETING VOTERS ($\mathcal{E}$-CCDV).

Given:
- A set C of candidates,
- a list V of votes over C,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is it possible to delete up to k voters from V such that p is the unique winner of the resulting $\mathcal{E}$ election?

Control by Partition of Voters: Gerrymandering

Constructive Control by Partition of Voters

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY PARTITION OF VOTERS ($\mathcal{E}$-CCPV).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition V into V_1 and V_2 such that p is the unique winner (with respect to the votes in V) of the final stage of the two-stage election in which the run-off is between

- the winners of (C, V_1) surviving the tie-handling rule and
- the winners of (C, V_2) surviving the tie-handling rule?

- "Ties eliminate" (TE): Only unique winners proceed to final stage.
- "Ties promote" (TP): All winners proceed to final stage.

Destructive Control

Remark:

- For each constructive control scenario, there is a corresponding destructive control type where the chair seeks to block the distinguished candidate's victory: $\mathcal{E}$-DCAUC, $\mathcal{E}$-DCAC, $\mathcal{E}$-DCDC, $\mathcal{E}$-DCPC-TE, $\mathcal{E}$-DCPC-TP, $\mathcal{E}$-DCRPC-TE, $\mathcal{E}$-DCRPC-TP, $\mathcal{E}$-DCAV, $\mathcal{E}$-DCDV, $\mathcal{E}$-DCPV-TE, and $\mathcal{E}$-DCPV-TP.

  In $\mathcal{E}$-DCDC it is not allowed to simply delete the distinguished candidate.

  $\Rightarrow$ This sums up to a total of 22 control types (and the corresponding control problems).

- While constructive control is due to Bartholdi, Tovey, and Trick (1992), the study of destructive control was initiated by Hemaspaandra, Hemaspaandra, and Rothe (2007).

Overview of the Common Control Problems

Table: Overview of candidate control problems for voting system $\mathcal{E}$

Control by	Constructive	Destructive
Adding candidates	$\mathcal{E}$-CCAC	$\mathcal{E}$-DCAC
Adding an unlimited number of candidates	$\mathcal{E}$-CCAUC	$\mathcal{E}$-DCAUC
Deleting candidates	$\mathcal{E}$-CCDC	$\mathcal{E}$-DCDC
Partition of candidates	$\mathcal{E}$-CCPC-TE	$\mathcal{E}$-DCPC-TE[1]
	$\mathcal{E}$-CCPC-TP	$\mathcal{E}$-DCPC-TP[2]
Partition of candidates with run-off	$\mathcal{E}$-CCRPC-TE	$\mathcal{E}$-DCRPC-TE[1]
	$\mathcal{E}$-CCRPC-TP	$\mathcal{E}$-DCRPC-TP[2]

[1] DCRPC-TE $=$ DCPC-TE in the unique-winner and the nonunique-winner model.

[2] DCRPC-TP $=$ DCPC-TP in the nonunique-winner model.

Overview of the Common Control Problems

Table: Overview of the voter control problems for voting system $\mathcal{E}$

Control by	Constructive	Destructive
Adding voters	$\mathcal{E}$-CCAV	$\mathcal{E}$-DCAV
Deleting voters	$\mathcal{E}$-CCDV	$\mathcal{E}$-DCDV
Partition of voters	$\mathcal{E}$-CCPV-TE $\mathcal{E}$-CCPV-TP	$\mathcal{E}$-DCPV-TE $\mathcal{E}$-DCPV-TP

Immunity and Susceptibility

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathfrak{CT}$ be a control type.

1. We say a voting system is *immune to $\mathfrak{CT}$* if it is impossible for the chair to make the given candidate
   - the unique winner in the constructive case and
   - not a unique winner in the destructive case,

   respectively, via exerting control of type $\mathfrak{CT}$.
2. We say a voting system is *susceptible to $\mathfrak{CT}$* if it is not immune to $\mathfrak{CT}$.

Resistance and Vulnerability

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let $\mathfrak{CT}$ be a control type.

A voting system that is susceptible to $\mathfrak{CT}$ is said to be

1. *vulnerable to $\mathfrak{CT}$* if the control problem corresponding to $\mathfrak{CT}$ can be solved in polynomial time, and
2. *resistant to $\mathfrak{CT}$* if the control problem corresponding to $\mathfrak{CT}$ is NP-hard.

Links Between Susceptibility Cases

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

1. *A voting system is susceptible to constructive control by adding candidates if and only if it is susceptible to destructive control by deleting candidates.*
2. *A voting system is susceptible to constructive control by deleting candidates if and only if it is susceptible to destructive control by adding candidates.*
3. *A voting system is susceptible to constructive control by adding voters if and only if it is susceptible to destructive control by deleting voters.*
4. *A voting system is susceptible to constructive control by deleting voters if and only if it is susceptible to destructive control by adding voters.*

Links Between Susceptibility Cases

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

1. *If a voting system is susceptible to constructive control by partition of voters (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
2. *If a voting system is susceptible to constructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
3. *If a voting system is susceptible to constructive control by partition of voters in model TE, then it is susceptible to constructive control by deleting voters.*
4. *If a voting system is susceptible to destructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to destructive control by deleting candidates.*

Links Between Susceptibility Cases

Definition

A voting system is *voiced* if in any election that has exactly one candidate, that candidate is always a (and thus, the unique) winner.

Theorem

1. *If a voiced voting system is susceptible to destructive control by partition of voters (in model TE or TP), then it is susceptible to destructive control by deleting voters.*
2. *Each voiced voting system is susceptible to constructive control by deleting candidates.*
3. *Each voiced voting system is susceptible to destructive control by adding candidates.*

Links Between Susceptibility Cases

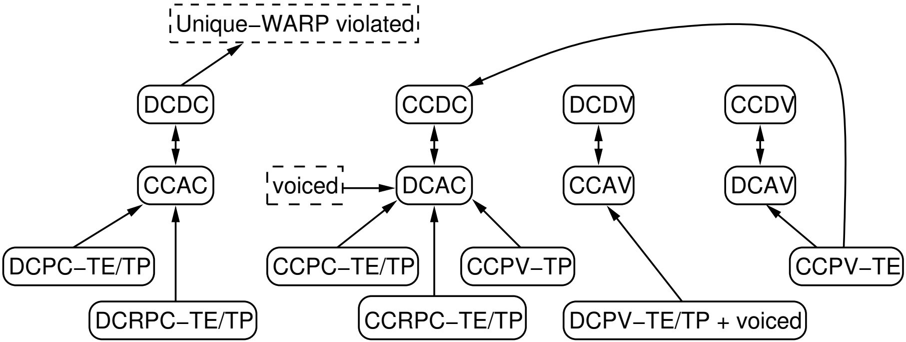

Figure: Links between susceptibility results for various control types

Control Complexity of Plurality and Condorcet Voting

	Plurality		Condorcet	
Control by	Constructive	Destructive	Constructive	Destructive
Adding Candidates	R(esistant)	**R**	I(mmune)	**V(ulnerable)**
Deleting Candidates	R	**R**	V	**I**
Partition of Candidates	TE: R TP: R	**TE: R** **TP: R**	V	**I**
Run-off Partition of Candidates	TE: R TP: R	**TE: R** **TP: R**	V	**I**
Adding Voters	V	**V**	R	**V**
Deleting Voters	V	**V**	R	**V**
Partition of Voters	**TE: V** **TP: R**	**TE: V** **TP: R**	R	**V**

Boldface results are due to Hemaspaandra, Hemaspaandra, and Rothe (2007).
Nonboldface results are due to Bartholdi, Tovey, and Trick (1992).

Some Susceptibility Cases for Plurality

Example (DCPC/DCRPC-TE/TP and DCDC)

Let $C = \{a, b, c, d\}$ be the candidate set, and let V consist of:

$3\times$	c	a	b	d
$2\times$	a	d	b	c
$2\times$	b	d	a	c

$\Longrightarrow$ c is the unique plurality winner in (C, V).

Partition C into $C_1 = \{a, c\}$ and $C_2 = \{b, d\}$.

Then a is the unique plurality winner in (C_1, V), so c is dethroned.

Thus plurality is susceptible to DCPC/DCRPC-TE/TP ...

... and, by our link among susceptibility cases, also to DCDC.

Some Susceptibility Cases for Plurality

Example (CCPV-TP/TE)

Let $C = \{a, b, c\}$ be the candidate set, and let V consist of:

$3\times$ (say, u_1, u_2, u_3)	a	c	b
$2\times$ (say, v_1, v_2)	b	a	c
$3\times$ (say, w_1, w_2, w_3)	c	a	b

$\Longrightarrow$ a and c are the plurality winners in (C, V).

Partition V into $V_1 = \{u_1, u_2, w_1, w_2, w_3\}$ and $V_2 = \{u_3, v_1, v_2\}$.

Then c is the unique plurality winner in (C, V_1), b is the unique plurality winner in (C, V_2), and c wins the run-off against b.

Thus plurality is susceptible to CCPV-TP/TE.

Some Susceptibility Cases for Plurality

Example (DCPV-TP/TE)

Let $C = \{a, b, c\}$ be the candidate set, and let $V' = V \cup \{v_3, w_4\}$ be:

$3\times$ (say, u_1, u_2, u_3)	a	c	b
$3\times$ (say, v_1, v_2, v_3)	b	a	c
$4\times$ (say, w_1, w_2, w_3, w_4)	c	a	b

$\Longrightarrow$ c is the unique plurality winner in (C, V').

Partition V' into $V_1' = \{u_1, u_2, u_3, w_1, w_2\}$ and $V_2' = \{v_1, v_2, v_3, w_3, w_4\}$.

Then a is the unique plurality winner in (C, V_1), b is the unique plurality winner in (C, V_2), and a wins the run-off against b, so c is dethroned.

Thus plurality is susceptible to DCPV-TP/TE.

Hitting Set

Definition

Name: HITTING SET.

Given:
- A set $B = \{b_1, b_2, \ldots, b_m\}$,
- a family $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets S_i of B, and
- a positive integer k.

Question: Does $\mathcal{S}$ have a hitting set of size at most k? That is, is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i, $S_i \cap B' \neq \emptyset$?

Hitting Set

Example

Suppose there are

- $m = 4$ students: $B = \{b_1, b_2, b_3, b_4\}$ and
- $n = 5$ courses: $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$ with

$$
\begin{aligned}
S_1 &= \{b_1, b_2\} & S_2 &= \{b_3, b_4\} \\
S_3 &= \{b_1, b_4\} & S_4 &= \{b_2, b_3\} \\
S_5 &= \{b_2, b_4\}
\end{aligned}
$$

$B' = \{b_2, b_4\}$ is a hitting set of size 2.

However, there is no hitting set of size 1 because S_1 and S_2 (respectively, S_3 and S_4) are disjoint.

Destructive Candidate Control in Plurality Voting

Construction: Given a HITTING SET instance $(B, \mathcal{S}, k)$, where $B = \{b_1, b_2, \ldots, b_m\}$, $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$, and $k \leq m$, construct the following election:

- The candidate set is $C = B \cup \{c, w\}$.
- The voter list V is defined as follows:
  1. $2(m-k) + 2n(k+1) + 4$ voters of the form $c\ w\ \cdots$, where "$\cdots$" means that the remaining candidates follow in an arbitrary order.
  2. $2n(k+1) + 5$ voters of the form $w\ c\ \cdots$.
  3. For each i, $1 \leq i \leq n$, there are $2(k+1)$ voters of the form $S_i\ c\ \cdots$, where "S_i" denotes the elements of S_i in some arbitrary order.
  4. For each j, $1 \leq j \leq m$, two voters of the form $b_j\ w\ \cdots$.

Destructive Candidate Control in Plurality Voting

Lemma (Hemaspaandra, Hemaspaandra, and Rothe (2007))

If B' is a hitting set of $\mathcal{S}$ of size k, then w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$.

Proof: If B' is a hitting set of $\mathcal{S}$ of size k, then in the election $(B' \cup \{c, w\}, V)$, we have

$$
\begin{array}{rcll}
\mathit{score}(c) & = & 2(m-k) + 2n(k+1) + 4, & \text{from voter group (1)} \\
\mathit{score}(w) & = & 2n(k+1) + 5 + 2(m-k), & \text{from voter groups (2) and (4)} \\
\mathit{score}(b_j) & \leq & 2n(k+1) + 2 \quad \text{for each } j, & \text{from voter groups (3) and (4).}
\end{array}
$$

It follows that w is the unique plurality winner of $(B' \cup \{c, w\}, V)$. ❑

Destructive Candidate Control in Plurality Voting

Lemma (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let $D \subseteq B \cup \{w\}$. If c is not a unique plurality winner of election $(D \cup \{c\}, V)$, then there exists a set $B' \subseteq B$ such that

1. $D = B' \cup \{w\}$,
2. *w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$, and*
3. *B' is a hitting set of $\mathcal{S}$ of size less than or equal to k.*

Destructive Candidate Control in Plurality Voting

Proof: Let $D \subseteq B \cup \{w\}$ and suppose that c is not a unique plurality winner of election $(D \cup \{c\}, V)$.

First note that for all $b \in D \cap B$, $score(b) < score(c)$ in $(D \cup \{c\}, V)$.

Since c is not a unique plurality winner of $(D \cup \{c\}, V)$, it follows that $w \in D$ and $score(w) \geq score(c)$.

Let $B' \subseteq B$ be such that $D = B' \cup \{w\}$.

Then $D \cup \{c\} = B' \cup \{c, w\}$.

Since $score(w)$ is odd and $score(c)$ is even, it follows that w is the unique plurality winner of $(B' \cup \{c, w\}, V)$.

This proves the first two properties stated.

Destructive Candidate Control in Plurality Voting

To prove the third property, note that in $(B' \cup \{c, w\}, V)$, we have

$$\begin{aligned} score(w) &= 2n(k+1) + 5 + 2(m - \|B'\|) \quad \text{and} \\ score(c) &= 2(m-k) + 2n(k+1) + 4 + 2(k+1)\ell, \end{aligned}$$

where ℓ is the number of sets in $\mathcal{S}$ that are not hit by B' (i.e., that have an empty intersection with B').

Since $score(c) \leq score(w)$, it follows that

$$2(m-k) + 2(k+1)\ell \leq 1 + 2(m - \|B'\|),$$

which implies $(k+1)\ell + \|B'\| - k \leq 0$. So $\ell = 0$.

Thus B' is a hitting set of $\mathcal{S}$ of size at most k. ❑

Destructive Candidate Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

$\mathcal{S}$ has a hitting set of size less than or equal to k if and only if destructive control by adding candidates can be executed for the election with qualified candidates $\{c, w\}$, spoiler candidates B, distinguished candidate c, and voter list V.

Proof: **($\Rightarrow$)** If $\mathcal{S}$ has a hitting set of size less than or equal to k, then since $k \leq m$, $\mathcal{S}$ has a hitting set of size k.

By the first lemma, w is the unique plurality winner of $(B' \cup \{c, w\}, V)$.

($\Leftarrow$) follows directly from the second lemma. ❑

Destructive Candidate Control in Plurality Voting

Corollary: Plurality voting is resistant to destructive control by adding candidates, i.e., Plurality-DCAUC and Plurality-DCAC are NP-hard.

Proof: By the previous theorem, our construction on slide 25 shows:

$$
\begin{aligned}
\text{HITTING SET} \quad &\leq_{\mathrm{m}}^{\mathrm{p}} \quad \text{Plurality-DCAUC}; \\
\text{HITTING SET} \quad &\leq_{\mathrm{m}}^{\mathrm{p}} \quad \text{Plurality-DCAC},
\end{aligned}
$$

which proves the corollary. ❑

Destructive Candidate Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

$\mathcal{S}$ has a hitting set of size at most k if and only if the election with candidate set C, distinguished candidate c, and voter list V can be destructively controlled by deleting at most $m - k$ candidates.

Proof: **($\Rightarrow$)** Let B' be a hitting set of $\mathcal{S}$ of size k. By the first lemma, c is not a unique plurality winner of the election $(B' \cup \{c, w\}, V)$.

Since $B' \cup \{c, w\} = C \setminus (B \setminus B')$, $\|B\| = m$, and $\|B'\| = k$, the right-hand side of the equivalence follows.

Destructive Candidate Control in Plurality Voting

(⇐) Let $D \subseteq B \cup \{w\}$ be such that $\|D\| \leq m - k$, and suppose that c is not a unique plurality winner of $(C \setminus D, V)$.

Since $c \in C \setminus D$, it follows from the second lemma that

$$(C \setminus D) \setminus \{c\} = B' \cup \{w\},$$

where B' is a hitting set of $\mathcal{S}$ of size less than or equal to k. ❑

Corollary: Plurality voting is resistant to destructive control by deleting candidates. That is, Plurality-DCDC is NP-hard.

Proof: By the previous theorem, our construction on slide 25 shows:

$$\text{Hitting Set} \quad \leq_{\mathrm{m}}^{\mathrm{p}} \quad \text{Plurality-DCDC},$$

which proves the corollary. ❑

Destructive Candidate Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

$\mathcal{S}$ has a hitting set of size at most k if and only if the election with candidate set C, distinguished candidate c, and voter list V can be destructively controlled by partition of candidates (with and without run-off, and for each both in model TE and TP).

Proof: **($\Rightarrow$)** Let B' be a hitting set of $\mathcal{S}$ of size k.

Partition C into $C_1 = B' \cup \{c, w\}$ and $C_2 = B \setminus B'$.

By the first lemma, w is the unique plurality winner of (C_1, V), and c thus cannot win the election (C, V).

Destructive Candidate Control in Plurality Voting

($\Leftarrow$) Suppose that there exists a partition of candidates such that c is not a unique plurality winner of the two-stage election corresponding to that partition.

Then, certainly, there exists a set $D \subseteq B \cup \{w\}$ such that c is not a unique plurality winner of $(D \cup \{c\}, V)$.

By the second lemma, $\mathcal{S}$ has a hitting set of size at most k. ❑

Corollary: Plurality voting is resistant to destructive control by partition of candidates (with and without run-off, and for each both in model TE and TP). That is, Plurality-DCPC-TE, Plurality-DCPC-TP, Plurality-DCRPC-TE, and Plurality-DCRPC-TP are NP-hard.

Recall: Control Complexity of Plurality and Condorcet

	Plurality		Condorcet	
Control by	Constructive	Destructive	Constructive	Destructive
Adding Candidates	R(esistant)	**R**	I(mmune)	**V(ulnerable)**
Deleting Candidates	R	**R**	V	**I**
Partition of Candidates	TE: R TP: R	**TE: R** **TP: R**	V	**I**
Run-off Partition of Candidates	TE: R TP: R	**TE: R** **TP: R**	V	**I**
Adding Voters	V	**V**	R	**V**
Deleting Voters	V	**V**	R	**V**
Partition of Voters	**TE: V** **TP: R**	**TE: V** **TP: R**	R	**V**

Boldface results are due to Hemaspaandra, Hemaspaandra, and Rothe (2007).
Nonboldface results are due to Bartholdi, Tovey, and Trick (1992).

Voter Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

1. *Plurality voting is vulnerable/certifiably-vulnerable to destructive control both by adding voters and by deleting voters.*

   "Certifiably-vulnerable" means the chair cannot only decide the problems Plurality-DCAV and Plurality-DCDV in polynomial time, but can even produce in polynomial time a "best possible" control action. Certifiable vulnerability implies vulnerability.
   (In particular, the "k" may be dropped from the problem instance.)

2. *In model TE, plurality voting is vulnerable/certifiably-vulnerable to constructive and destructive control by partition of voters.*
   That is, Plurality-CCPV-TE and Plurality-DCPV-TE is in P.

Destructive Control by Adding Voters in Plurality

Proof:

1. (a) Plurality voting is certifiably-vulnerable to destructive control by adding voters: "Smart Greedy"

   Given (C, c, V, W) as in DCAV (without k):

   - If c already is not a unique plurality winner in (C, V), adding no voters accomplishes our goal, and we are done.
   - Otherwise, sort all $d_i \in C - \{c\}$ by decreasing deficit, i.e., letting $\mathit{diff}(d_i)$ denote d_i's deficit of first-place votes needed to tie c, we have

   $$\mathit{diff}(d_1) \leq \mathit{diff}(d_2) \leq \cdots \leq \mathit{diff}(d_{\|C\|-1}).$$

   - For $i = 1, 2, \ldots, \|C\| - 1$, if

   $$\|\{w \in W \mid w\text{'s first choice is } d_i\}\| \geq \mathit{diff}(d_i),$$

   then add $\mathit{diff}(d_i)$ of these unregistered voters to ensure that d_i ties c (and c thus is not a unique winner) and halt.
   - If no iteration was successful, output "control impossible" and halt.

Destructive Control by Deleting Voters in Plurality Voting

(b) Plurality voting is certifiably-vulnerable to destructive control by deleting voters: "Dumb Greedy"

Given (C, c, V) as in DCDV (without k):

- If $C = \{c\}$, then output "control impossible" and halt;
- else if c already is not a unique plurality winner in (C, V), deleting no voters accomplishes our goal, and we are done.
- If every candidate other than c gets zero first-place votes, then output "control impossible" and halt.
- Otherwise, let d be the candidate closest to c in first-place votes, and let $diff(d)$ denote d's deficit of first-place votes needed to tie c.

  Deleting $diff(d)$ voters whose first choice is c assures that c is not a unique winner, and this is the fewest deletions that can achieve that.

Constructive Control by Partition of Voters (TE) in Plurality Voting

2. Plurality voting is certifiably-vulnerable to constructive control by partition of voters in model TE: Plurality-CCPV-TE is in P

   - Let (C, c, V) be given as in CCPV-TE.
   - For any partition (V_1, V_2) of V, let *Nominees*(C, V_i), $i \in \{1, 2\}$, denote the set of candidates who are nominated by the subcommittee V_i (with candidates C) for the run-off in model TE.

Constructive Control by Partition of Voters (TE) in Plurality Voting

Consider the following cases (Cases 3 and 5 need not be disjoint):

Case 1: $\mathit{Nominees}(C, V_1) = \{c\}$ and $\mathit{Nominees}(C, V_2) = \emptyset$ due to $V_2 = \emptyset$.

Case 2: $\mathit{Nominees}(C, V_1) = \{c\}$ and $\mathit{Nominees}(C, V_2) = \{c\}$.

Case 3: $\mathit{Nominees}(C, V_1) = \{c\}$ and $\mathit{Nominees}(C, V_2) = \emptyset$ due to c and d (and possibly additional other candidates) tying, where $c \neq d$.

Case 4: $\mathit{Nominees}(C, V_1) = \{c\}$ and $\mathit{Nominees}(C, V_2) = \{d\}$, $c \neq d$.

Case 5: $\mathit{Nominees}(C, V_1) = \{c\}$ and $\mathit{Nominees}(C, V_2) = \emptyset$ due to d and e (and possibly additional other candidates) tying, where $c \neq d \neq e \neq c$.

Constructive Control by Partition of Voters (TE) in Plurality Voting

Given (C, c, V) as in CCPV-TE:

- If c is the unique plurality winner in (C, V) (thus catching Cases 1, 2, and 3), then output $(V, \emptyset)$ as a successful partition and halt;
- else if $\|C\| = 2$, then output "control impossible" (which in this context means that making c a unique winner is impossible) and halt.
- Otherwise, first try to make Case 4 hold in the Case 4 Loop;
- and then, if that fails, try to make Case 5 hold in the Case 5 Loop.
- Otherwise (i.e., if the Case 5 Loop was not successful either), c cannot win, so we output "control impossible" and halt.

Constructive Control by Partition of Voters (TE) in Plurality Voting

Case 4 Loop:

- For each $d \in C$, $d \neq c$, such that c beats d in a pairwise plurality election by the voters in V, do the following:
  If it holds that, for each $e \in C$ with $c \neq e \neq d$,

  $$score(e) \leq score(c) + score(d) - 2,$$

  then output (V_1, V_2) as a successful partition and halt, where
  - V_1 consists of
    - all $score(c)$ voters whose first choice is c and
    - exactly $\min(score(e), score(c) - 1)$ of the voters whose first choice is e, and
  - where $V_2 = V \setminus V_1$.

Constructive Control by Partition of Voters (TE) in Plurality Voting

Explanation of the Case 4 Loop:

- V_1 has
  - $score(c)$ votes for c and
  - $\min(score(e), score(c) - 1) < score(c)$ votes for e.

$\Longrightarrow c$ is nominated in (C, V_1).

- In V_2, in the worst case we have

$$score(e) \leq score(d) - 1.$$

$\Longrightarrow d$ is nominated in (C, V_2).

- And c beats d in the run-off and is the only winner.

Constructive Control by Partition of Voters (TE) in Plurality Voting

Case 5 Loop:

- For each $d \in C$ and for each $e \in C$ such that $\|\{c, d, e\}\| = 3$ and $score(d) \leq score(e)$, do the following:
  If it holds that, for each $f \in C \setminus \{c\}$,

  $$score(f) \leq score(c) + score(d) - 1,$$

  then output (V_1, V_2) as a successful partition and halt, where
  - V_1 consists of
    - all $score(c)$ voters whose first choice is c,
    - exactly $score(e) - score(d)$ of the voters whose first choice is e, and
    - for all $f \in C \setminus \{c, d, e\}$, exactly $\min(score(f), score(c) - 1)$ of the voters whose first choice is f, and
  - where $V_2 = V \setminus V_1$.

Constructive Control by Partition of Voters (TE) in Plurality Voting

Explanation of the Case 5 Loop:

- V_1 has
  - $score(c)$ votes for c,
  - $score(e) - score(d) \leq score(c) - 1$ votes for e (because of $score(f) \leq score(c) + score(d) - 1$ with $f = e$), and
  - $\min(score(f), score(c) - 1) < score(c)$ votes for f.

  $\Longrightarrow$ c is nominated in (C, V_1).
- In V_2, there are
  - $score(d)$ votes for d,
  - $score(d)$ votes for e, and
  - in the worst case $\leq score(d)$ votes for f (again because of $score(f) \leq score(c) + score(d) - 1$).

  $\Longrightarrow$ Due to TE, no one is nominated in (C, V_2).
- So, c alone takes part in and wins the run-off. ❑

Destructive Control by Partition of Voters (TE) in Plurality Voting

2. Plurality voting is certifiably-vulnerable to destructive control by partition of voters in model TE: Plurality-DCPV-TE is in P

   - Let (C, c, V) be given as in DCPV-TE.
   - If $C = \{c\}$, output "control impossible" and halt, as c must win;
   - else if c already is not a unique plurality winner, output $(V, \emptyset)$ as a successful partition and halt.
   - Otherwise, check if every voter's first choice is c or if $\|C\| = 2$, and if one of these two conditions is true, output "control impossible" and halt, since c cannot help but win.

Destructive Control by Partition of Voters (TE) in Plurality Voting

- Let d be a candidate who other than c got the most first-place votes, and let e be a candidate who other than c and d got the most first-place votes.
- We can certainly dethrone c if

$$score(c) \leq score(d) + score(e). \tag{1}$$

  Namely, if (1) holds, we output (V_1, V_2) as a successful partition and halt, where
  - V_1 consists of
    - all $score(d)$ voters whose first choice is d and
    - exactly $score(d)$ voters whose first choice is c (recall that in the current case we already know that $score(c) > score(d)$), and
  - where $V_2 = V \setminus V_1$.

Destructive Control by Partition of Voters (TE) in Plurality Voting

- In (C, V_1), there are
  - $score(d)$ votes for d,
  - $score(d)$ votes for c, and
  - no one has a higher score.

  According to TE, no one will be nominated in (C, V_1).

- In (C, V_2), there are
  - $score(c) - score(d) \leq score(e)$ votes for c by (1),
  - $score(e)$ votes for e, and
  - no one has a higher score.

  $\Longrightarrow$ Either e or no one will be nominated in (C, V_2).

- $\Longrightarrow$ c is dethroned.

Destructive Control by Partition of Voters (TE) in Plurality Voting

- On the other hand, if Equation (1) is not satisfied, we have

$$score(c) > score(d) + score(e),$$

so in any partition (V_1, V_2), c wins in one of (C, V_1) or (C, V_2).

Thus, it is impossible to make c lose in both subcommittees.

- If c is nominated by both subcommittees (in model TE), c trivially is the unique winner of the final run-off.
- So, we now check if it is possible for c to fulfill Property A:
  - c wins in *exactly* one subcommittee, and
  - yet can be made to *not be the unique winner* of the final run-off.

Destructive Control by Partition of Voters (TE) in Plurality Voting

For this to happen, it is (given the case we are in) a necessary and sufficient condition that there exists some candidate $d \neq c$ such that:

- d ties or beats c in a pairwise plurality election, and
- for each candidate e, $c \neq e \neq d$, we have that $score(e) < score(c) + score(d) - 2$.

Call this Property B. We show Property A $\iff$ Property B.

(⇒) That some $d \neq c$ ties or beats c in $(\{c, d\}, V)$ is clear.

But if $score(e) \geq score(c) + score(d) - 2$ for some e, $c \neq e \neq d$, then with $score(c) > score(d) + score(e)$ (i.e., (1) does not hold), we have:

$$score(e) \geq score(c) + score(d) - 2 > 2 \cdot score(d) + score(e) - 2,$$

so $1 > score(d)$, i.e., $score(d) = 0$. However, such a d cannot be the unique winner of the other subelection: **contradiction!**

Destructive Control by Partition of Voters (TE) in Plurality Voting

($\Leftarrow$) By Property B, there is some $d \neq c$ that ties or beats c in the run-off, so c *is not the unique winner* of the run-off.

We show that there is a partition (V_1, V_2), $V = V_1 \cup V_2$, such that c wins in (C, V_1) and d wins in (C, V_2).

- Define V_1 to consist of
  - all $score(c)$ votes for c and
  - for each e, $c \neq e \neq d$, $\min(score(c) - 1, score(e))$ votes for e, and
- let $V_2 = V \setminus V_1$.

Then c wins in (C, V_1).

In (C, V_2), it follows from $score(e) < score(c) + score(d) - 2$ that $score(e) < score(d) - 1$, so d wins in (C, V_2).

Hence, c wins in *exactly* one subelection, proving Property A.

Destructive Control by Partition of Voters (TE) in Plurality Voting

We can in polynomial time brute-force check whether Property B holds for some candidate d, and if so, output (V_1, V_2) as a successful partition and halt, where V_1 consists of

- all $score(c)$ voters whose first choice is c and,
- for each candidate e with $c \neq e \neq d$, of exactly

  $$\min(score(c) - 1, score(e))$$

  voters whose first choice is e,

and where $V_2 = V \setminus V_1$.

Finally, if Property B cannot be satisfied for any d, output "control impossible" and halt. ❑

Voter Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

1. *Plurality voting is resistant to constructive control by partition of voters in model TP. That is, Plurality-*CCPV-TP *is* NP*-hard.*
2. *Plurality voting is resistant to destructive control by partition of voters in model TP. That is, Plurality-*DCPV-TP *is* NP*-hard.*

Proof: **Idea:**

- We use a restricted version of HITTING SET and
- apply the construction from the candidate control cases to it.

Restricted Hitting Set

Definition

Name: RESTRICTED HITTING SET (RHS).

Given:
- A set $B = \{b_1, b_2, \ldots, b_m\}$,
- a family $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets S_i of B, and
- a positive integer k such that $n(k+1)+1 \leq m-k$.

Question: Does $\mathcal{S}$ have a hitting set of size at most k?
That is, is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i, $S_i \cap B' \neq \emptyset$?

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Restricted Hitting Set is NP*-complete.*

Proof: Exercise. ❑

Restricted Hitting Set

Example

The inequality of RESTRICTED HITTING SET:

$$n(k+1)+1 \leq m-k$$

is satisfied, for example, by $k = 2$, $m = 12$, $n = 3$.

Positive RHS Instance

$B = \{b_1, b_2, \ldots, b_{12}\}$, $k = 2$, and $\mathcal{S} = \{S_1, S_2, S_3\}$ with

$$\begin{aligned} S_1 &= \{b_1, b_2, b_3, b_4\}, \\ S_2 &= \{b_4, b_5, b_6, b_7, b_8\}, \\ S_3 &= \{b_9, b_{10}, b_{11}, b_{12}\}. \end{aligned}$$

Hitting set: $B' = \{b_4, b_9\}$.

Negative RHS Instance

$B = \{b_1, b_2, \ldots, b_{12}\}$, $k = 2$, and $\mathcal{S} = \{S_1, S_2, S_3\}$ with

$$\begin{aligned} S_1 &= \{b_1, b_2, b_3, b_4\}, \\ S_2 &= \{b_5, b_6, b_7, b_8\}, \\ S_3 &= \{b_9, b_{10}, b_{11}, b_{12}\}. \end{aligned}$$

No hitting set.

Voter Control in Plurality Voting

Construction: Given a HITTING SET instance $(B, \mathcal{S}, k)$, where $B = \{b_1, b_2, \ldots, b_m\}$, $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$, and $k \leq m$, construct the following election:

- The candidate set is $C = B \cup \{c, w\}$.
- The voter list V is defined as follows:
  1. $2(m-k) + 2n(k+1) + 4$ voters of the form $c\ w\ \cdots$, where "$\cdots$" means that the remaining candidates follow in an arbitrary order.
  2. $2n(k+1) + 5$ voters of the form $w\ c\ \cdots$.
  3. For each i, $1 \leq i \leq n$, there are $2(k+1)$ voters of the form $S_i\ c\ \cdots$, where "S_i" denotes the elements of S_i in some arbitrary order.
  4. For each j, $1 \leq j \leq m$, two voters of the form $b_j\ w\ \cdots$.

Voter Control in Plurality Voting: Copying Is Stealing!

Voter Control in Plurality Voting

Lemma

In the election (C, V) from this construction, if $n(k+1)+1 \leq m-k$ then for every partition of V into V_1 and V_2, c is a plurality winner of (C, V_1) or of (C, V_2).

Proof: For a contradiction, suppose that c is a winner of neither (C, V_1) nor (C, V_2).

For each $U \subseteq V$ and for each $i \in C$, let $score_U(i)$ denote the number of first-place votes that i has in (C, U).

Let $x \in B \cup \{w\}$ be a winner of (C, V_1), and let $y \in B \cup \{w\}$ be a winner of (C, V_2).

Voter Control in Plurality Voting

Then

$$score_{V_1}(x) + score_{V_2}(y) \geq score_V(c) + 2. \qquad (2)$$

Since c's score in (C, V) is greater than that of any other candidate, we have $x \neq y$. It follows that

$$\begin{aligned} score_{V_1}(x) + score_{V_2}(y) &\leq score_V(w) + score_V(b_i) \\ &\leq 2n(k+1) + 5 + 2n(k+1) + 2 \\ &\leq 2n(k+1) + 5 + 2(m-k) \\ &= score_V(c) + 1, \end{aligned}$$

which contradicts Equation (2).

Thus, c is a winner of (C, V_1) or of (C, V_2). ❑

Voter Control in Plurality Voting

Lemma

In the election (C, V) from this construction, if $n(k+1)+1 \leq m-k$ then the following three statements are equivalent:

1. *$\mathcal{S}$ has a hitting set of size at most k.*
2. *V can be partitioned such that w is the unique plurality winner in the TP model.*
3. *V can be partitioned such that c is not a unique plurality winner in the TP model.*

Both lemmas together imply the theorem. ❑

Voter Control in Plurality Voting

Proof: of Lemma.

(1) $\Rightarrow$ (2): Let B' be a hitting set of $\mathcal{S}$ of size k.

Partition V into V_1 and V_2, where V_1 consists

- of one voter of the form $w\ c\ \cdots$ (group 2) and
- for every $b \in B'$ one voter of the form $b\ w\ \cdots$ (group 4),

and where $V_2 = V \setminus V_1$.

Then the candidates in $B' \cup \{w\}$ are the winners of (C, V_1) and move forward to the run-off in the TP model, and c is the winner of (C, V_2).

By the lemma on slide 26, w is the unique plurality winner of the final election $(B' \cup \{c, w\}, V)$.

Recall the Lemmas from Slides 26 and 27

Lemma (Slide 26: Hemaspaandra, Hemaspaandra, and Rothe (2007))

If B' is a hitting set of $\mathcal{S}$ of size k, then w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$.

Lemma (Slide 27: Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let $D \subseteq B \cup \{w\}$. If c is not a unique plurality winner of election $(D \cup \{c\}, V)$, then there exists a set $B' \subseteq B$ such that

1. $D = B' \cup \{w\}$,
2. *w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$, and*
3. *B' is a hitting set of $\mathcal{S}$ of size less than or equal to k.*

Voter Control in Plurality Voting

(2) $\Rightarrow$ (3): If w is the unique plurality winner for some partition of V in TP, then c cannot be the unique plurality winner for the same partition.

(3) $\Rightarrow$ (1): Suppose there is a partition of V such that c is not the unique plurality winner of the election in the TP model.

By the lemma on slide 59, c is a winner of one of the subelections and will thus participate in the final run-off.

It follows that c is not the unique winner of a run-off election involving c, i.e., c is not the unique winner in $(D \cup \{c\}, V)$, for some $D \subseteq B \cup \{w\}$.

By the lemma on slide 27, $\mathcal{S}$ has a hitting set of size at most k. ❑

Recall: Control Complexity of Plurality and Condorcet

	Plurality		Condorcet	
Control by	Constructive	Destructive	Constructive	Destructive
Adding Candidates	R(esistant)	**R**	I(mmune)	**V(ulnerable)**
Deleting Candidates	R	**R**	V	**I**
Partition of Candidates	TE: R TP: R	**TE: R** **TP: R**	V	**I**
Run-off Partition of Candidates	TE: R TP: R	**TE: R** **TP: R**	V	**I**
Adding Voters	V	**V**	R	**V**
Deleting Voters	V	**V**	R	**V**
Partition of Voters	**TE: V** **TP: R**	**TE: V** **TP: R**	R	**V**

Boldface results are due to Hemaspaandra, Hemaspaandra, and Rothe (2007).
Nonboldface results are due to Bartholdi, Tovey, and Trick (1992).

Immunity for Condorcet Voting

The unique version of the Weak Axiom of Revealed Preference (Unique-WARP): A unique winner among a collection of candidates always remains a unique winner among every subcollection of candidates that includes him or her.

Theorem (Bartholdi, Tovey, and Trick (1992))

Any voting system that satisfies Unique-WARP is immune to constructive control by adding candidates.

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Any voting system that satisfies Unique-WARP is immune to destructive control by deleting candidates and (in both TE and TP) to destructive control by partition and run-off partition of candidates.

Immunity for Condorcet Voting

Corollary

Condorcet voting is immune to

- *constructive control by adding candidates,*
- *destructive control by deleting candidates,*
- *destructive control by partition of candidates, and*
- *destructive control by run-off partition of candidates.*

Proof: Condorcet voting satisfies Unique-WARP. ❑

Susceptibility for Condorcet Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Condorcet voting is susceptible to all other types of control.

Proof: **Exercise.** ❑

Destructive Control in Condorcet Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Condorcet voting is vulnerable/certifiably-vulnerable to destructive control by

1. *adding candidates,*
2. *adding voters,*
3. *deleting voters, and*
4. *partition of voters.*

Destructive Control in Condorcet Voting: DCAC

Proof:

1. **Destructive control by adding candidates:**
   We are given
   - a set C of qualified candidates and
   - a distinguished candidate $c \in C$,
   - a set D of possible spoiler candidates, and
   - a list V of voters with preferences over $C \cup D$.

   If c already is not a Condorcet winner, adding no candidates accomplishes our goal, and we are done.

   Otherwise, if any spoiler candidate ties or beats c, add one such candidate and halt.

   Otherwise, output "control impossible" and halt.

Destructive Control in Condorcet Voting: DCAV

2. **Destructive control by adding voters:**
We are given
   - a set C of candidates and
   - a distinguished candidate $c \in C$,
   - a list V of registered voters, and
   - an additional list W of as yet unregistered voters (both V and W have preferences over C).

If $C = \{c\}$, then output “control impossible” and halt;

else if c already is not a Condorcet winner in the election (C, V), adding no voters accomplishes our goal, and we are done.

Destructive Control in Condorcet Voting: DCAV

Otherwise, define *surplus*(c, i) as the number of registered voters who prefer c to i minus the number of registered voters who prefer i to c.

For each candidate $i \neq c$, call i *lucky* if and only if *surplus*(c, i) is less than or equal to the number of unregistered voters who prefer i to c.

Example

Registered voters V			
1 voter	a	c	b
2 voters	c	a	b

Unregistered voters W			
1 voter	a	b	c
1 voter	b	c	a

surplus$(c, a) = 2 - 1 = 1$
surplus$(c, b) = 3 - 0 = 3$

Thus a is lucky, b is not. Hence, adding $a\ b\ c$ makes a and c tie.

Destructive Control in Condorcet Voting: DCAV

If there is at least one lucky candidate, then

- let d be a lucky candidate such that $\mathit{surplus}(c, d)$ is minimum, and
- add $\mathit{surplus}(c, d)$ unregistered voters who prefer d to c.

If there exists no lucky candidate, output “control impossible” and halt.

Destructive Control in Condorcet Voting: DCDV

3. **Destructive control by deleting voters:**
   We are given
   - a set C of candidates and
   - a distinguished candidate $c \in C$,
   - a list V of voters with preferences over C.

   If $C = \{c\}$, then output "control impossible" and halt;

   else if c already is not a Condorcet winner in the election (C, V), deleting no voters accomplishes our goal, and we are done.

   Otherwise, find a candidate d who comes closest to c (i.e., relative to whom the surplus of c is minimum), and delete $surplus(c, d)$ voters from V who prefer c to d.

   Now c and d tie, so c is dethroned.

Destructive Control in Condorcet Voting: DCPV

4. **Destructive control by partition of voters:**
   We are given
   - a set C of candidates and
   - a distinguished candidate $c \in C$,
   - a list V of voters with preferences over C.

   Checking the trivial cases: If $C = \{c\}$, output "control impossible" and halt, as c must win.

   Otherwise, if c already is not a Condorcet winner, output $(V, \emptyset)$ as a successful partition and halt.

   Otherwise, if $\|C\| = 2$, output "control impossible" and halt, since in this case c *is* the Condorcet winner, so c is preferred by a strict majority of votes to the other candidate and thus will win at least one subcommittee and also the run-off.

Destructive Control in Condorcet Voting: DCPV

Loop: If none of the trivial cases applies, for each $a, b \in C$ with $\|\{a, b, c\}\| = 3$, we test whether we can

- make a tie or beat c in (C, V_1) and
- make b tie or beat c in (C, V_2).

For each voter, we will now focus just on the ordering of a, b, and c. Denote the number of voters

- with order $c\ a\ b$ or $c\ b\ a$ by W_c,
- with order $a\ b\ c$ or $b\ a\ c$ by L_c,
- with order $a\ c\ b$ by S_a, and
- with order $b\ c\ a$ by S_b.

Destructive Control in Condorcet Voting: DCPV

If $W_c - L_c > S_a + S_b$, then this *a* and *b* are hopeless, so move on to consider the next *a* and *b* in the loop.

Otherwise, we have

$$W_c - L_c \quad \leq \quad S_a + S_b. \tag{3}$$

Output (V_1, V_2) as a successful partition and halt, where

- V_1 contains
  - all the S_a voters with order $a\ c\ b$, and
  - also $\min(W_c, S_a)$ voters contributing to W_c, and
- where $V_2 = V \setminus V_1$.

Destructive Control in Condorcet Voting: DCPV

In (C, V_1), a ties or beats c, since

- a gets S_a votes and
- c gets $\min(W_c, S_a)$ votes.

And in (C, V_2), b ties or beats c, since

- there are $S_b + L_c$ voters who prefer b to c, and
- there are $W_c - \min(W_c, S_a)$ voters who prefer c to b.

Thus, to prove that the construction works, we need that

$$S_b + L_c \quad \geq \quad W_c - \min(W_c, S_a),$$

which is equivalent to

$$S_b + \min(W_c, S_a) \quad \geq \quad W_c - L_c. \tag{4}$$

Destructive Control in Condorcet Voting: DCPV

But

- if $S_a \leq W_c$ then $\min(W_c, S_a) = S_a$, so by Equation (3),

$$S_b + \min(W_c, S_a) \geq W_c - L_c.$$

  That is, Equation (4) is implied by Equation (3).

- If $S_a > W_c$ then $\min(W_c, S_a) = W_c$, so Equation (4) follows immediately from the fact that $S_b + L_c \geq 0$.

Thus, b indeed ties or beats c in (C, V_2). ❑

Approval Voting

Definition

In *approval voting*, votes are represented by approval vectors in $\{0, 1\}^{\|C\|}$ (with respect to a fixed order of the candidates in C), where

- 0 stands for disapproval and
- 1 stands for approval.

Given an election (C, V) and a candidate $c \in C$, define

- the *approval score of c in (C, V)* (denoted by $score_{(C,V)}(c)$) as the number of c's approvals in (C, V), and
- all candidates with a largest approval score are the *approval winners in (C, V)*.

Approval Voting

Remark:

- An election may have more than one approval winner.
- Approval voting is a voiced voting system.
- Approval voting satisfies Unique-WARP. Hence, it is immune to
  - constructive control by adding candidates,
  - destructive control by deleting candidates,
  - and (in both TE and TP) to destructive control by partition and run-off partition of candidates.
- In addition, approval voting is immune to constructive control by partition and run-off partition of candidates in model TP: **Exercise.**

Control Complexity of Approval Voting

	Approval	
Control by	Construct.	Destruct.
Adding Candidates	**I**	**V**
Deleting Candidates	**V**	**I**
Partition of Candidates	**TE: V** **TP: I**	**TE: I** **TP: I**
Run-off Partition of Candidates	**TE: V** **TP: I**	**TE: I** **TP: I**
Adding Voters	**R**	**V**
Deleting Voters	**R**	**V**
Partition of Voters	**TE: R** **TP: R**	**TE: V** **TP: V**

All results are due to Hemaspaandra, Hemaspaandra, and Rothe (2007).

Susceptibility in Approval Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Approval voting is susceptible to

1. (a) *destructive control by partition of voters in models TE and TP,*
   (b) *destructive control by deleting voters, and*
   (c) *constructive control by adding voters.*
2. (a) *destructive control by adding voters and*
   (b) *constructive control by deleting voters.*
   (c) *constructive control by partition of voters in models TE and TP.*
3. (a) *destructive control by adding candidates and*
   (b) *constructive control by deleting candidates.*
   (c) *constructive control by partition of candidates and run-off partition of candidates, both in model TE.*

Susceptibility in Approval Voting

Proof: Let $C = \{a, b, c\}$. Votes are strings in $\{0, 1\}^3$, e.g.,

a	b	c
1	0	1
0	1	1

$\Longrightarrow$ $score(a) = 1$, $score(b) = 1$, $score(c) = 2$ $\Longrightarrow$ c alone wins.

1. (a) There are 10 votes in V:

	a	b	c
v_1, v_2, v_3, v_4	0	0	1
v_5, v_6, v_7	1	0	0
v_8, v_9, v_{10}	0	1	0

c is the unique approval winner with $score(c) = 4$. Partition V into $V_1 = \{v_1, v_2, v_5, v_6, v_7\}$ and $V_2 = V \setminus V_1$. Then a wins (C, V_1), b wins (C, V_2), and both the run-off. Susceptibility to DCPV-TE/TP.

Susceptibility in Approval Voting

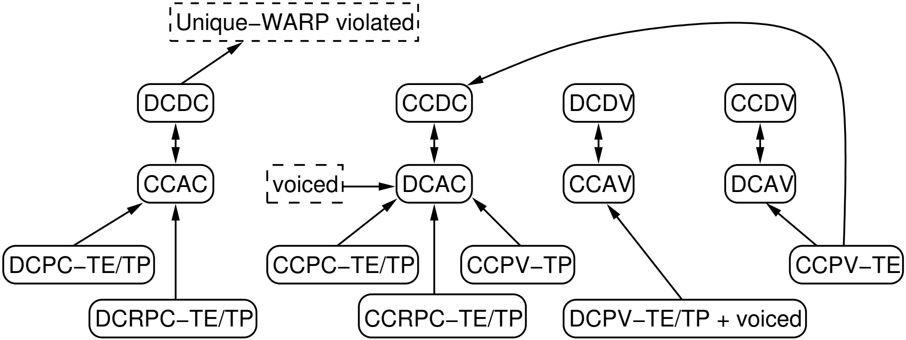

Figure: Links between susceptibility results for various control types

1. (b) follows from 1(a): Susceptibility to DCDV.

   (c) is equivalent to 1(b): Susceptibility to CCAV.

Susceptibility in Approval Voting

2. (a) One registered voter $v = 0\ 0\ 1$, one unregistered voter $w = 1\ 0\ 0$. Susceptibility to DCAV.

   (b) Dito: Susceptibility to CCDV.

   (c) There are 8 votes in V:

	a	b	c
v_1, v_2, v_3	1	0	0
v_4, v_5	0	1	0
v_6, v_7, v_8	0	0	1

a and c win in (C, V). Partition V into $V_1 = \{v_1, v_2, v_6, v_7, v_8\}$ and $V_2 = V \setminus V_1 = \{v_3, v_4, v_5\}$. Then c wins (C, V_1), b wins (C, V_2), and c the run-off.

Susceptibility to CCPV-TE/TP.

Susceptibility in Approval Voting

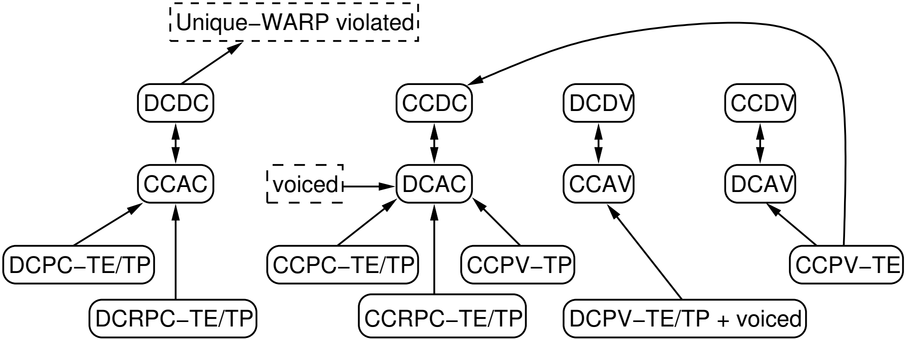

Figure: Links between susceptibility results for various control types

3. (a) follows from 2(c): Susceptibility to DCAC.
   (b) is equivalent to 3(a): Susceptibility to CCDC.

Susceptibility in Approval Voting

3. (c) There are 2 votes in V:

	a	b	c
v_1	1	1	1
v_2	1	1	0

In (C, V), a and b win, c doesn't.

Partition C into $C_1 = \{a, b\}$ and $C_2 = C \setminus C_1 = \{c\}$.

By the TE rule, no one from (C_1, V) enters the run-off, so c alone wins.

Susceptibility to CCPC-TE and CCRPC-TE. ❏

Candidate Control in Approval Voting: CCDC, CCPC-TE, CCRPC-TE, and DCAC

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Approval voting is vulnerable/certifiably-vulnerable to

1. *constructive control by deleting candidates,*
2. *constructive control by partition of candidates in model TE,*
3. *constructive control by run-off partition of candidates in model TE,*
4. *destructive control by adding candidates.*

Candidate Control in Approval Voting: CCDC

Proof: In the first three cases, we are given

- a set C of candidates and
- a distinguished candidate $c \in C$,
- a list V of voters represented by their approval vectors over C.

1. In the deleting candidates case,
   - if c already is the unique approval winner in the election (C, V), deleting no candidates accomplishes our goal, and we are done.
   - Otherwise, delete every candidate other than c who has at least as many Yes votes as c has in V and halt.

Candidate Control in Approval Voting: CCPC-TE

2. In the partition of candidates case,
   - if c already is the unique approval winner, then output $(\emptyset, C)$ as a successful partition and halt.
   - Otherwise, for each candidate $a \in C$, let y_a denote the number of Yes votes cast for a in V, and let

     $$Y = \max\{y_a \mid a \in C\}.$$

     Now, if there exists *exactly* one $a \in C \setminus \{c\}$ such that $y_a = Y$, then output "control impossible" and halt, since c cannot be made the unique winner in this case.

Candidate Control in Approval Voting: CC(R)PC-TE

- On the other hand, if there exist at least two distinct candidates in $C \setminus \{c\}$ whose number of Yes votes is Y, then output (C_1, C_2) with
  - $C_1 = C \setminus \{c\}$ and
  - $C_2 = \{c\}$

  as a successful partition and halt.

  This works, since in subelection (C_1, V) all candidates are eliminated according to the TE rule.

3. Note that the same algorithm also works for the run-off partition of candidates case in model TE.

Candidate Control in Approval Voting: DCAC

4. In this case, we are given
   - a set C of qualified candidates and
   - a distinguished candidate $c \in C$,
   - a set D of possible spoiler candidates, and
   - a list V of voters represented by their approval vectors over $C \cup D$.

   If c already is not a unique approval winner in (C, V), adding no candidates accomplishes our goal, and we are done.

   Otherwise, if there exists a spoiler candidate d who ties or beats c among the voters in V in Yes votes, add one such spoiler candidate and halt.

   Otherwise, output "control impossible" and halt. ❑

Recall: Control Complexity of Approval Voting

	Approval	
Control by	Construct.	Destruct.
Adding Candidates	**I**	**V**
Deleting Candidates	**V**	**I**
Partition of Candidates	**TE: V** **TP: I**	**TE: I** **TP: I**
Run-off Partition of Candidates	**TE: V** **TP: I**	**TE: I** **TP: I**
Adding Voters	**R**	**V**
Deleting Voters	**R**	**V**
Partition of Voters	**TE: R** **TP: R**	**TE: V** **TP: V**

All results are due to Hemaspaandra, Hemaspaandra, and Rothe (2007).

Constructive Voter Control in Approval Voting: CCAV, CCDV, CCPV-TP, and CCPV-TE

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Approval voting is resistant to constructive control by

1. *adding voters,*
2. *deleting voters,*
3. *partition of voters in model TP, and*
4. *partition of voters in model TE.*

Constructive Voter Control in Approval Voting

Proof: To prove NP-hardness of these four control problems, we reduce from the following NP-complete problem:

Name: EXACT COVER BY THREE-SETS (X3C).

Given:
- A set $B = \{b_1, b_2, \ldots, b_m\}$, $m = 3k$, $k \geq 1$, and
- a collection $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$.

Question: Is there a subcollection $\mathcal{S}' \subseteq \mathcal{S}$ such that each element of B occurs in exactly one set in $\mathcal{S}'$?
In other words, does there exist an index set $I \subseteq \{1, 2, \ldots, n\}$ with $\|I\| = k$ such that $\bigcup_{i \in I} S_i = B$?

Constructive Voter Control in Approval Voting: CCAV

1. Given an instance $(B, \mathcal{S})$ of X3C, where
   - $B = \{b_1, b_2, \ldots, b_m\}$, $m = 3k$, $k > 1$,
   - $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$, and
   - $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$,

   construct the following instance of CCAV for approval voting:
   - The candidate set is $C = B \cup \{w\}$, where w is the distinguished candidate.
   - V consists of $k - 2$ registered voters who each approve of $b_1, b_2, \ldots, b_m$ and disapprove of w.
   - W consists of n unregistered voters: For each i, $1 \leq i \leq n$, there is one voter in W who approves of w and the three candidates in S_i, and who disapproves of all other candidates.

Constructive Voter Control in Approval Voting: CCAV

We claim that $\mathcal{S}$ contains an exact cover for B if and only if w can be made the unique approval winner by adding at most k voters.

(⇒) Simply add the k voters from W that correspond to the exact cover for B. Then

- w has k Yes votes and
- every $b \in B$ has $(k-2)+1 = k-1$ Yes votes,

so w is the unique approval winner.

Constructive Voter Control in Approval Voting: CCAV

($\Leftarrow$) Suppose that w can be made the unique approval winner by adding at most k voters.

Then we clearly

- need to add exactly k voters and
- every $b \in B$ can gain at most one Yes vote.

Since each voter in W casts three Yes votes for candidates in B, it follows that every $b \in B$ gains exactly one Yes vote.

Thus, the k added voters correspond to an exact cover for B.

Constructive Voter Control in Approval Voting: CCDV

2. Let an instance $(B, \mathcal{S})$ of X3C be given, where
   - $B = \{b_1, b_2, \ldots, b_m\}$, $m = 3k$, $k > 0$,
   - $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$, and
   - $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$.

   For each j, $1 \leq j \leq m$, let

$$\ell_j = \|\{S_i \in \mathcal{S} \mid b_j \in S_i\}\|.$$

   Construct the following election: The candidate set is

$$C = B \cup \{w\},$$

   where w is the distinguished candidate.

Constructive Voter Control in Approval Voting: CCDV

The voter list V consists of the following voters:

- For each i, $1 \leq i \leq n$, there is one voter in V who approves of all candidates in S_i and who disapproves of all other candidates.
- There are n voters $v_1, v_2, \ldots, v_n$ in V such that, for each i, $1 \leq i \leq n$,
  - v_i approves of w, and
  - v_i approves of b_j if and only if $i \leq n - \ell_j$.

Note that the election (C, V) has the property that all candidates have n Yes votes.

Constructive Voter Control in Approval Voting: CCDV

We claim that $\mathcal{S}$ contains an exact cover for B if and only if w can be made the unique approval winner by deleting at most k voters.

($\Rightarrow$) Simply delete the k voters from V that correspond to an exact cover for B.

Then every $b \in B$ loses one Yes vote, leaving w the unique approval winner.

Constructive Voter Control in Approval Voting: CCDV

($\Leftarrow$) Suppose that w can be made the unique approval winner by deleting at most k voters.

Without loss of generality, we may assume that none of the deleted voters approves of w.

So, we assume that only voters corresponding to S_i's have been deleted.

For w to have become the unique winner, every $b \in B$ must have lost at least one Yes vote.

It follows that the deleted voters correspond to a cover, and since the cover has size at most k, this must be an exact cover for B.

Constructive Voter Control in Approval: CCPV-TP

3. Let an instance $(B, \mathcal{S})$ of X3C be given, where
   - $B = \{b_1, b_2, \ldots, b_m\}$, $m = 3k$, $k > 0$,
   - $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$, and
   - $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$.

   Modify the construction from the CCDV proof as follows.

   The candidate set is

$$C = B \cup \{w, x, y\},$$

   where w is the distinguished candidate.

   This election will have the property that all candidates other than x have $n + k + 2$ Yes votes.

Constructive Voter Control in Approval: CCPV-TP

The voter list V consists of the following voters:

- For each i, $1 \leq i \leq n$, there is one voter in V who approves of y and of all members of S_i and who disapproves of all other candidates.
- There are n voters $v_1, v_2, \ldots, v_n$ in V such that, for each $i \leq n$,
  - v_i approves of w,
  - v_i disapproves of x and y, and
  - v_i approves of b_j if and only if $i \leq n - \ell_j$.
- There are $k + 1$ voters in V who approve of x and disapprove of all other candidates.
- Finally, there are $k + 2$ voters in V who disapprove of x and approve of all other candidates.

Constructive Voter Control in Approval: CCPV-TP

We claim that $\mathcal{S}$ contains an exact cover for B if and only if w can be made the unique approval winner by partition of voters in model TP.

(⇒) If $\mathcal{S}$ contains an exact cover for B, then

- let V_2 consist of the k voters corresponding to the sets in the cover and of all the $k+1$ voters who approve of only x, and
- let $V_1 = V \setminus V_2$.

Then

- w is the unique approval winner of (C, V_1),
- x is the unique approval winner of (C, V_2), and
- w wins the run-off against x.

Constructive Voter Control in Approval: CCPV-TP

($\Leftarrow$) Suppose that w can be made the unique approval winner by partition of voters in model TP.

Since w is the unique winner in the run-off, and since every candidate other than x is tied with w (each having $n + k + 2$ Yes votes in V), the only candidates that can participate in the run-off are w and x.

Since we are in the TP model,

- w must be the unique winner of one of the subelections and
- x must be the unique winner of the other subelection.

Constructive Voter Control in Approval: CCPV-TP

Let (V_1, V_2) be a partition of V such that w is the unique winner of (C, V_1) and such that x is the unique winner of (C, V_2).

As in the proof of the CCDV case, it follows that the voters corresponding to S_i's that are not in V_1 (i.e., that are in V_2) correspond to a cover.

Since x is the unique winner of (C, V_2) and x has $k + 1$ Yes votes, y can have at most k Yes votes in V_2.

It follows that there are at most k voters corresponding to S_i's in V_2.

Thus, there are exactly k such voters, and these voters correspond to an exact cover.

Constructive Voter Control in Approval: CCPV-TE

4. Let an instance $(B, \mathcal{S})$ of X3C be given, where
   - $B = \{b_1, b_2, \ldots, b_m\}$, $m = 3k$, $k > 0$,
   - $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$, and
   - $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$.

   Modify the construction from the CCPV-TP proof as follows.

   The candidate set is

$$C = B \cup \{w, x, y\} \cup \{z_1, \ldots, z_n\},$$

   where w is the distinguished candidate.

   This election will have the property that all candidates other than x and y have n Yes votes.

Constructive Voter Control in Approval: CCPV-TE

The voter set V consists of the following voters:

- For each i, $1 \leq i \leq n$, there is one voter in V who approves of y and of all members of S_i and disapproves of all other candidates.
- For each i, $1 \leq i \leq n$, there is one voter in V who approves of y and z_i and who disapproves of all other candidates.
- There are n voters $v_1, v_2, \ldots, v_n$ in V such that, for each $i \leq n$,
  - v_i approves of w,
  - v_i disapproves of x and y,
  - v_i approves of b_j if and only if $i \leq n - \ell_j$, and
  - v_i approves of z_j if and only if $i \neq n$.
- There are $n + k$ voters in V who approve of x and who disapprove of all other candidates.

Constructive Voter Control in Approval: CCPV-TE

We claim that $\mathcal{S}$ contains an exact cover for B if and only if w can be made the unique approval winner by partition of voters in model TE.

(⇒) If $\mathcal{S}$ contains an exact cover for B, then

- let V_2 consist of
  - the k voters corresponding to the sets in the cover and
  - of all the $n + k$ voters who approve of only x and
  - for each i, $1 \leq i \leq n$, of the voter who approves of only y and z_i.
- Let $V_1 = V \setminus V_2$.

Then w is the unique approval winner of (C, V_1), and x and y are tied for first place in (C, V_2) with $n + k$ Yes votes each.

Since we are in model TE, no candidates are nominated by (C, V_2), and w wins the run-off (and thus the election) by default.

Constructive Voter Control in Approval: CCPV-TE

(⇐) Suppose that w can be made the unique approval winner by partition of voters in model TE.

Since we are in model TE, w must be the unique winner of one of the subelections.

Let (V_1, V_2) be a partition of V such that w is the unique winner of (C, V_1).

As in the proof of the CCDV case, it follows that the voters corresponding to S_i's that are not in V_1 (i.e., that are in V_2) correspond to a cover.

Constructive Voter Control in Approval: CCPV-TE

Suppose there are more than k voters that correspond to S_i's in V_2.

Note that for each i, $1 \leq i \leq n$, the voter that approves of only y and z_i must also be in V_2 (for if it weren't, z_i would have at least as many Yes votes in V_1 as w). It follows that y has more than $n + k$ Yes votes in V_2.

But then y is the unique approval winner in V_2, since no other candidate has more than $n + k$ Yes votes in V.

Since y beats w in the run-off, this contradicts the fact that w wins the election.

It follows that there are at most k voters corresponding to S_i's in V_2.

Thus, there are exactly k such voters, and these voters correspond to an exact cover. ❑

Destructive Voter Control in Approval Voting: DCAV, DCDV, DCPV-TP, and DCPV-TE

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Approval voting is vulnerable/certifiably-vulnerable to destructive control by

1. *adding voters,*
2. *deleting voters, and*
3. *partition of voters in model TP and in model TE,*

Destructive Voter Control in Approval Voting: DCAV

Proof:

1. We are given
   - a set C of candidates,
   - a distinguished candidate $c \in C$,
   - a list V of registered voters, and
   - a list W of unregistered voters, both represented by their approval vectors over C.

   Checking the trivial cases:

   If $C = \{c\}$, then output "control impossible" and halt.

   Otherwise, if c already is not the unique approval winner in the election (C, V), adding no voters accomplishes our goal, and we are done.

Destructive Voter Control in Approval Voting: DCAV

Otherwise, for each candidate $i \neq c$, again define *surplus*(c, i) to be the number of Yes votes for c in V minus
the number of Yes votes for i in V.

Among all candidates j other than c (if any) such that there exist at least *surplus*(c, j) voters in W who vote Yes for j and No for c,

- let d be any such j for which *surplus*(c, j) is minimum, and
- add *surplus*(c, d) unregistered voters who vote Yes for d and No for c.

If no j satisfying the above conditions exists, then output “control impossible” and halt.

Destructive Voter Control in Approval Voting: DCDV

2. We are given
   - a set C of candidates,
   - a distinguished candidate $c \in C$, and
   - a list V of voters, represented by their approval vectors over C.

Checking the trivial cases:

If $C = \{c\}$, then output "control impossible" and halt.

Otherwise, if c already is not the unique approval winner in the election (C, V), deleting no voters accomplishes our goal, and we are done.

Destructive Voter Control in Approval Voting: DCDV

Otherwise,

- let d be a candidate among $C \setminus \{c\}$ for whom *surplus*(c, d) is minimum, and
- delete *surplus*(c, d) voters from V who vote Yes for c and No for d (such voters must exist, as they are what is causing the surplus in the first place).

Destructive Voter Control in Approval Voting: DCPV

3. We describe two polynomial-time algorithms for these two control problems, one for TE and one for TP. We are given
   - a set C of candidates,
   - a distinguished candidate $c \in C$, and
   - a list V of voters, represented by their approval vectors over C.

   Both algorithms proceed in the following three phases:

   1. **Checking the trivial cases:**
      If $C = \{c\}$, output "control impossible" and halt, as c must win.

      Otherwise, if c already is not the unique winner, output $(V, \emptyset)$ as a successful partition and halt.

      Otherwise, if $\|C\| = 2$, output "control impossible" and halt, since in this case c *is* the unique winner, so c will win in at least one subcommittee and will also win the run-off.

Destructive Voter Control in Approval Voting: DCPV

2. **Loop:**
   In this phase, if none of the trivial cases applies, we try to find a pair of candidates, *a* and *b*, that allows us to determine a successful partition of voters.

   This phase is described below, separately for TE and TP.

3. **Termination:**
   If in no loop iteration did we find an *a* and *b* that allowed us to output a partition of voters dethroning *c*, then output "control impossible" and halt.

Destructive Voter Control in Approval Voting: DCPV

For each voter in V, we focus just on his/her approval of a, b, and c, represented (in that order) as a vector from $\{0,1\}^3$.
Denote the number of voters with approval vector

- 001 by W_c,
- 110 by L_c,
- 100 by S_a,
- 010 by S_b,
- 101 by S_{ac}, and
- 011 by S_{bc}.

Voters with approval vectors 000 or 111 need not be considered, since they do not affect the difference of Yes votes among a, b, and c.

Destructive Voter Control in Approval Voting: DCPV

Loop in model TE:

For each $a, b \in C$ with $\|\{a, b, c\}\| = 3$, we test whether we can

- make a tie or beat c in (C, V_1) and
- make b tie or beat c in (C, V_2).

If $W_c - L_c > S_a + S_b$, then this a and b are hopeless, so move on to consider the next a and b in the loop.

Otherwise, we have

$$W_c - L_c \quad \leq \quad S_a + S_b. \tag{5}$$

Destructive Voter Control in Approval Voting: DCPV

Output (V_1, V_2) as a successful partition and halt, where

- V_1 contains all voters contributing to S_{ac} and S_a, and also $\min(W_c, S_a)$ voters contributing to W_c, and
- $V_2 = V \setminus V_1$.

In (C, V_1), a ties or beats c, since a gets

$$S_a - \min(W_c, S_a) \geq 0$$

more Yes votes than c.

And in (C, V_2), b ties or beats c, since b receives

$$S_b + L_c - (W_c - \min(W_c, S_a))$$

more Yes votes than c.

Destructive Voter Control in Approval Voting: DCPV

So, for the construction to work, we must argue that

$$S_b + L_c + \min(W_c, S_a) - W_c \geq 0.$$

That is, we need

$$W_c - L_c \quad \leq \quad \min(W_c, S_a) + S_b. \tag{6}$$

If $W_c < S_a$, Equation (6) follows trivially from the fact that $0 \leq L_c + S_b$.

And if $W_c \geq S_a$, Equation (6) follows immediately from Equation (5).

Destructive Voter Control in Approval Voting: DCPV

Loop in model TP:

For each $a, b \in C$ with $\|\{a, b, c\}\| = 3$, we test whether we can

- make a strictly beat c in (C, V_1) and
- make b strictly beat c in (C, V_2).

If $W_c - L_c > S_a + S_b - 2$ or $S_a = 0$ or $S_b = 0$, then this a and b are hopeless, so move on to consider the next a and b in the loop.

Otherwise, we have

$$W_c - L_c \quad \leq \quad S_a + S_b - 2 \tag{7}$$

and $S_a > 0$ and $S_b > 0$.

Destructive Voter Control in Approval Voting: DCPV

Output (V_1, V_2) as a successful partition and halt, where

- V_1 contains all voters contributing to S_{ac} and S_a, and also $\min(W_c, S_a - 1)$ voters contributing to W_c, and
- $V_2 = V \setminus V_1$.

In (C, V_1), a (strictly) beats c, since a gets

$$S_a - \min(W_c, S_a - 1) > 0$$

more Yes votes than c.

And in (C, V_2), b (strictly) beats c, since b has

$$S_b + L_c - (W_c - \min(W_c, S_a - 1))$$

more Yes votes than c.

Destructive Voter Control in Approval Voting: DCPV

So, for the construction to work, we must argue that

$$S_b + L_c + \min(W_c, S_a - 1) - W_c > 0.$$

That is, we need

$$W_c - L_c \quad < \quad \min(W_c, S_a - 1) + S_b. \tag{8}$$

If $W_c \leq S_a - 1$, Equation (8) reduces to $0 < L_c + S_b$, which follows from the fact that in the current case $S_b > 0$.

And if $W_c > S_a - 1$, Equation (8) follows immediately from Equation (7). ❑

Resistance to Electoral Control by Hybridization

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2009))

There is a hybridization method that takes candidate-anonymous voting systems as input and outputs a hybrid voting system such that

1. *the hybrid voting system possesses all the resistances to control possessed by any of its constituents, and*
2. *the hybrid voting system has an easy winner determination problem if all its constituents do so.* **without proof**

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2009))

There exists a neutral, anonymous voting system that is resistant to all standard types of electoral control and whose winners can be determined in polynomial time. **without proof**

Sincere-Strategy Preference-Based Approval Voting

Control by	Plurality		SP-AV		AV	
	Constr.	Destr.	Constr.	Destr.	Constr.	Destr.
AUC & AC	R	R	**R**	**R**	I	V
DC	R	R	**R**	**R**	V	I
PC	TE: R TP: R	TE: R TP: R	**TE: R** **TP: R**	**TE: R** **TP: R**	TE: V TP: I	TE: I TP: I
RPC	TE: R TP: R	TE: R TP: R	**TE: R** **TP: R**	**TE: R** **TP: R**	TE: V TP: I	TE: I TP: I
AV	V	V	**R**	**V**	R	V
DV	V	V	**R**	**V**	R	V
PV	TE: V TP: R	TE: V TP: R	**TE: R** **TP: R**	**TE: V** **TP: R**	TE: R TP: R	TE: V TP: V

Results for SP-AV are due to Erdélyi, Nowak, and Rothe (2009)

Sincere-Strategy Preference-Based Approval Voting

Number of	Condorcet	AV	Llull	Copeland	Plurality	SP-AV
resistances	3	4	14	15	16	19
immunities	4	9	0	0	0	0
vulnerabilities	7	9	8	7	6	3

Table: Number of resistances, immunities, and vulnerabilities

Best Wishes from St. Nicholas

Bucklin Voting

Definition

- The *strict majority threshold for a list V of voters* is
$$maj(V) = \lfloor \|V\|/2 \rfloor + 1.$$
- Given an election (C, V) and a candidate $c \in C$, define the *level i score of c in (C, V)* (denoted by $score^i_{(C,V)}(c)$) as the number of votes in V that rank c among their top i positions.
- The *Bucklin score of c in (C, V)* is the smallest i such that
$$score^i_{(C,V)}(c) \geq maj(V).$$
- All candidates with a smallest Bucklin score, say k, and a largest level k score are the *Bucklin winners (BV winners) in (C, V)*.

Bucklin Voting

Example (Bucklin Voting)

$C = \{a, b, c, d\}$ and $V = (v_1, v_2, v_3, v_4, v_5)$, so $\mathrm{maj}(V) = 3$

$v_1 : \quad b\ c\ a\ d$

$v_2 : \quad c\ d\ a\ b$

$v_3 : \quad a\ d\ c\ b$

$v_4 : \quad c\ a\ d\ b$

$v_5 : \quad b\ d\ c\ a$

Bucklin Voting

Example (Bucklin Voting)

$C = \{a, b, c, d\}$ and $V = (v_1, v_2, v_3, v_4, v_5)$, so $\text{maj}(V) = 3$

v_1 : $b\ c\ a\ d$

v_2 : $c\ d\ a\ b$

v_3 : $a\ d\ c\ b$

v_4 : $c\ a\ d\ b$

v_5 : $b\ d\ c\ a$

	a	b	c	d
score^1	1	2	2	0

Bucklin Voting

Example (Bucklin Voting)

$C = \{a, b, c, d\}$ and $V = (v_1, v_2, v_3, v_4, v_5)$, so $\mathrm{maj}(V) = 3$

v_1 : $b\ c\ a\ d$

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v_3 : $a\ d\ c\ b$

v_4 : $c\ a\ d\ b$

v_5 : $b\ d\ c\ a$

	a	b	c	d
score^1	1	2	2	0
score^2	2	2	3	3

Bucklin Voting

Example (Bucklin Voting)

$C = \{a, b, c, d\}$ and $V = (v_1, v_2, v_3, v_4, v_5)$, so maj$(V) = 3$

v_1 : $b\ c\ a\ d$

v_2 : $c\ d\ a\ b$

v_3 : $a\ d\ c\ b$

v_4 : $c\ a\ d\ b$

v_5 : $b\ d\ c\ a$

	a	b	c	d
score^1	1	2	2	0
score^2	2	2	3	3

$\Rightarrow$ c and d are level 2 Bucklin winners in (C, V)

Fallback Voting

Fallback Voting is a hybrid system due to Brams and Sanver (2009) that combines Bucklin with approval voting.

Definition

- Each voter provides both an approval vector and a linear ordering of all approved candidates.
- The subset of candidates approved of by a voter is also called his or her *approval strategy*.
- Given an election (C, V) and a candidate $c \in C$, the notions of
  - *level i score of c in (C, V)* and
  - *level k fallback voting winner (level k FV winner) in (C, V)*

  are defined analogously to the case of Bucklin voting.

Fallback Voting

- If there exists a level k FV winner for some $k \leq \|C\|$, he or she is called a *fallback winner (FV winner) in* (C, V).
- If, however, for no $k \leq \|C\|$ a level k FV winner exists, every candidate with a largest (approval) score is an *FV winner in* (C, V).

Remark: Bucklin voting is the special case of fallback voting where each voter approves of all candidates. Consequently,

- FV inherits NP-hardness lower bounds from BV.
- BV inherits polynomial-time upper bounds from FV.

Fallback Voting

Example (Fallback Voting)

$C = \{a, b, c, d\}$ and $V = (v_1, v_2, v_3, v_4, v_5)$, so $\mathrm{maj}(V) = 3$

$$
\begin{aligned}
v_1 &: \quad b\ c \mid \{a, d\} \\
v_2 &: \quad c \mid \{a, b, d\} \\
v_3 &: \quad a \mid \{b, c, d\} \\
v_4 &: \quad \mid \{a, b, c, d\} \\
v_5 &: \quad \mid \{a, b, c, d\}
\end{aligned}
$$

Fallback Voting

Example (Fallback Voting)

$C = \{a, b, c, d\}$ and $V = (v_1, v_2, v_3, v_4, v_5)$, so $\text{maj}(V) = 3$

$v_1 : \quad b\ c \mid \{a, d\}$

$v_2 : \quad c \mid \{a, b, d\}$

$v_3 : \quad a \mid \{b, c, d\}$

$v_4 : \quad \mid \{a, b, c, d\}$

$v_5 : \quad \mid \{a, b, c, d\}$

	a	b	c	d
score^1	1	1	1	0

Fallback Voting

Example (Fallback Voting)

$C = \{a, b, c, d\}$ and $V = (v_1, v_2, v_3, v_4, v_5)$, so maj$(V) = 3$

$v_1 : \quad b\ c \mid \{a, d\}$

$v_2 : \quad c \mid \{a, b, d\}$

$v_3 : \quad a \mid \{b, c, d\}$

$v_4 : \quad \mid \{a, b, c, d\}$

$v_5 : \quad \mid \{a, b, c, d\}$

	a	b	c	d
score^1	1	1	1	0
score^2	1	1	2	0

Fallback Voting

Example (Fallback Voting)

$C = \{a, b, c, d\}$ and $V = (v_1, v_2, v_3, v_4, v_5)$, so $\text{maj}(V) = 3$

$v_1 : \quad b\ c \mid \{a, d\}$

$v_2 : \quad c \mid \{a, b, d\}$

$v_3 : \quad a \mid \{b, c, d\}$

$v_4 : \quad \mid \{a, b, c, d\}$

$v_5 : \quad \mid \{a, b, c, d\}$

	a	b	c	d
score^1	1	1	1	0
score^2	1	1	2	0

$\Rightarrow$ c is the unique fallback winner by approval score in (C, V)

Results for Fallback Voting and Bucklin Voting

	FV		BV		SP-AV	
Control by	Constr.	Destr.	Constr.	Destr.	Constr.	Destr.
Adding Candidates (unlim.)						
Adding Candidates (lim.)						
Deleting Candidates						
Partition of Candidates						
Run-off Partition of Candidates						
Adding Voters						
Deleting Voters						
Partition of Voters						

Results for Fallback Voting and Bucklin Voting

Control by	FV		BV		SP-AV	
	Constr.	Destr.	Constr.	Destr.	Constr.	Destr.
Adding Candidates (unlim.)	R	R				
Adding Candidates (lim.)	R	R				
Deleting Candidates	R	R				
Partition of Candidates	TE: R TP: R	TE: R TP: R				
Run-off Partition of Candidates	TE: R TP: R	TE: R TP: R				
Adding Voters	R	V				
Deleting Voters	R	V				
Partition of Voters	TE: R TP: S	TE: S TP: S				

Results due to [ER10]

[ER10] G. Erdélyi and J. Rothe: Control complexity in fallback voting. In: *Proceedings of Computing: the 16th Australasian Theory Symposium*, 32(4): 39–48, 2010.

Results for Fallback Voting and Bucklin Voting

Control by	FV		BV		SP-AV	
	Constr.	Destr.	Constr.	Destr.	Constr.	Destr.
Adding Candidates (unlim.)	R	R				
Adding Candidates (lim.)	R	R				
Deleting Candidates	R	R				
Partition of Candidates	TE: R TP: R	TE: R TP: R				
Run-off Partition of Candidates	TE: R TP: R	TE: R TP: R				
Adding Voters	R	V				
Deleting Voters	R	V				
Partition of Voters	TE: R TP: R	TE: R TP: R				

Results due to [ER10], [EFRS15]

[ER10] G. Erdélyi and J. Rothe: Control complexity in fallback voting. In: *Proceedings of Computing: the 16th Australasian Theory Symposium*, 32(4): 39–48, 2010.

[EFRS15] G. Erdélyi, M. Fellows, J. Rothe, and L. Schend: Control Complexity in Bucklin and Fallback Voting: A Theoretical Analysis. Journal of Computer and System Sciences 81(4): 632–660, 2015. (Also in Proc. AAMAS-2011.)

Results for Fallback Voting and Bucklin Voting

Control by	FV		BV		SP-AV	
	Constr.	Destr.	Constr.	Destr.	Constr.	Destr.
Adding Candidates (unlim.)	R	R	R	R		
Adding Candidates (lim.)	R	R	R	R		
Deleting Candidates	R	R	R	R		
Partition of Candidates	TE: R TP: R	TE: R TP: R	TE: R TP: R	TE: R TP: R		
Run-off Partition of Candidates	TE: R TP: R	TE: R TP: R	TE: R TP: R	TE: R TP: R		
Adding Voters	R	V	R	V		
Deleting Voters	R	V	R	V		
Partition of Voters	TE: R TP: R	TE: R TP: R	TE: R TP: R	TE: R TP: S		

Results due to [ER10], [EFRS15]

[ER10] G. Erdélyi and J. Rothe: Control complexity in fallback voting. In: *Proceedings of Computing: the 16th Australasian Theory Symposium*, 32(4): 39–48, 2010.

[EFRS15] G. Erdélyi, M. Fellows, J. Rothe, and L. Schend: Control Complexity in Bucklin and Fallback Voting: A Theoretical Analysis. Journal of Computer and System Sciences 81(4): 632–660, 2015. (Also in Proc. AAMAS-2011.)

Results for Fallback Voting and Bucklin Voting

Control by	FV		BV		SP-AV	
	Constr.	Destr.	Constr.	Destr.	Constr.	Destr.
Adding Candidates (unlim.)	R	R	R	R	R	R
Adding Candidates (lim.)	R	R	R	R	R	R
Deleting Candidates	R	R	R	R	R	R
Partition of Candidates	TE: R TP: R	TE: R TP: R	TE: R TP: R	TE: R TP: R	TE: R TP: R	TE: R TP: R
Run-off Partition of Candidates	TE: R TP: R	TE: R TP: R	TE: R TP: R	TE: R TP: R	TE: R TP: R	TE: R TP: R
Adding Voters	R	V	R	V	R	V
Deleting Voters	R	V	R	V	R	V
Partition of Voters	TE: R TP: R	TE: R TP: R	TE: R TP: R	TE: R TP: S	TE: R TP: R	TE: V TP: R

Results due to [ER10], [EFRS15], and [ENR09].

[ER10] G. Erdélyi and J. Rothe: Control complexity in fallback voting. In: *Proceedings of Computing: the 16th Australasian Theory Symposium*, 32(4): 39–48, 2010.

[EFRS15] G. Erdélyi, M. Fellows, J. Rothe, and L. Schend: Control Complexity in Bucklin and Fallback Voting: A Theoretical Analysis. Journal of Computer and System Sciences 81(4): 632–660, 2015. (Also in Proc. AAMAS-2011.)

[ENR09] G. Erdélyi, M. Nowak, and J. Rothe: Sincere-strategy preference-based approval voting fully resists constructive control and broadly resists destructive control. *Mathematical Logic Quarterly*, 55(4): 425–443, 2009.

Control Complexity of FV and BV: Summary

- FV has 20 resistances to electoral control and is vulnerable to 2 types of electoral control.
- BV is fully resistant to both candidate control and constructive control.
- BV behaves almost as good as FV, with 19 (possibly even 20) resistances and 2 (at most 3) vulnerabilities.
- Resistance results for BV strengthen those for FV.
- FV has the most known resistances among natural voting systems with polynomial-time winner determination.

Bucklin Voting: Control by Partition of Voters

Example (Bucklin Voting and CCPV)

$C = \{a, b, c, d, e\}$, $V = (v_1, \ldots, v_5)$, $V_1 = (v_1, v_2)$, $V_2 = (v_3, v_4, v_5)$

	(C, V)
v_1	$b\ a\ c\ d\ e$
v_2	$b\ d\ c\ a\ e$
v_3	$c\ a\ d\ b\ e$
v_4	$a\ d\ c\ b\ e$
v_5	$c\ e\ b\ a\ d$
	$\to a$

Bucklin Voting: Control by Partition of Voters

1^{st} stage: (C, V_1) (C, V_2)

Example (Bucklin Voting and CCPV)

$C = \{a, b, c, d, e\}$, $V = (v_1, \ldots, v_5)$, $V_1 = (v_1, v_2)$, $V_2 = (v_3, v_4, v_5)$

	(C, V)	(C, V_1)	(C, V_2)
v_1	$b\ a\ c\ d\ e$	$b\ a\ c\ d\ e$	
v_2	$b\ d\ c\ a\ e$	$b\ d\ c\ a\ e$	
v_3	$c\ a\ d\ b\ e$		$c\ a\ d\ b\ e$
v_4	$a\ d\ c\ b\ e$		$a\ d\ c\ b\ e$
v_5	$c\ e\ b\ a\ d$		$c\ e\ b\ a\ d$
	$\rightarrow a$		

Bucklin Voting: Control by Partition of Voters

1^{st} stage: (C, V_1) (C, V_2)

W_1 W_2

Example (Bucklin Voting and CCPV)

$C = \{a, b, c, d, e\}$, $V = (v_1, \ldots, v_5)$, $V_1 = (v_1, v_2)$, $V_2 = (v_3, v_4, v_5)$

	(C, V)	(C, V_1)	(C, V_2)
v_1	$b\ a\ c\ d\ e$	$b\ a\ c\ d\ e$	
v_2	$b\ d\ c\ a\ e$	$b\ d\ c\ a\ e$	
v_3	$c\ a\ d\ b\ e$		$c\ a\ d\ b\ e$
v_4	$a\ d\ c\ b\ e$		$a\ d\ c\ b\ e$
v_5	$c\ e\ b\ a\ d$		$c\ e\ b\ a\ d$
	$\rightarrow a$	$W_1 = \{b\}$	$W_2 = \{c\}$

Bucklin Voting: Control by Partition of Voters

1^{st} stage: (C, V_1) (C, V_2)

W_1 W_2

2^{nd} stage: $(W_1 \cup W_2, V)$

Example (Bucklin Voting and CCPV)

$C = \{a, b, c, d, e\}$, $V = (v_1, \ldots, v_5)$, $V_1 = (v_1, v_2)$, $V_2 = (v_3, v_4, v_5)$

	(C, V)	(C, V_1)	(C, V_2)	$(W_1 \cup W_2, V)$
v_1	$b\ a\ c\ d\ e$	$b\ a\ c\ d\ e$		$b\ c$
v_2	$b\ d\ c\ a\ e$	$b\ d\ c\ a\ e$		$b\ c$
v_3	$c\ a\ d\ b\ e$		$c\ a\ d\ b\ e$	$c\ b$
v_4	$a\ d\ c\ b\ e$		$a\ d\ c\ b\ e$	$c\ b$
v_5	$c\ e\ b\ a\ d$		$c\ e\ b\ a\ d$	$c\ b$
	$\rightarrow a$	$W_1 = \{b\}$	$W_2 = \{c\}$	

Bucklin Voting: Control by Partition of Voters

1^{st} stage: (C, V_1) (C, V_2)

W_1 W_2

2^{nd} stage: $(W_1 \cup W_2, V)$

Example (Bucklin Voting and CCPV)

$C = \{a, b, c, d, e\}$, $V = (v_1, \ldots, v_5)$, $V_1 = (v_1, v_2)$, $V_2 = (v_3, v_4, v_5)$

	(C, V)	(C, V_1)	(C, V_2)	$(W_1 \cup W_2, V)$
v_1	$b\ a\ c\ d\ e$	$b\ a\ c\ d\ e$		$b\ c$
v_2	$b\ d\ c\ a\ e$	$b\ d\ c\ a\ e$		$b\ c$
v_3	$c\ a\ d\ b\ e$		$c\ a\ d\ b\ e$	$c\ b$
v_4	$a\ d\ c\ b\ e$		$a\ d\ c\ b\ e$	$c\ b$
v_5	$c\ e\ b\ a\ d$		$c\ e\ b\ a\ d$	$c\ b$
	$\to a$	$W_1 = \{b\}$	$W_2 = \{c\}$	$\to c$

Bucklin: Constructive Control by Partition of Voters

Theorem (Erdélyi, Fellows, Rothe, and Schend (2015))

Bucklin voting is resistant to constructive control by partition of voters in both tie-handling models, TE and TP.

Corollary (Erdélyi, Fellows, Rothe, and Schend (2015))

Fallback voting is resistant to constructive control by partition of voters in both tie-handling models, TE and TP.

Bucklin: Constructive Control by Partition of Voters

Proof: To show NP-hardness we reduce X3C to our control problems.

Let $(B, \mathcal{S})$ be an X3C instance with

- $B = \{b_1, b_2, \ldots, b_{3m}\}$, $m > 1$, and
- a collection $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$.

We define the election (C, V), where w is the distinguished candidate and

$$C = B \cup \{c, w, x\} \cup D \cup E \cup F \cup G$$

is the set of candidates with

$$\begin{aligned} D &= \{d_1, \ldots, d_{3nm}\}, & E &= \{e_1, \ldots, e_{(3m-1)(m+1)}\}, \\ F &= \{f_1, \ldots, f_{(3m+1)(m-1)}\}, & G &= \{g_1, \ldots, g_{n(3m-3)}\}. \end{aligned}$$

Bucklin: Constructive Control by Partition of Voters

For each j, $1 \leq j \leq 3m$, define

$$\ell_j = \|\{S_i \in \mathcal{S} \mid b_j \in S_i\}\|,$$

and for each i, $1 \leq i \leq n$, define

$$\begin{aligned}
B_i &= \{b_j \in B \mid i \leq n - \ell_j\}, \\
D_i &= \{d_{(i-1)3m+1}, \ldots, d_{3im-\|B_i\|}\}, & \|D_i\| &= 3m - \|B_i\| \\
G_i &= \{g_{(i-1)(3m-3)+1}, \ldots, g_{i(3m-3)}\}. & \|G_i\| &= 3m - 3
\end{aligned}$$

Also, for each k, $1 \leq k \leq m+1$, and for each l, $1 \leq l \leq m-1$, define

$$\begin{aligned}
E_k &= \{e_{(3m-1)(k-1)+1}, \ldots, e_{(3m-1)k}\}, & \|E_k\| &= 3m - 1 \\
F_l &= \{f_{(3m+1)(l-1)+1}, \ldots, f_{(3m+1)l}\}. & \|F_l\| &= 3m + 1
\end{aligned}$$

Bucklin: Constructive Control by Partition of Voters

Let V consist of the following $2n+2m$ voters:

#	For each ...	number of voters	ranking of candidates
1	$i \in \{1, \ldots, n\}$	1	$c \;\; S_i \;\; G_i \;\; (G \setminus G_i) \;\; F \;\; D \;\; E \;\; (B \setminus S_i) \;\; w \;\; x$
2	$i \in \{1, \ldots, n\}$	1	$B_i \;\; D_i \;\; w \;\; G \;\; E \;\; (D \setminus D_i) \;\; F \;\; (B \setminus B_i) \;\; c \;\; x$
3	$k \in \{1, \ldots, m+1\}$	1	$x \;\; c \;\; E_k \;\; F \;\; (E \setminus E_k) \;\; G \;\; D \;\; B \;\; w$
4	$l \in \{1, \ldots, m-1\}$	1	$F_l \;\; c \;\; (F \setminus F_l) \;\; G \;\; D \;\; E \;\; B \;\; w \;\; x$

$$\|V\| = 2n + 2m \implies \text{maj}(V) = n + m + 1.$$

Bucklin: Constructive Control by Partition of Voters

Purpose of the padding candidates:

- D_i, $1 \leq i \leq n$: ensure that w is always placed at position $3m + 1$ in the second voter group of V.
- E_k, $1 \leq k \leq m + 1$: ensure that no other candidate besides c and x gains more than one point up to the $(3m + 1)$st level in the third voter group of V.
- F_l, $1 \leq l \leq m - 1$: ensure that c does not gain any points up to level $3m + 1$ in the fourth voter group of V.
- G_i, $1 \leq i \leq n$: ensure that no other candidate besides c and those in S_i gains more than one point up to level $3m + 1$ in the first voter group of V.

Bucklin: Constructive Control by Partition of Voters

	c	b_j	w	x
$score^1$	n	$\leq n$	0	$m+1$
$score^2$	$\mathbf{n+m+1}$	$\leq n$	0	$m+1$
$score^{3m}$	$n+m+1$	n	0	$m+1$
$score^{3m+1}$	$n+m+1$	n	n	$m+1$

Table: Level i scores in (C, V) for $i \in \{1, 2, 3m, 3m+1\}$ and $B \cup \{c, w, x\}$.

In (C, V), candidate c is the unique level 2 BV winner with a level 2 score of $n+m+1$.

Bucklin: Constructive Control by Partition of Voters

We claim that $\mathcal{S}$ has an exact cover $\mathcal{S}'$ for B if and only if w can be made the unique BV winner of the resulting election by partition of voters (regardless of the tie-handling model used).

(⇒) Suppose $\mathcal{S}$ has an exact cover $\mathcal{S}'$ for B.

Partition V as follows. Let V_1 consist of:

- the m voters of the first group that correspond to the exact cover (i.e., those m voters of the form

  $c \;\; S_i \;\; G_i \;\; (G - G_i) \;\; F \;\; D \;\; E \;\; (B - S_i) \;\; w \;\; x$ for which $S_i \in \mathcal{S}'$)

  and
- the $m+1$ voters of the third group (i.e., all voters of the form

  $x \;\; c \;\; E_k \;\; F \;\; (E - E_k) \;\; G \;\; D \;\; B \;\; w$).

Let $V_2 = V \setminus V_1$.

Bucklin: Constructive Control by Partition of Voters

	(C, V_1)			(C, V_2)		
	c	b_j	x	c	b_j	w
$score^1$	m	0	$\mathbf{m+1}$	$n-m$	$\leq n-1$	0
$score^2$	$2m+1$	≤ 1	$m+1$	$n-m$	$\leq n-1$	0
$score^{3m}$	$2m+1$	1	$m+1$	$n-m$	$n-1$	0
$score^{3m+1}$	$2m+1$	1	$m+1$	$n-m$	$n-1$	$\mathbf{n}$

Table: Level i scores in (C, V_1) and (C, V_2) for $i \in \{1, 2, 3m, 3m+1\}$ and the candidates in $B \cup \{c, w, x\}$.

- $\|V_1\| = 2m+1 \implies \text{maj}(V_1) = m+1$. Hence, x wins in (C, V_1).
- $\|V_2\| = 2n-1 \implies \text{maj}(V_2) = n$. Hence, w wins in (C, V_2).

Bucklin: Constructive Control by Partition of Voters

	x	w
$score^1$	$m+1$	$\mathbf{2n+m-1}$
$score^2$	$2n+2m$	$2n+2m$

Table: Level i scores of w and x in the final election $(\{w,x\},V)$ for $i \in \{1,2\}$.

In the run-off, w wins with a strict majority on the first level.

Since both subelections, (C, V_1) and (C, V_2), have unique BV winners, candidate w can be made the unique BV winner by partition of voters, regardless of the tie-handling model used.

Bucklin: Constructive Control by Partition of Voters

(⇐) Suppose that w can be made the unique BV winner by exerting control by partition of voters (for concreteness, say in TP).

Let (V_1, V_2) be such a successful partition. Since w wins the resulting two-stage election, w has to win at least one of the subelections (say, w wins (C, V_2)).

If candidate c participates in the final round, he or she wins the election with a strict majority no later than on the second level, no matter which other candidates move forward to the final election.

That means that in both subelections, (C, V_1) and (C, V_2), c must not be a BV winner.

Bucklin: Constructive Control by Partition of Voters

Only in the second voter group candidate w (who has to be a BV winner in election (C, V_2)) gets points earlier than on the second-to-last level. So w has to be a level $3m+1$ BV winner in election (C, V_2) via votes from the second voter group in V_2.

As c scores already on the first two levels in voter groups 1 and 3, only x and the candidates in B can prevent c from winning in (C, V_1).

However, since voters from the second voter group have to be in V_2 (as stated above), in subelection (C, V_1) only candidate x can prevent c from moving forward to the final round.

Since x is always placed behind c in all votes except those votes from the third voter group, x has to be a level 1 BV winner in election (C, V_1).

Bucklin: Constructive Control by Partition of Voters

In (C, V_2) candidate w gains all the points on exactly the $(3m + 1)$st level, whereas the other candidates scoring more than one point up to this level receive their points on either earlier or later levels, so no candidate can tie with w on the $(3m + 1)$st level and w is the unique level $3m + 1$ BV winner in election (C, V_2).

As both subelections, (C, V_1) and (C, V_2), have unique BV winners other than c, the construction works in model TE as well.

It remains to show that $\mathcal{S}$ has an exact cover $\mathcal{S}'$ for B.

Since w has to win (C, V_2) with the votes from the second voter group, not all voters from the first voter group can be in V_2 (otherwise c would have n points already on the first level).

Bucklin: Constructive Control by Partition of Voters

On the other hand, there can be at most m voters from the first voter group in V_1 because otherwise x would not be a level 1 BV winner in election (C, V_1).

To ensure that no candidate contained in B has the same score as w, namely n points, and gets these points on an earlier level than w in (C, V_2), there have to be exactly m voters from the first group in V_1 and these voters correspond to an exact cover for B.

Without further adaptions, this reduction also covers the nonunique-winner case. ❑

Bucklin: Destructive Control by Partition of Voters

Theorem (Erdélyi, Fellows, Rothe, and Schend (2015))

Bucklin voting is resistant to destructive control by partition of voters in tie-handling model TE.

Corollary (Erdélyi, Fellows, Rothe, and Schend (2015))

Fallback voting is resistant to destructive control by partition of voters in tie-handling model TE.

Dominating Set

Definition

1. A *dominating set* of an undirected graph $G = (B, A)$ is a subset $D \subseteq B$ such that for each $x \in B \setminus D$ there exists a vertex $y \in D$ such that $\{x, y\} \in A$.
2. The *size of a dominating set* is the number of its vertices.
3.
   - The *neighborhood of a vertex $b_i \in B$* is defined by $\mathrm{N}(b_i) = \{b_j \in B \mid \{b_i, b_j\} \in A\}$;
   - the *closed neighborhood of $b_i \in B$* is defined by $\mathrm{N}[b_i] = \mathrm{N}(b_i) \cup \{b_i\}$;
   - for a subset $S \subseteq B$, the *neighborhood of S* is defined as $\mathrm{N}(S) = \bigcup_{b_i \in S} \mathrm{N}(b_i)$; and
   - the *closed neighborhood of S* is defined as $\mathrm{N}[S] = \bigcup_{b_i \in S} \mathrm{N}[b_i]$.

Dominating Set

Name: DOMINATING SET (DS).

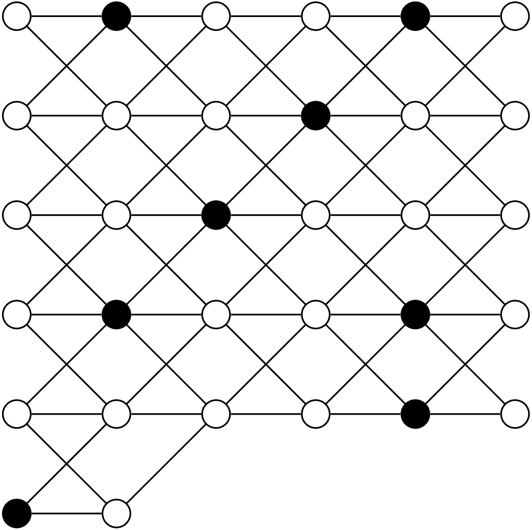
Given: A graph $G = (B, A)$ and a positive integer $k \leq \|B\|$.

Question: Is there a dominating set of size at most k in G?

In other words, the dominating set problem tests, given a graph $G = (B, A)$ and an integer k, whether there is a subset $B' \subseteq B$ of size at most k such that

$$B = \mathrm{N}[B'].$$

Dominating Set



Bucklin: Destructive Control by Partition of Voters

Proof: To show NP-hardness we reduce the NP-complete problem DOMINATING SET to our control problem.

Let $((B, A), k)$ be a given instance of DOMINATING SET with $B = \{b_1, b_2, \ldots, b_n\}$ and $n \geq 1$.

Define the election (C, V) with candidate set

$$C = B \cup D \cup E \cup F \cup H \cup \{c, u, v, w, x, y\}, \quad \text{where}$$

$$\begin{aligned} D &= \{d_1, d_2, \ldots, d_{(k-1)(n+4)}\}, & E &= \{e_1, e_2, \ldots, e_{2(k+n)}\}, \\ F &= \{f_1, f_2, \ldots, f_{3n}\}, \text{ and} & H &= \{h_1, h_2, \ldots, h_{n^2}\}. \end{aligned}$$

Let c be the distinguished candidate.

Bucklin: Destructive Control by Partition of Voters

V consists of $2k + 2n$ votes over C, arranged in four groups:

1. For each i, $1 \leq i \leq n$, there is one voter of the form:

$$F_i \;\; (B \setminus \mathrm{N}[b_i]) \;\; H_i \;\; y \;\; w \;\; (\mathrm{N}[b_i] \cup D \cup E \cup (F \setminus F_i) \cup (H \setminus H_i)) \;\; u \;\; v \;\; c \;\; x,$$

   where $F_i = \{f_{3(i-1)+1}, f_{3(i-1)+2}, f_{3i}\}$ and $H_i = \{h_{(i-1)n+1}, \ldots, h_{(i-1)n+\|\mathrm{N}[b_i]\|}\}$.
   Note that $\|H_i\| = \|\mathrm{N}[b_i]\|$ and $\|F_i\| = 3$, so candidate w is always placed on the $(n+5)$th position.

2. There is one voter of the form:

$$x \;\; w \;\; c \;\; B \;\; u \;\; v \;\; (D \cup E \cup F \cup H) \;\; y.$$

Bucklin: Destructive Control by Partition of Voters

3. For each i, $1 \leq i \leq k-1$, there is one voter of the form:

$$x \;\; D_i \;\; (B \cup (D \setminus D_i) \cup E \cup F \cup H) \;\; u \;\; v \;\; y \;\; w \;\; c,$$

   where $D_i = \{d_{(i-1)(n+4)+1}, \ldots, d_{i(n+4)}\}$, so $\|D_i\| = n+4$.

4. For each i, $1 \leq i \leq k+n$, there is one voter of the form:

$$c \;\; E_i \;\; x \;\; y \;\; (B \cup D \cup (E \setminus E_i) \cup F \cup H) \;\; u \;\; v \;\; w,$$

   where $E_i = \{e_{2i-1}, e_{2i}\}$, so $\|E_i\| = 2$.

Bucklin: Destructive Control by Partition of Voters

	c	w	x
$score^1$	$k+n$	0	k
$score^2$	$k+n$	1	k
$score^3$	$\boldsymbol{k+n+1}$	1	k

Table: Level i scores of c, w, and x in (C, V) for $i \in \{1, 2, 3\}$.

None of the other candidates scores more than one point up to the third level.

Note that c reaches a strict majority on this level and thus is the unique level 3 BV winner in this election.

Bucklin: Destructive Control by Partition of Voters

Lemma

In the election (C, V) from the above construction, for every partition of V into V_1 and V_2, candidate c is a unique BV winner of at least one of the subelections, (C, V_1) and (C, V_2).

Proof: of Lemma. For a contradiction, we assume that in both subelections, (C, V_1) and (C, V_2), c is *not a unique BV winner*.

The previous table shows that half of the voters in V place c already on the first level. Thus,

- for $i \in \{1, 2\}$, $\|V_i\|$ must be an even number and
- $score^1_{(C,V_i)}(c) = \frac{\|V_i\|}{2}$.

Bucklin: Destructive Control by Partition of Voters

Due to the voter in the second voter group, candidate c will get a strict majority on the third level in one of the subelections, say in (C, V_1).

So there has to be a candidate beating or tieing with candidate c on the second or third level in (C, V_1). The candidates in B, D, E, F, H and u, v, w, y do not score more than one point up to the third level.

Thus only candidate x can possibly beat or tie with c on the second or third level in (C, V_1). However, since x does not score more than k points in total until the fourth level, c is the unique level 3 BV winner in subelection (C, V_1), a contradiction.

Thus c is a unique BV winner of at least one of the subelections. ❑

Bucklin: Destructive Control by Partition of Voters

We claim that $G = (B, A)$ has a dominating set B' of size k if and only if candidate c can be prevented from being a unique BV winner by partition of voters in model TE.

Bucklin: Destructive Control by Partition of Voters

(⇒) Let B' be a dominating set for G of size k. Partition V into V_1 and V_2 as follows. Let V_1 consist of the following $2k$ voters:

- The k voters of the first voter group corresponding to the dominating set, i.e., for those i with $b_i \in B'$, we have one voter:
  $F_i\ (B \setminus \mathrm{N}[b_i])\ H_i\ y\ w\ (\mathrm{N}[b_i] \cup D \cup E \cup (F \setminus F_i) \cup (H \setminus H_i))\ u\ v\ c\ x,$
- the one voter from the second group:
  $x\ \ w\ \ c\ \ B\ \ u\ \ v\ \ (D \cup E \cup F \cup H)\ \ y,$
- the entire third voter group, i.e., for each j, $1 \leq j \leq k-1$, there is one voter of the form:
  $x\ \ D_j\ \ (B \cup (D \setminus D_j) \cup E \cup F \cup H \cup)\ \ u\ \ v\ \ y\ \ w\ \ c.$

Let $V_2 = V \setminus V_1$.

Bucklin: Destructive Control by Partition of Voters

Note that the strict majority threshold in V_1 is $\text{maj}(V_1) = k + 1$.

	c	w	x	y	$b_i \in B$
$score^{n+5}$	1	$\mathbf{k+1}$	k	k	$\leq k$

Table: Level $n + 5$ scores of the relevant candidates in (C, V_1).

Note that w reaches a strict majority of $k + 1$ on this level (and no other candidate reaches a strict majority on this or an earlier level).

Hence, w is the unique level $n + 5$ BV winner in subelection (C, V_1) and thus participates in the final round.

Bucklin: Destructive Control by Partition of Voters

From the previous lemma it follows that candidate c is the unique winner in subelection (C, V_2).

So the final-stage election is $(\{c, w\}, V)$ and we have the following scores on the first two levels:

$$\begin{aligned} score^1_{(\{c,w\},V)}(c) = score^1_{(\{c,w\},V)}(w) &= k + n, \\ score^2_{(\{c,w\},V)}(c) = score^2_{(\{c,w\},V)}(w) &= 2k + 2n. \end{aligned}$$

Since none of c and w have a strict majority on the first level, both candidates are level 2 BV winners in this two-candidate final-stage election. Hence, c has been prevented from being a unique BV winner by partition of voters in model TE.

Bucklin: Destructive Control by Partition of Voters

($\Leftarrow$) Assume that c can be prevented from being a unique BV winner by partition of voters in model TE.

From the previous lemma we know that candidate c must participate in the final-stage election.

Since we are in model TE, at most two candidates participate in the final run-off.

To prevent c from being a unique BV winner of the final election, there must be another finalist and this other candidate has to beat or tie with c.

Since w is the only candidate that can beat or tie with c in a two-candidate election, w has to move on to the final round to run against c.

Bucklin: Destructive Control by Partition of Voters

Let us say that c is the unique winner of subelection (C, V_2) and w is the unique winner of subelection (C, V_1).

For w to win subelection (C, V_1) alone, V_1 has to contain voters from the first voter group and w can win only on the $(n + 5)$th level:

- In particular, x is placed before w in all voter groups except the first, so w can win in (C, V_1) only via voters from the first voter group participating in (C, V_1).
- Moreover, since w is placed in the last or second-to-last position in all voters from the third and fourth groups, and
- since there is only one voter in the second group, w can win only on the $(n + 5)$th level (which is w's position in the votes from the first voter group).

Bucklin: Destructive Control by Partition of Voters

Let $I \subseteq \{1, \ldots, n\}$ be the set of indices i such that first-group voter

$$F_i \ (B \setminus \mathrm{N}[b_i]) \ H_i \ y \ w \ (\mathrm{N}[b_i] \cup D \cup E \cup (F \setminus F_i) \cup (H \setminus H_i)) \ u \ v \ c \ x$$

belongs to V_1. Let $\ell = \|I\|$.

Since w is the unique level $n+5$ BV winner of subelection (C, V_1) but y is placed before w in every vote in the first group, the one voter from the second group (which is the only voter who prefers w to y) must belong to V_1. Thus we know that

$$score^{n+5}_{(C,V_1)}(w) = \ell + 1 \quad \text{and} \quad score^{n+4}_{(C,V_1)}(y) = score^{n+5}_{(C,V_1)}(y) = \ell.$$

Bucklin: Destructive Control by Partition of Voters

For the candidates in B, we have

$$score^{n+4}_{(C,V_1)}(b_j) = score^{n+5}_{(C,V_1)}(b_j) = 1 + \|\{b_i \mid i \in I \text{ and } b_j \notin \mathrm{N}[b_i]\}\|,$$

since each b_j scores one point up to the $(n+4)$th level from the voter in the second group and one point from the first group for every b_i with $i \in I$ such that $b_j \notin \mathrm{N}[b_i]$ in graph G.

Again, since w is the unique level $n+5$ BV winner of subelection (C, V_1), no $b_j \in B$ can score a point in *each* of the ℓ votes from the first voter group that belong to V_1.

This implies that for each $b_j \in B$ there has to be at least one b_i with $i \in I$ that is adjacent to b_j in G. Thus, the set B' of candidates b_i with $i \in I$ corresponds to a dominating set in G.

Bucklin: Destructive Control by Partition of Voters

Recall that $score^{n+5}_{(C,V_1)}(w) = \ell + 1$ and $score^{n+4}_{(C,V_1)}(y) = \ell$.

Note also that $score^{n+4}_{(C,V_1)}(b_j) \leq \ell$ for $1 \leq j \leq n$.

Since w needs a strict majority to be a BV winner in subelection (C, V_1), it must hold that $\text{maj}(V_1) \leq \ell + 1$.

Since y and the $b_j \in B$ have a score of ℓ already one level earlier than w, it must hold that $\text{maj}(V_1) = \ell + 1$, which implies $\|V_1\| = 2\ell$ or $\|V_1\| = 2\ell + 1$.

To ensure this cardinality of V_1, other votes have to be added.

Bucklin: Destructive Control by Partition of Voters

Since y must not gain additional points from these votes up to the $(n+5)$th level, they cannot come from the fourth voter group.

The remaining votes from the third voter group total up to $k-1$.

Thus, since w is the unique BV winner in subelection (C, V_1), it must hold that $\ell \leq k$.

So $\|B'\| = \ell \leq k$ and this means that there exists a dominating set of size at most k. ❑

Fallback Voting: Control by Partition of Voters

Theorem (Erdélyi, Fellows, Rothe, and Schend (2015))

Fallback voting is resistant to destructive control by partition of voters in tie-handling model TP.

Remark: For Bucklin voting the complexity of destructive control by partition of voters in tie-handling model TP is open.

Fallback Voting: Control by Partition of Voters

Proof: To show NP-hardness, we reduce ARHS to DCPV-TP.

Name: Another Restricted Hitting Set (ARHS).

Given:
- A set $B = \{b_1, b_2, \ldots, b_m\}$,
- a family $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of nonempty subsets S_i of B such that $n > m$, and
- a positive integer k such that $1 < k < m$.

Question: Does $\mathcal{S}$ have a hitting set of size at most k? That is, is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i, $S_i \cap B' \neq \emptyset$?

Lemma (Erdélyi, Piras, and Rothe (2011))

Another Restricted Hitting Set is NP-complete.

Proof: Exercise. ❑

Fallback Voting: Control by Partition of Voters

Construction: Let $(B, \mathcal{S}, k)$ be a given instance of ARHS, where

- $B = \{b_1, b_2, \ldots, b_m\}$ is a set,
- $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$ such that $n > m$, and
- k is a positive integer such that $1 < k < m$.

Define the election (C, V), where

$$C = B \cup D \cup E \cup \{c, w\}$$

is the candidate set with

$$\begin{aligned} D &= \{d_1, \ldots, d_{2(m+1)}\} \\ E &= \{e_1, \ldots, e_{2(m-1)}\} \end{aligned}$$

Fallback Voting: Control by Partition of Voters

and where V consists of the following $2n(k+1)+4m+2mk$ voters:

#	For each …	number of voters	ranking (approved)
1	$i \in \{1, \ldots, n\}$	$k+1$	$w \;\; S_i \;\; c$
2	$j \in \{1, \ldots, m\}$	1	$c \;\; b_j \;\; w$
3	$j \in \{1, \ldots, m\}$	$k-1$	b_j
4	$p \in \{1, \ldots, m+1\}$	1	$d_{2(p-1)+1} \;\; d_{2p} \;\; w$
5	$r \in \{1, \ldots, 2(m-1)\}$	1	e_r
6		$n(k+1)+m-k+1$	c
7		$mk+k-1$	$c \;\; w$
8		1	$w \;\; c$

Fallback Voting: Control by Partition of Voters

Level i scores for $i \in \{1, 2, m+2\}$ in (C, V):

	c	w
score^1	$n(k+1)+2m+mk$	$n(k+1)+1$
score^2	$n(k+1)+2m+mk+1$	$n(k+1)+mk+k$
score^{m+2}	$2n(k+1)+2m+mk+1$	$n(k+1)+2m+mk+k+1$

	$b_j \in B$	$d_p \in D$	$e_r \in E$
score^1	$k-1$	≤ 1	1
score^2	$\leq k+n(k+1)$	1	1
score^{m+2}	$\leq k+n(k+1)$	1	1

Fallback Voting: Control by Partition of Voters

We have

$$\text{maj}(V) = n(k+1) + 2m + mk + 1.$$

In election (C, V), only c and w reach a strict majority,

- w on the third level and
- c on the second level.

Thus c is the unique level 2 FV winner of election (C, V).

Lemma

In the election (C, V) thus constructed, for every partition of V into V_1 and V_2, candidate c is an FV winner of (C, V_1) or (C, V_2).

Fallback Voting: Control by Partition of Voters

Proof: For a contradiction, suppose that in both subelections, (C, V_1) and (C, V_2), candidate c is not an FV winner.

Since $score^1_{(C,V)}(c) = \frac{\|V\|}{2}$, the two subelections must satisfy that both $\|V_1\|$ and $\|V_2\|$ are even numbers, and

$$\begin{aligned} score^1_{(C,V_1)}(c) &= \frac{\|V_1\|}{2}, \\ score^1_{(C,V_2)}(c) &= \frac{\|V_2\|}{2}. \end{aligned}$$

Otherwise, c would have a strict majority already on the first level in one of the subelections and would win that subelection.

Fallback Voting: Control by Partition of Voters

For each $i \in \{1, 2\}$,

- c already on the first level has only one point less than the strict majority threshold $\text{maj}(V_i)$ in subelection (C, V_i), and
- c will get a strict majority in (C, V_i) no later than on the $(m+2)$nd level.

Thus, for $i \in \{1, 2\}$, there must be candidates whose level $m+2$ scores in (C, V_i) are higher than the level $m+2$ score of c in (C, V_i).

The previous tables show the level $m+2$ scores of all candidates in (C, V): Only w and some $b_j \in B$ have a chance to beat c on that level in (C, V_i), $i \in \{1, 2\}$.

Fallback Voting: Control by Partition of Voters

Suppose that c is defeated in both subelections by two distinct candidates from B (say,

- b_x defeats c in (C, V_1) and
- b_y defeats c in (C, V_2)).

Thus the following must hold (for the left-hand sides of the inequalities, note that each vote occurs in only one of the two subelections; and to avoid double-counting, we substract the double-counted points):

$$
\begin{aligned}
score^{m+2}_{(C,V_1)}(b_x) + score^{m+2}_{(C,V_2)}(b_y) &\geq score^{m+2}_{(C,V)}(c) + 2 \\
2n(k+1) + 2k - n(k+1) &\geq 2n(k+1) + mk + 2m + 3 \\
2k &\geq n(k+1) + mk + 2m + 3.
\end{aligned}
$$

Fallback Voting: Control by Partition of Voters

By our basic assumption $m > k > 1$, this implies the following contradiction:

$$\begin{aligned} 0 &\geq n(k+1) + (m-2)k + 2m + 3 \\ &> n(k+1) + (k-2)k + 2k + 3 \\ &= n(k+1) + k^2 + 3 > 0. \end{aligned}$$

Thus the only possibility for c to not win any of the two subelections is that

- c is defeated in one subelection, say (C, V_1), by a candidate from B, say b_x, and
- in the other subelection, (C, V_2), by candidate w.

Fallback Voting: Control by Partition of Voters

Then it must hold that (again avoiding double-counting):

$$\begin{aligned} score^{m+2}_{(C,V_1)}(b_x) + score^{m+2}_{(C,V_2)}(w) &\geq score^{m+2}_{(C,V)}(c) + 2 \\ 2n(k+1) + 2k + 2m + mk + 1 - n(k+1) - 1 &\geq 2n(k+1) + mk + 2m + 3 \\ 2k &\geq n(k+1) + 3. \end{aligned}$$

Since $n > 1$, this cannot hold, so c must be an FV winner in one of the two subelections. ❑

Fallback Voting: Control by Partition of Voters

We claim that $\mathcal{S}$ has a hitting set $B' \subseteq B$ of size k if and only if c can be prevented from being a unique FV winner by partition of voters in model TP.

(⇒) Suppose $B' \subseteq B$ is a hitting set of size k for $\mathcal{S}$.

Partition V into V_1 and V_2 as follows:

- Let V_1 consist of those voters of the
  - second group where $b_j \in B'$ and
  - third group where $b_j \in B'$.
- Let $V_2 = V \setminus V_1$.

Fallback Voting: Control by Partition of Voters

Consider the following score table in (C, V_1):

	c	w	$b_j \in B'$	$b_j \notin B'$
score^1	k	0	$k-1$	0
score^2	k	0	k	0
score^3	k	k	k	0

Table: Level i scores in (C, V_1) for $i \in \{1, 2, 3\}$ and candidates in $B \cup \{c, w\}$.

Note that $\text{maj}(V_1) = \lfloor \frac{k^2}{2} \rfloor + 1$. Hence, no candidate reaches a strict majority in (C, V_1), and candidates c, w, and each $b_j \in B'$ win the election with an approval score of k.

Fallback Voting: Control by Partition of Voters

Consider the following score tables in (C, V_2):

	c	w
score[1]	$n(k+1)+2m-k+mk$	$n(k+1)+1$
score[2]	$n(k+1)+2m-k+mk+1$	$n(k+1)+mk+k$
score[3]	$\geq n(k+1)+2m-k+mk+1$	$n(k+1)+mk+2m+1$

	$b_j \notin B'$	$b_j \in B'$
score[1]	$k-1$	0
score[2]	$\leq k+n(k+1)$	$\leq n(k+1)$
score[3]	$\leq k+n(k+1)$	$\leq n(k+1)$

Table: Level i scores in (C, V_2) for $i \in \{1, 2, 3\}$ and candidates in $B \cup \{c, w\}$.

Fallback Voting: Control by Partition of Voters

Since in (C, V_2) no candidate from B wins, the candidates participating in the final round are $B' \cup \{c, w\}$, with scores:

	c	w	$b_j \in B'$
score^1	$n(k+1)+2m+mk$	$n(k+1)+m+2$	$k-1$
score^2	$n(k+1)+2m+mk+1$	$n(k+1)+2m+mk+1$	$\leq k+n(k+1)$

Table: Level i scores in the final-stage election $(B' \cup \{c, w\}, V)$ for $i \in \{1, 2\}$.

Since both c and w are level 2 FV winners, c has been prevented from being a unique FV winner by partition of voters in model TP.

Fallback Voting: Control by Partition of Voters

($\Leftarrow$) Suppose c can be prevented from being a unique FV winner by partition of voters in model TP. From our lemma it follows that c participates in the final round.

Since c has a strict majority of approvals, c has to be tied with or lose against another candidate by a strict majority at some level.

Only candidate w has a strict majority of approvals, so w has to tie or beat c at some level in the final round.

Because of the low scores of the candidates in D and E we may assume that only candidates from B are participating in the final round besides c and w.

Let $B' \subseteq B$ be the set of candidates who also participate in the final round.

Fallback Voting: Control by Partition of Voters

Let ℓ be the number of sets in $\mathcal{S}$ not hit by B'.

As w cannot reach a strict majority of approvals on the first level, we consider the level 2 scores of c and w:

$$\begin{aligned} score^2_{(B' \cup \{c,w\},V)}(c) &= n(k+1) + 2m + mk + 1 + \ell(k+1), \\ score^2_{(B' \cup \{c,w\},V)}(w) &= n(k+1) + 2m + mk + k - \|B'\| + 1. \end{aligned}$$

Since c has a strict majority already on the second level, w must tie or beat c on this level, so the following must hold:

Fallback Voting: Control by Partition of Voters

$$
\begin{aligned}
score^2_{(B' \cup \{c,w\},V)}(c) - score^2_{(B' \cup \{c,w\},V)}(w) &\leq 0 \\
n(k+1) + 2m + mk + 1 + \ell(k+1) & \\
-n(k+1) - 2m - mk - k + \|B'\| - 1 &\leq 0 \\
\|B'\| - k + \ell(k+1) &\leq 0.
\end{aligned}
$$

This is possible only if $\ell = 0$ (i.e., all sets in $\mathcal{S}$ are hit by B'), which implies $\|B'\| \leq k$.

Thus $\mathcal{S}$ has a hitting set of size at most k. ❑

Complexity of Candidate Control

													Copeland$^\alpha$											
	Plurality		Condorcet		Approval		Bucklin		Fallback		SP-AV		$\alpha = 0$		$0 < \alpha < 1$		$\alpha = 1$ (Llull)		Range voting		NRV		Schulze	
	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D
CAUC	R	R	I	V	I	V	R	R	R	R	R	R	V	V	R	V	V	V	I	V	R	R	R	?
CAC	R	R	I	V	I	V	R	R	R	R	R	R	R	V	R	V	R	V	I	V	R	R	R	?
CDC	R	R	V	I	V	I	R	R	R	R	R	R	R	V	R	V	R	V	V	I	R	R	R	?
CPC-TE	R	R	V	I	V	I	R	R	R	R	R	R	R	V	R	V	R	V	V	I	R	R	R	V
CPC-TP	R	R			I	I	R	R	R	R	R	R	R	V	R	V	R	V	I	I	R	R	R	V
CRPC-TE	R	R	V	I	V	I	R	R	R	R	R	R	R	V	R	V	R	V	V	I	R	R	R	V
CRPC-TP	R	R			I	I	R	R	R	R	R	R	R	V	R	V	R	V	I	I	R	R	R	V

Complexity of Candidate Control: **Challenge!**

															Copeland$^\alpha$									
	Plurality		Condorcet		Approval		Bucklin		Fallback		SP-AV		$\alpha = 0$		$0 < \alpha < 1$		$\alpha = 1$ (Llull)		Range voting		NRV		Schulze	
	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D
CAUC	R	R	I	V	I	V	R	R	R	R	R	R	V	V	R	V	V	V	I	V	R	R	R	?
CAC	R	R	I	V	I	V	R	R	R	R	R	R	R	V	R	V	R	V	I	V	R	R	R	?
CDC	R	R	V	I	V	I	R	R	R	R	R	R	R	V	R	V	R	V	V	I	R	R	R	?
CPC-TE	R	R	V	I	V	I	R	R	R	R	R	R	R	V	R	V	R	V	V	I	R	R	R	V
CPC-TP	R	R			I	I	R	R	R	R	R	R	R	V	R	V	R	V	I	I	R	R	R	V
CRPC-TE	R	R	V	I	V	I	R	R	R	R	R	R	R	V	R	V	R	V	V	I	R	R	R	V
CRPC-TP	R	R			I	I	R	R	R	R	R	R	R	V	R	V	R	V	I	I	R	R	R	V

Complexity of Voter Control

<table>
<tr><th></th><th></th><th></th><th></th><th></th><th></th><th></th><th colspan="3">Copeland$^\alpha$</th><th></th><th></th><th></th></tr>
<tr><th></th><th>Plurality</th><th>Condorcet</th><th>Approval</th><th>Bucklin</th><th>Fallback</th><th>SP-AV</th><th>$\alpha = 0$</th><th>$0 < \alpha < 1$</th><th>$\alpha = 1$ (Llull)</th><th>Range voting</th><th>NRV</th><th>Schulze</th></tr>
<tr><th></th><th>C D</th><th>C D</th><th>C D</th><th>C D</th><th>C D</th><th>C D</th><th>C D</th><th>C D</th><th>C D</th><th>C D</th><th>C D</th><th>C D</th></tr>
<tr><td>CAV</td><td>V V</td><td>R V</td><td>R V</td><td>R V</td><td>R V</td><td>R V</td><td>R R</td><td>R R</td><td>R R</td><td>R V</td><td>R V</td><td>R V</td></tr>
<tr><td>CDV</td><td>V V</td><td>R V</td><td>R V</td><td>R V</td><td>R V</td><td>R V</td><td>R R</td><td>R R</td><td>R R</td><td>R V</td><td>R V</td><td>R V</td></tr>
<tr><td>CPV-TE</td><td>V V</td><td rowspan="2">R V</td><td>R V</td><td>R R</td><td>R R</td><td>R V</td><td>R R</td><td>R R</td><td>R R</td><td>R V</td><td>R R</td><td>R R</td></tr>
<tr><td>CPV-TP</td><td>R R</td><td>R V</td><td>R ?</td><td>R R</td><td>R R</td><td>R R</td><td>R R</td><td>R R</td><td>R V</td><td>R R</td><td>R R</td></tr>
</table>

Complexity of Voter Control: **Challenge!**

	Plurality		Condorcet		Approval		Bucklin		Fallback		SP-AV		Copeland$^\alpha$ $\alpha = 0$		Copeland$^\alpha$ $0 < \alpha < 1$		Copeland$^\alpha$ $\alpha = 1$ (Llull)		Range voting		NRV		Schulze	
	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D
CAV	V	V	R	V	R	V	R	V	R	V	R	V	R	R	R	R	R	R	R	V	R	V	R	V
CDV	V	V	R	V	R	V	R	V	R	V	R	V	R	R	R	R	R	R	R	V	R	V	R	V
CPV-TE	V	V	R	V	R	V	R	R	R	R	R	V	R	R	R	R	R	R	R	V	R	R	R	R
CPV-TP	R	R			R	V	R	?	R	R	R	R	R	R	R	R	R	R	R	V	R	R	R	R

Complexity of Control by Partition in Veto Elections

control type	CPC-TE		CPC-TP		CRPC-TE		CRPC-TP		CPV-TE		CPV-TP	
	C	D	C	D	C	D	C	D	C	D	C	D
unique-winner	R	R	R	R	R	R	R	R	V	V	R	R
nonunique-winner	R	R	R	R	R	R	R	R	V	V	R	R

Table: Overview of complexity results for control by partition in veto elections:

Complexity of Control by Partitioning Veto Elections and of Control by Adding Candidates to Plurality Elections, C. Maushagen and J. Rothe. *Annals of Mathematics and Artificial Intelligence* 82(4):219–244, 2018.

Candidate Control in Maximin Elections

CAUC		CAC		CDC		CPC-TE		CPC-TP		CRPC-TE		CRPC-TP	
C	D	C	D	C	D	C	D	C	D	C	D	C	D
V♡	V♡	R♡	V♡	V♡	V♡	**R**	V♠	**R**	V♠	**R**	V♠	**R**	V♠

Table: Complexity results for candidate control in maximin elections:

Boldface: The Last Voting Rule Is Home: Complexity of Control by Partition of Candidates or Voters in Maximin Elections, C. Maushagen and J. Rothe. Proc. ECAI'20, IOS Press, pp. 163–170, 2020.

♡ Multimode Control Attacks on Elections, P. Faliszewski and E. Hemaspaandra and L. Hemaspaandra. *Journal of Artificial Intelligence Research* 40:305–351, 2011.

♠ Complexity of Control by Partitioning Veto and Maximin Elections and of Control by Adding Candidates to Plurality Elections, C. Maushagen and J. Rothe. Proc. ECAI'16, IOS Press, pp. 277–285, 2016.

Voter Control in Maximin Elections

CAV		CDV		CPV-TE		CPV-TP	
C	D	C	D	C	D	C	D
R♡	R♡	R♡	R♡	**R**	**R**	**R**	**R**

Table: Overview of complexity results for voter control in maximin elections:

Boldface: The Last Voting Rule Is Home: Complexity of Control by Partition of Candidates or Voters in Maximin Elections, C. Maushagen and J. Rothe. Proc. ECAI'20, IOS Press, pp. 163–170, 2020.

♡ Multimode Control Attacks on Elections, P. Faliszewski and E. Hemaspaandra and L. Hemaspaandra. *Journal of Artificial Intelligence Research* 40:305–351, 2011.

Constructive Control in Borda Elections

CCAUC	CCAC	CCDC	CCPC-TE	CCPC-TP	CCRPC-TE	CCRPC-TP	CCAV	CCDV	CCPV-TE	CCPV-TP
R	R$	R❄	**R**	**R**	**R**	**R**	R§	R♠	**R**	**R**

Table: Complexity results for constructive control in Borda elections:

Boldface: Control Complexity in Borda Elections: Solving All Open Cases of Offline Control and Some Cases of Online Control, M. Neveling and J. Rothe. *Artificial Intelligence* 298, 2021

§ Complexity of Control of Borda Count Elections, N. Russel. Master's Thesis, Rochester Institute of Technology, 2007

$ Cloning in Elections: Finding the Possible Winners, E. Elkind, P. Faliszewski, and A. Slinko. *Journal of Artificial Intelligence Research* 42:529–573, 2011

❄ Elections with Few Voters: Candidate Control Can Be Easy, J. Chen, P. Faliszewski, R. Niedermeier, and N. Talmon. Proc. AAAI'15, AAAI Press, pp. 2045–2051, 2015

♠ Dichotomy for Pure Scoring Rules Under Manipulative Electoral Actions, E. Hemaspaandra and H. Schnoor. Proc. ECAI'16, IOS Press, pp. 1071–1079, 2016.

Destructive Control in Borda Elections

DCAUC	DCAC	DCDC	DCPC-TE	DCPC-TP	DCRPC-TE	DCRPC-TP	DCAV	DCDV	DCPV-TE	DCPV-TP
V	V†	V†	**V**	**R**	**V**	**R**	V§	V§	V§	**R**

Table: Complexity results for constructive control in Borda elections:

Boldface: Control Complexity in Borda Elections: Solving All Open Cases of Offline Control and Some Cases of Online Control, M. Neveling and J. Rothe. *Artificial Intelligence* 298, 2021

§ Complexity of Control of Borda Count Elections, N. Russel. Master's Thesis, Rochester Institute of Technology, 2007

† Controlling Elections by Replacing Candidates or Votes, A. Loreggia, N. Narodytska, F. Rossi, B. Venable, and T. Walsh. Proc. AAMAS'15, IFAAMAS, pp. 1737–1738, 2015

Merry Christmas from Santa Claus!