Manipulation: Strategic Voting

Example

Consider the Borda election with candidates a, b, and c and the following votes:

	Sincere Votes				Strategic Votes		
points :	2	1	0		2	1	0
5 votes :	a	b	c		a	b	c
5 votes :	b	a	c	$\Rightarrow$	b	c	a
1 vote :	c	a	b		c	a	b
	Borda winner a				Borda winner b		

Variants of the Manipulation Problem

Definition (Constructive Coalitional Manipulation)

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE COALITIONAL MANIPULATION ($\mathcal{E}$-CCM).

Given:
- A set C of candidates,
- a list V of nonmanipulative voters over C,
- a list S of manipulative voters (whose votes over C are still unspecified) with $V \cap S = \emptyset$, and
- a distinguished candidate $c \in C$.

Question: Is there a way to set the preferences of the voters in S such that, under election system $\mathcal{E}$, c is a winner of election $(C, V \cup S)$?

Variants of the Manipulation Problem

Remark: Variants:

- $\mathcal{E}$-Destructive Coalitional Manipulation ($\mathcal{E}$-DCM) is the same with "c is not a winner of $(C, V \cup S)$."
- If $\|S\| = 1$, we obtain the single-manipulator problems:
  - $\mathcal{E}$-Constructive Manipulation ($\mathcal{E}$-CM) and
  - $\mathcal{E}$-Destructive Manipulation ($\mathcal{E}$-DM).
- Voters can also be weighted (see next slide).
- These problems can also be defined in the "unique-winner" model.

Variants of the Manipulation Problem

Definition (Constructive Coalitional Weighted Manipulation)

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE (DESTRUCTIVE) COALITIONAL WEIGHTED MANIPULATION ($\mathcal{E}$-CCWM / $\mathcal{E}$-DCWM).

Given:
- A set C of candidates,
- a list V of nonmanipulative voters over C each having a nonnegative integer weight,
- a list of the weights of the manipulators in S (whose votes over C are still unspecified) with $V \cap S = \emptyset$, and
- a distinguished candidate $c \in C$.

Question: Can the preferences of the voters in S be set such that c is a $\mathcal{E}$-winner (is not an $\mathcal{E}$-winner) of $(C, V \cup S)$?

Some Basic Complexity Classes

Definition

1. FP denotes the *class of polynomial-time computable total functions* mapping from Σ^* to Σ^*.
2. P denotes the *class of problems that can be decided in polynomial time* (i.e., via a deterministic polynomial-time Turing machine).
3. NP denotes the *class of problems that can be accepted in polynomial time* (i.e., via a nondeterministic polynomial-time Turing machine).

NP in Ancient Times

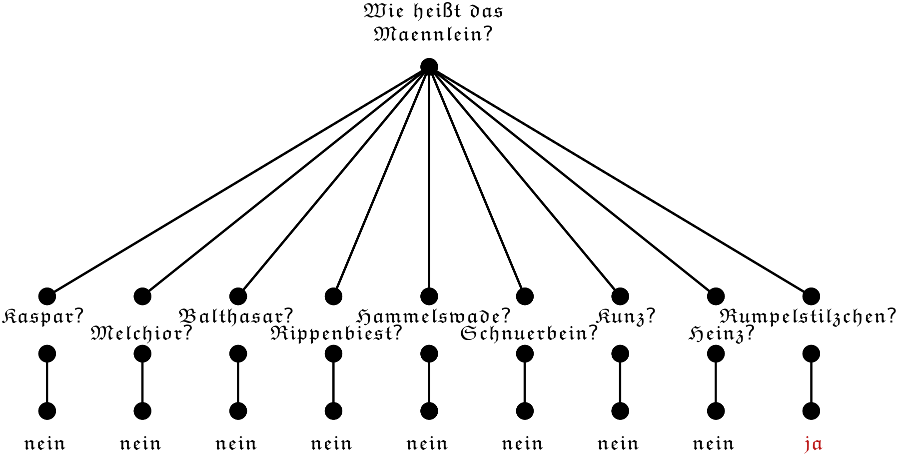


Figure: Nondeterministic Guessing and Deterministic Checking

NP Today

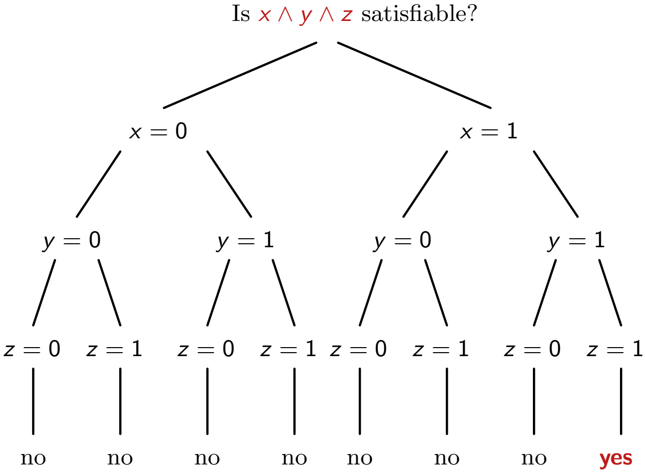


Figure: Nondeterministic Guessing and Deterministic Checking

Some Basic Complexity Classes

Remark:

- Intuitively, FP and P, respectively, capture feasibility/efficiency of computing functions and solving decision problems.
- $A \in \mathrm{NP}$ if and only if there exist a set $B \in \mathrm{P}$ and a polynomial p such that for each $x \in \Sigma^*$,

$$x \in A \iff (\exists w)\,[|w| \leq p(|x|) \text{ and } (x, w) \in B].$$

  That is, NP is the class of problems whose YES instances can be easily checked.

- Central open question of TCS: $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$
- Examples of problems in NP: SAT, Traveling Salesperson Problem, Vertex Cover, Clique, Hamilton Circuit, ...

Pol-Time Many-One Reducibility and Completeness

Definition

Let Σ be an alphabet and $A, B \subseteq \Sigma^*$. Let $\mathcal{C}$ be any complexity class.

1. Define the *polynomial-time many-one reducibility*, denoted by $\leq_{\mathrm{m}}^{\mathrm{p}}$, as follows: $A \leq_{\mathrm{m}}^{\mathrm{p}} B$ if there is a function $f \in \mathrm{FP}$ such that
$$(\forall x \in \Sigma^*)\,[x \in A \iff f(x) \in B].$$
2. A set B is *$\leq_{\mathrm{m}}^{\mathrm{p}}$-hard for $\mathcal{C}$* (or *$\mathcal{C}$-hard*) if $A \leq_{\mathrm{m}}^{\mathrm{p}} B$ for each $A \in \mathcal{C}$.
3. A set B is *$\leq_{\mathrm{m}}^{\mathrm{p}}$-complete for $\mathcal{C}$* (or *$\mathcal{C}$-complete*) if
   1. B is $\leq_{\mathrm{m}}^{\mathrm{p}}$-hard for $\mathcal{C}$ (lower bound) and
   2. $B \in \mathcal{C}$ (upper bound).
4. $\mathcal{C}$ is *closed under the $\leq_{\mathrm{m}}^{\mathrm{p}}$-reducibility* (*$\leq_{\mathrm{m}}^{\mathrm{p}}$-closed*, for short) if
$$(A \leq_{\mathrm{m}}^{\mathrm{p}} B \text{ and } B \in \mathcal{C}) \implies A \in \mathcal{C}.$$

Properties of $\leq_{\mathrm{m}}^{\mathrm{p}}$

1. $A \leq_{\mathrm{m}}^{\mathrm{p}} B$ implies $\overline{A} \leq_{\mathrm{m}}^{\mathrm{p}} \overline{B}$, yet in general it is not true that $A \leq_{\mathrm{m}}^{\mathrm{p}} \overline{A}$.
2. $\leq_{\mathrm{m}}^{\mathrm{p}}$ is a reflexive and transitive, yet not antisymmetric relation.
3. P and NP are $\leq_{\mathrm{m}}^{\mathrm{p}}$-closed.
   That is, upper bounds are inherited downward with respect to $\leq_{\mathrm{m}}^{\mathrm{p}}$.
4. If $A \leq_{\mathrm{m}}^{\mathrm{p}} B$ and A is $\leq_{\mathrm{m}}^{\mathrm{p}}$-hard for some complexity class $\mathcal{C}$, then B is $\leq_{\mathrm{m}}^{\mathrm{p}}$-hard for $\mathcal{C}$.
   That is, lower bounds are inherited upward with respect to $\leq_{\mathrm{m}}^{\mathrm{p}}$.
5. Let $\mathcal{C}$ and $\mathcal{D}$ be any complexity classes. If $\mathcal{C}$ is $\leq_{\mathrm{m}}^{\mathrm{p}}$-closed and B is $\leq_{\mathrm{m}}^{\mathrm{p}}$-complete for $\mathcal{D}$, then $\mathcal{D} \subseteq \mathcal{C} \iff B \in \mathcal{C}$.
   In particular, if B is NP-complete, then
   $$\mathrm{P} = \mathrm{NP} \iff B \in \mathrm{P}.$$

Plurality and Regular Cup Are Easy to Manipulate

Theorem (Conitzer, Sandholm, and Lang (2007))

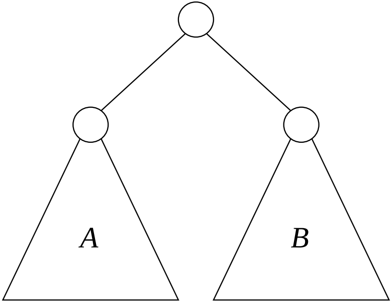
Plurality-CCWM *and Regular-Cup-*CCWM *are in* P *(for any number of candidates, in both the unique-winner and nonunique-winner model).*

Proof Sketch:

1. For plurality, the manipulators simply check if the distinguished candidate c wins when they all rank c first.
   - If so, they have found a successful strategy.
   - If not, no strategy can make c win.
2. For the regular cup protocol, let the assignment of candidates to the leaves of the binary balanced tree be given.

Plurality and Regular Cup Are Easy to Manipulate

- Every inner vertex represents a subelection T:
- Only the order of the candidates in A and B is relevant for the outcome of the subelection.
- **Goal:** Determine the potential winners of each subelection.



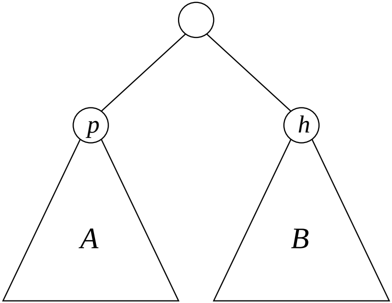
Proposition: A candidate p can win a subelection T $\iff$

1. p can win the subelection in one of the two children of T's root and
2. p can defeat any potential winner, say h, in the subelection of the other child of T's root by pairwise comparison.

Plurality and Regular Cup Are Easy to Manipulate

Proof of Proposition: **(⇒)** is obvious.

(⇐) Assume that, using the manipulators' votes, (1) and (2) are true:



- Let $\vec{A}$ be a manipulator's order of the candidates in A that makes p win and
- $\vec{B}$ be this manipulator's order of the candidates in B that makes h win.

This manipulator then votes $\vec{A}\vec{B}$ over $A \cup B$, the others accordingly. Then p and h are in this subelection's final round, which p wins. ❑ Proposition

From this proposition, we can design a recursive algorithm running in time $\mathcal{O}(m^3 n)$, where m is the number of candidates and n the number of voters. ❑

Copeland with three Candidates Is Easy to Manipulate

Copeland voting: For each $c, d \in C$, $c \neq d$,

- let $N(c, d)$ be the number of voters who prefer c to d,
- let $Z(c, d) = 1$ if $N(c, d) > N(d, c)$ and
- $Z(c, d) = 1/2$ if $N(c, d) = N(d, c)$.
- The *Copeland score of c* is $CScore(c) = \sum_{d \neq c} Z(c, d)$.
- Whoever has the maximum Copeland score wins.

Theorem (Conitzer, Sandholm, and Lang (2007))

Copeland-CCWM *for three candidates is in* P *(in the unique-winner model only).*

Proof: We show that: If Copeland with three candidates has a successful CCWM strategy, then it has a successful CCWM strategy where all manipulators vote identically.

Copeland with three Candidates Is Easy to Manipulate

Let $C = \{a, b, c\}$, with distinguished candidate c.

Further, we are given

- the preferences and weights of the honest voters in V and
- the weights (but not the preferences) of the manipulators in S.

Let $w : V \cup S \to \mathbb{N}$ be the weight function.

Let $K = \sum_{s \in S} w(s)$ be the total weight of the manipulators.

For a sublist $U \subseteq V \cup S$ and any two candidates $x, y \in C$, we define

$$\begin{aligned} N_U(x, y) &= \sum_{u \in U \,:\, x >_u y} w(u) \quad \text{and} \\ D_U(x, y) &= N_U(x, y) - N_U(y, x). \end{aligned}$$

Copeland with three Candidates Is Easy to Manipulate

- $N_U(x, y)$ is the sum of the weights of those voters in U that prefer x to y.
- The difference $D_U(x, y)$ is positive if the total weight of x's supporters is greater than that of y's supporters,
- $D_U(x, y)$ is negative if y's supporters outweigh x's supporters (since $D_U(x, y) = -D_U(y, x)$), and
- $D_U(x, y) = 0$ if both groups balance each other out.

Now consider the following four cases.

Copeland with three Candidates Is Easy to Manipulate

Case 1: $K > D_V(a,c)$ and $K > D_V(b,c)$.

In this case, if all manipulators in S cast the vote $c > a > b$, c is the one and only winner of $(C, V \cup S)$.

Case 2: $K > D_V(a,c)$ and $K = D_V(b,c)$.

It may be assumed, without loss of generality, that all manipulators put their favorite candidate c on top of their votes.

However, who of them votes $c > a > b$ and who votes $c > b > a$?

Since c is on top of every vote from S, we have from the case assumption that

$$D_{V\cup S}(c,a) = K - D_V(a,c) > 0, \quad (1)$$

$$D_{V\cup S}(c,b) = K - D_V(b,c) = 0. \quad (2)$$

Copeland with three Candidates Is Easy to Manipulate

Due to (1), c gets one point from the pairwise comparison with a in $(C, V \cup S)$, and a gets no points from this comparison.

Due to (2), the pairwise comparison between b and c in $(C, V \cup S)$ ends up with a tie, so both get half a point.

Without the last pairwise comparison, a versus b,

- c already has one and a half points in $(C, V \cup S)$,
- b half a point, and
- a has no point at all.

In order to make c a unique winner in $(C, V \cup S)$, b must not get a whole point from the comparison with a, i.e., it must hold that

$$D_{V \cup S}(a, b) \geq 0.$$

Copeland with three Candidates Is Easy to Manipulate

This, however, is true exactly if

$$K \geq D_V(b, a).$$

Also in this case, all manipulators cast the vote $c > a > b$, seeking to ensure that $D_{V \cup S}(a, b) \geq 0$.

If this is not enough to make c a unique winner—namely, because

$$K < D_V(b, a),$$

then there exists no successful manipulation in this case.

Copeland with three Candidates Is Easy to Manipulate

Case 3: $K = D_V(a,c)$ and $K > D_V(b,c)$.

This case can be handled analogously to Case 2, with the roles of a and b reversed.

Case 4: $K < D_V(a,c)$ or $K < D_V(b,c)$ or $(K \leq D_V(a,c)$ and $K \leq D_V(b,c))$.

In this case, the Copeland score of c in $(C, V \cup S)$ cannot be greater than 1, regardless of how the manipulators vote.

Thus, they are doomed to fail:
c cannot be made a unique winner. ❑

Maximin Voting

Maximin voting (a.k.a. the Simpson or Simpson-Kramer rule):
For each $c, d \in C$, $c \neq d$, let again $N(c, d)$ be the number of voters who prefer c to d.

- The *maximin score of c* is

$$MScore(c) = \min_{d \neq c} N(c, d).$$

- Whoever has the maximum *MScore* wins.

That is, the maximin winners are those candidates whose worst pairwise comparison with other candidates is best.

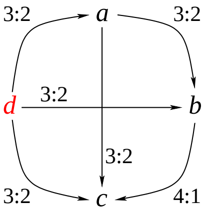
Maximin voting satisfies the Condorcet criterion.

Determining Maximin Winners

Example

Consider the election (C, V) with $C = \{a, b, c, d\}$ and V:

v_1 :	a	b	c	d
v_2 :	b	a	d	c
v_3 :	c	d	a	b
v_4 :	d	a	b	c
v_5 :	d	b	c	a



$\Longrightarrow$ d is the Condorcet winner.

	a	b	c	d	$MScore$	
a	$\times$	3	3	2	2	
b	2	$\times$	4	2	2	
c	2	1	$\times$	2	1	
d	3	3	3	$\times$	**3**	$\leftarrow$ max

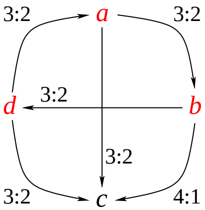
$\Longrightarrow$ d is the maximin winner.

Determining Maximin Winners

Example

Consider the election (C, V) with $C = \{a, b, c, d\}$ and V:

v_1 :	a	b	c	d
v_2 :	b	a	d	c
v_3 :	c	d	a	b
v_4 :	d	a	b	c
v_5 :	b	d	c	a



$\Longrightarrow$ there is no Condorcet winner.

	a	b	c	d	$MScore$	
a	$\times$	3	3	2	**2**	$\leftarrow$ max
b	2	$\times$	4	3	**2**	$\leftarrow$ max
c	2	1	$\times$	2	1	
d	3	2	3	$\times$	**2**	$\leftarrow$ max

$\Longrightarrow$ a, b, d are the maximin winners.

Maximin with three Candidates Is Easy to Manipulate

Theorem (Conitzer, Sandholm, and Lang (2007))

Maximin-CCWM *for three candidates is in* P *(in both the unique-winner and nonunique-winner model).*

Proof: We show that: If maximin with three candidates has a successful CCWM strategy, then it has a successful CCWM strategy where all manipulators vote identically.

Let $C = \{a, b, c\}$, with distinguished candidate c.

Further, we are given

- the preferences and weights of the honest voters in V and
- the weights (but not the preferences) of the manipulators in S.

Maximin with three Candidates Is Easy to Manipulate

Let $w : V \cup S \to \mathbb{N}$ be the weight function.

Let $W = \sum_{v \in V \cup S} w(v)$ be the total weight of all voters.

For any two candidates $x, y \in C$, recall that

$$N(x, y) = \sum_{v \in V \cup S \,:\, x > y} w(v).$$

It may be assumed, without loss of generality, that all manipulators put their favorite candidate c on top of their votes.

Suppose the manipulators $s \in S$ can make c win by individually voting either $c\ a\ b$ or $c\ b\ a$.

We show: Then c also wins if all manipulators $s \in S$ simultaneously vote either $c\ a\ b$ or $c\ b\ a$.

Maximin with three Candidates Is Easy to Manipulate

Case 1: $N(a,b) < N(a,c)$ and $N(b,a) < N(b,c)$.

In this case, since $N(a,b) \geq N(b,a)$ and $N(b,a) \geq N(a,b)$, one of a and b (say a) has weight at least $W/2$ against the other (i.e., $N(a,b) \geq N(b,a)$).

Hence, $MScore(a) \geq W/2$ (because $N(a,c) > N(a,b)$).

Since $N(a,c) > N(a,b)$, c's weight against a is less than $W/2$.

Thus $MScore(c) < W/2 \leq MScore(a)$.

It follows that c does not win, a contradiction.

So this case cannot occur.

Maximin with three Candidates Is Easy to Manipulate

Case 2: $N(a, b) \geq N(a, c)$ or $N(b, a) \geq N(b, c)$.

Without loss of generality, assume $N(a, b) \geq N(a, c)$ (the other case is analogous).

Thus c is a's strongest rival.

Then all manipulators $s \in S$ can simultaneously vote $c\ a\ b$. This

- modifies neither $MScore(a)$
- nor $MScore(c)$;
- it can only decrease $MScore(b)$ (but cannot increase it). ❑

Upper bounds are inherited downward w.r.t. $\leq_{\mathrm{m}}^{\mathrm{p}}$

Corollary

All more restrictive variants of the manipulation problem are in P *for:*

- *plurality (for any number of candidates),*
- *regular cup (for any number of candidates),*
- *Copeland (for at most three candidates), and*
- *maximin (for at most three candidates).*

STV-CM is NP-complete

Single Transferable Vote (STV) for m candidates proceeds in $m-1$ rounds. In each round:

- A candidate with lowest plurality score is eliminated (using some tie-breaking rule if needed) and
- all votes for this candidate transfer to the next remaining candidate in this vote's order.

The last remaining candidate wins.

Theorem (Bartholdi and Orlin (1991))

STV-CONSTRUCTIVE MANIPULATION *is* NP*-complete.*

STV-CM is NP-complete: Reduction from X3C

Proof: Membership in NP is clear.

To prove NP-hardness of STV-Constructive Manipulation, we reduce from the following NP-complete problem:

Name: Exact Cover by Three-Sets (X3C).

Given:
- A set $B = \{b_1, b_2, \ldots, b_{3m}\}$, $m \geq 1$, and
- a collection $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$.

Question: Is there a subcollection $\mathcal{S}' \subseteq \mathcal{S}$ such that each element of B occurs in exactly one set in $\mathcal{S}'$?

In other words, does there exist an index set $I \subseteq \{1, 2, \ldots, n\}$ with $\|I\| = m$ such that $\bigcup_{i \in I} S_i = B$?

STV-CM is NP-complete: The Candidates

Given an instance $(B, \mathcal{S})$ of $\mathrm{X3C}$ with

$$
\begin{aligned}
B &= \{b_1, b_2, \ldots, b_{3m}\} \\
\mathcal{S} &= \{S_1, S_2, \ldots, S_n\}
\end{aligned}
$$

where $m \geq 1$, $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$, construct an election $(C, V \cup \{s\})$ with manipulator s and $5n + 3(m+1)$ candidates:

1. "possible winners": c and w;
2. "first losers": $a_1, a_2, \ldots, a_n$ and $\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n$;
3. "w-bloc": $b_0, b_1, \ldots, b_{3m}$;
4. "second line": $d_1, d_2, \ldots, d_n$ and $\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_n$;
5. "garbage collectors": $g_1, g_2, \ldots, g_n$.

STV-CM is NP-complete: The Properties

Property 1: $a_1, a_2, \dots, a_n$ and $\overline{a}_1, \overline{a}_2, \dots, \overline{a}_n$ are among the first $3n$ candidates to be eliminated.

Property 2: Let $I = \{i \mid \overline{a}_i \text{ is eliminated prior to } a_i\}$. Then

$$c \text{ can be made win } (C, V \cup \{s\}) \iff I \text{ is a 3-cover.}$$

Property 3:

1. For any $I \subseteq \{1, 2, \dots, n\}$, there is a preference for s such that

$$\overline{a}_i \text{ is eliminated prior to } a_i \iff i \in I.$$

2. Such a preference for s is constructed as follows:
   - If $i \in I$ then place a_i in the ith position of s.
   - If $i \notin I$ then place $\overline{a}_i$ in the ith position of s.

STV-CM is NP-complete: The Nonmanipulative Voters

(1)	$12n$	votes:	c	$\cdots$				
(2)	$12n-1$	votes:	w	c	$\cdots$			
(3)	$10n+2m$	votes:	b_0	w	c	$\cdots$		
(4) For each $i \in \{1,2,\dots,3m\}$,	$12n-2$	votes:	b_i	w	c	$\cdots$		
(5) For each $j \in \{1,2,\dots,n\}$,	$12n$	votes:	g_j	w	c	$\cdots$		
(6) For each $j \in \{1,2,\dots,n\}$,	$6n+4j-5$	votes:	d_j	$\overline{d}_j$	w	c	$\cdots$	
and if $S_j = \{b_x, b_y, b_z\}$ then	2	votes:	d_j	b_x	w	c	$\cdots$	
	2	votes:	d_j	b_y	w	c	$\cdots$	
	2	votes:	d_j	b_z	w	c	$\cdots$	
(7) For each $j \in \{1,2,\dots,n\}$,	$6n+4j-1$	votes:	$\overline{d}_j$	d_j	w	c	$\cdots$	
	2	votes:	$\overline{d}_j$	b_0	w	c	$\cdots$	
(8) For each $j \in \{1,2,\dots,n\}$,	$6n+4j-3$	votes:	a_j	g_j	w	c	$\cdots$	
	1	vote:	a_j	d_j	g_j	w	c	$\cdots$
	2	votes:	a_j	$\overline{a}_j$	g_j	w	c	$\cdots$
(9) For each $j \in \{1,2,\dots,n\}$,	$6n+4j-3$	votes:	$\overline{a}_j$	g_j	w	c	$\cdots$	
	1	vote:	$\overline{a}_j$	$\overline{d}_j$	g_j	w	c	$\cdots$
	2	votes:	$\overline{a}_j$	a_j	g_j	w	c	$\cdots$

STV-CM is NP-complete: Votes for c and w in (1) and (2)

- Initially, $score(c) = 12n$ and $score(w) = 12n - 1$.
- Since all voters except (1) vote $w > c$, c can get further votes only when w is eliminated.
- If w gets two more votes, w cannot be eliminated before c.
- Hence, manipulator s must ensure that w is eliminated before c.

STV-CM is NP-complete: w-Bloc (3) and (4)

- Initially,

$$\begin{aligned} score(b_0) &= 10n + 2m \\ score(b_i) &= 12n - 2, \quad 1 \leq i \leq 3m. \end{aligned}$$

- For each voter in (3) and (4), w is directly behind b_0 or b_i, $1 \leq i \leq 3m$.

- If b_0 or b_i, $1 \leq i \leq 3m$, is eliminated, w gets more than two more votes and c doesn't win.

- Hence, manipulator s must ensure that $score(b_i) \geq 12n$, $0 \leq i \leq 3m$. Therefore: "second line" candidates are needed.

STV-CM is NP-complete: Second Line

- Initially, $score(d_j) = score(\overline{d}_j) = 6n + 4j + 1$, $1 \leq j \leq n$.
- If d_j is eliminated, then
  - two votes each go to b_x, b_y, b_z, where $S_j = \{b_x, b_y, b_z\}$;
  - the remaining votes go to $\overline{d}_j$.
- If $\overline{d}_j$ is eliminated, then
  - two votes go to b_0;
  - the remaining votes go to d_j.
- If d_j is eliminated before $\overline{d}_j$, then $score(\overline{d}_j) > 12n$.
- If $\overline{d}_j$ is eliminated before d_j, then $score(d_j) > 12n$.
- Hence, at most one of d_j and $\overline{d}_j$ can be eliminated before c or w.

STV-CM is NP-complete: First Losers

- Initially, for $1 \leq j \leq n$,

$$score(a_j) = score(\overline{a}_j) = 6n + 4j = score(d_j) - 1 = score(\overline{d}_j) - 1.$$

- If a_j is eliminated, then
  - one vote goes to d_j,
  - two votes go to $\overline{a}_j$, and
  - the remaining votes go to g_j.
- If $\overline{a}_j$ is eliminated, then
  - one vote goes to $\overline{d}_j$,
  - two votes go to a_j, and
  - the remaining votes go to g_j.

STV-CM is NP-complete: Garbage Collectors

- Initially, $score(g_j) = 12n$, $1 \leq j \leq n$.
- Hence, they are safe against being eliminated too early.
- Their purpose is, e.g., in (5), to ensure that w doesn't score.

STV-CM is NP-complete: Elimination Sequence Encodes a 3-Cover

Lemma (Bartholdi and Orlin (1991))

1. *Exactly one of d_j and $\overline{d}_j$ will be among the first $3n$ candidates to be eliminated.*

2. *Candidate c will win if and only if*

$$J = \{j \mid d_j \text{ is among the first } 3n \text{ candidates to be eliminated}\}$$

is the index set of an exact 3-cover for $\mathcal{S}$.

STV-CM is NP-complete: Elimination Sequence Encodes a 3-Cover

Proof:

1. The first $3n$ candidates to be eliminated belong to the set

$$\{a_j, \overline{a}_j, d_j, \overline{d}_j \mid 1 \leq j \leq n\},$$

   and

   - if d_j is eliminated then $score(\overline{d}_j) > 12n$,
   - if $\overline{d}_j$ is eliminated then $score(d_j) > 12n$,

2. **($\Leftarrow$)** Let J be the index set of an exact 3-cover, i.e., $|J| = m$.

   Consider the $(3n+1)st$ round, right after the $3n$-th candidate has been eliminated.

STV-CM is NP-complete: Elimination Sequence Encodes a 3-Cover

Let $j \in J$ and $b_i \in S_j$.

Then d_j has been eliminated and b_i has received two of d_j's votes in (6).

$\Rightarrow$ $\mathit{score}(b_i) \geq 12n > \mathit{score}(w)$.

Since J is the index set of an *exact* 3-cover, we have

$$\mathit{score}(b_i) \geq 12n \quad \text{for } \mathit{each}\ i,\ 1 \leq i \leq 3m.$$

For $j \notin J$, $\overline{d}_j$ has been eliminated.

Thus $n - m$ candidates $\overline{d}_j$ have transferred two votes to b_0.

$\Rightarrow$ $\mathit{score}(b_0) \geq 10n + 2m + 2(n-m) = 12n > \mathit{score}(w)$.

STV-CM is NP-complete: Elimination Sequence Encodes a 3-Cover

Thus, in round $3n+1$, there remain c, w, all b_i, all g_j, d_j for $j \notin J$, and $\overline{d}_j$ for $j \in J$. All these candidates except w have a score of at least $12n$.

Now, w is eliminated and w's $12n-1$ votes from (2) are transferred to c.

Next, the b_i, $0 \leq i \leq 3m$, are eliminated, whose votes from (3), (4), and (6) are transferred to c.

$\Rightarrow$ $score(c) \geq 24n - 1 + (3m+1)12n = (m+1)36n - 1$.

Only in (5), (8), and (9), g_j receives votes before c. Hence, no g_j has more votes than $12n + 10n + 10n = 32n$.

STV-CM is NP-complete: Elimination Sequence Encodes a 3-Cover

Only in (6) and (7) and for one voter in (8) and (9) is d_j or $\overline{d}_j$ placed before c. Hence, none of the remaining d_j and $\overline{d}_j$ has more votes than

$$2(10n+1)+2=20n+4.$$

Thus all g_j and all the remaining d_j and $\overline{d}_j$ will be eliminated before c.

It follows that c alone wins.

($\Rightarrow$) Suppose that J is *not* the index set of an exact 3-cover.

Case 1: $|J| > m$. Then fewer than $n-m$ candidates $\overline{d}_j$ will be eliminated in the first $3n$ rounds. Thus $\mathit{score}(b_0) \leq 12n-2 < \mathit{score}(w)$.

STV-CM is NP-complete: Elimination Sequence Encodes a 3-Cover

Hence, b_0 will be eliminated before w and transfers its votes from (3) to w. Thus w cannot be eliminated before c and so c cannot win.

Case 2: $|J| \geq m$. Then there is some uncovered b_i, $1 \leq i \leq 3m$, with

$$score(b_i) \leq 12 - 2,$$

since b_i did not receive the two votes from d_j in (6).

Hence, b_i is eliminated before w and transfers its two votes from (4) to w. Thus $score(w) \geq 12n + 1 > 12n = score(c)$.

It follows that w cannot be eliminated before c and so c cannot win. ❑ Lemma

STV-CM is NP-complete: The Manipulor's Preference

Lemma (Bartholdi and Orlin (1991))

Let $I \subseteq \{1, 2, \dots, n\}$ and consider the strategic preference of manipulator s in which the i-th candidate is

- *a_i if $i \in I$ and*
- *$\overline{a}_i$ if $i \notin I$.*

Then the order in which the first $3n$ candidates are eliminated is:

1. *The $(3i-2)$nd candidate to be eliminated is*
   - *$\overline{a}_i$ if $i \in I$ and*
   - *a_i if $i \notin I$.*
2. *The $(3i-1)$st candidate to be eliminated is*
   - *d_i if $i \in I$ and*
   - *$\overline{d}_i$ if $i \notin I$.*
3. *The $3i$-th candidate to be eliminated is*
   - *a_i if $i \in I$ and*
   - *$\overline{a}_i$ if $i \notin I$.*

STV-CM is NP-complete: Recall the Nonmanipulaters

(1)	$12n$	votes:	c	$\cdots$				
(2)	$12n-1$	votes:	w	c	$\cdots$			
(3)	$10n+2m$	votes:	b_0	w	c	$\cdots$		
(4) For each $i \in \{1,2,\ldots,3m\}$,	$12n-2$	votes:	b_i	w	c	$\cdots$		
(5) For each $j \in \{1,2,\ldots,n\}$,	$12n$	votes:	g_j	w	c	$\cdots$		
(6) For each $j \in \{1,2,\ldots,n\}$,	$6n+4j-5$	votes:	d_j	$\overline{d}_j$	w	c	$\cdots$	
and if $S_j = \{b_x, b_y, b_z\}$ then	2	votes:	d_j	b_x	w	c	$\cdots$	
	2	votes:	d_j	b_y	w	c	$\cdots$	
	2	votes:	d_j	b_z	w	c	$\cdots$	
(7) For each $j \in \{1,2,\ldots,n\}$,	$6n+4j-1$	votes:	$\overline{d}_j$	d_j	w	c	$\cdots$	
	2	votes:	$\overline{d}_j$	b_0	w	c	$\cdots$	
(8) For each $j \in \{1,2,\ldots,n\}$,	$6n+4j-3$	votes:	a_j	g_j	w	c	$\cdots$	
	1	vote:	a_j	d_j	g_j	w	c	$\cdots$
	2	votes:	a_j	$\overline{a}_j$	g_j	w	c	$\cdots$
(9) For each $j \in \{1,2,\ldots,n\}$,	$6n+4j-3$	votes:	$\overline{a}_j$	g_j	w	c	$\cdots$	
	1	vote:	$\overline{a}_j$	$\overline{d}_j$	g_j	w	c	$\cdots$
	2	votes:	$\overline{a}_j$	a_j	g_j	w	c	$\cdots$

STV-CM is NP-complete: The Manipulor's Preference

Proof: Induction on i. Assume the first $3i-3$ candidates have been eliminated, and for each $j<i$, these are:

$$a_j, \quad \overline{a}_j, \quad \text{and exactly one of } d_j \text{ and } \overline{d}_j.$$

Case 1: $i \in I$. Then a_i is the i-th candidate in the preference of manipulator s, i.e.,

$$score(a_i) = 6n + 4i + 1.$$

$$\text{Since} \quad score(\overline{a}_i) = 6n + 4i, \quad \overline{a}_i \text{ is eliminated.}$$

$$\begin{aligned} \Longrightarrow \; score(\overline{d}_i) &= 6n + 4i + 2 && \text{and} \\ score(a_i) &= 6n + 4i + 3 && \text{as } \overline{a}_i\text{'s votes in (9) are transferred} \\ score(d_i) &= 6n + 4i + 1 && \text{as before from voter group (6).} \end{aligned}$$

STV-CM is NP-complete: The Manipulor's Preference

So d_i is eliminated now. Hence, d_i's votes in group (6) are transferred to $\overline{d}_i$:

$$score(\overline{d}_i) = 6n + 4i + 2 + 6n + 4i - 5 = 12n + 8i - 3.$$

Therefore, a_i is eliminated next.

Case 2: $i \notin I$. Then $\overline{a}_i$ is the i-th candidate in the preference of manipulator s, i.e.,

$$score(\overline{a}_i) = 6n + 4i + 1.$$

Since $score(a_i) = 6n + 4i$, a_i is eliminated.

$$\begin{aligned} \Longrightarrow \quad score(d_i) &= 6n + 4i + 2 && \text{and} \\ score(\overline{a}_i) &= 6n + 4i + 3 && \text{as } a_i\text{'s votes in (8) are transferred} \\ score(\overline{d}_i) &= 6n + 4i + 1 && \text{as before from voter group (7).} \end{aligned}$$

STV-CM is NP-complete: The Manipulor's Preference

So $\overline{d}_i$ is eliminated now. Hence, $\overline{d}_i$'s votes in group (7) are transferred to d_i:

$$score(d_i) = 6n + 4i + 2 + 6n + 4i - 1 = 12n + 8i + 1.$$

Therefore, $\overline{a}_i$ is eliminated next. ❑ Lemma

From these two lemmas, it follows that

$$c \text{ wins due to manipulator } s \iff (B, \mathcal{S}) \in X3C$$

This proves the theorem. ❑

Scoring-Protocols$\setminus${Trivial, Plurality}-CCWM

Theorem (Conitzer, Sandholm, and Lang (2007))

Scoring-Protocols$\setminus${*Trivial, Plurality*}-Constructive Coalitional Weighted Manipulation *for three candidates is* NP*-complete.*

Remark:

1. For two candidates, every scoring protocol is easy to manipulate.
2. Plurality is easy to manipulate for any number of candidates, and trivially, $(0, \ldots, 0)$-CCWM is in P as well.
3. In particular, Veto-CCWM and Borda-CCWM for three candidates are NP-complete.
4. The above theorem was independently proven by Hemaspaandra & Hemaspaandra (2007) and Procaccia & Rosenschein (2006).

Scoring-Protocols$\setminus$\{Trivial, Plurality\}-CCWM: Reduction from Partition

Proof: Membership in NP is clear.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a scoring protocol other than the trivial one or plurality. Without loss of generality, we may assume that

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 \quad \text{with} \quad \alpha_3 = 0 \quad \text{and} \quad \alpha_2 \geq 2.$$

Because: If $\alpha_2 = 0$, then α

- were either trivial (if $\alpha = (0, 0, 0)$)
- or plurality (if $\alpha_1 \geq 1$, i.e., $\alpha = (\alpha_1, 0, 0)$), both a contradiction.

But $\alpha_2 \geq 1$ can be scaled to $\alpha_2 \geq 2$ without any problem.

Scoring-Protocols$\setminus${Trivial, Plurality}-CCWM: Reduction from Partition

To prove NP-hardness of α-CCWM, we reduce from the following NP-complete problem:

Name: Partition.

Given: A nonempty sequence $(k_1, k_2, \ldots, k_n)$ of positive integers such that $\sum_{i=1}^{n} k_i$ is an even number.

Question: Does there exist a subset $A \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in A} k_i = \sum_{i \in \{1,2,\ldots,n\} \setminus A} k_i \ ?$$

Scoring-Protocols$\setminus\{$Trivial, Plurality$\}$-CCWM: Reduction from PARTITION

Given an instance $(k_1, k_2, \ldots, k_n)$ of PARTITION with $\sum_{i=1}^{n} k_i = 2K$ for some integer K, construct an election $(C, V \cup S)$ with

$$C = \{a, b, p\} \quad \text{with distinguished candidate } p$$

and

	Vote Weight	Preference		
V :	$(2\alpha_1 - \alpha_2)K - 1$	a	b	p
	$(2\alpha_1 - \alpha_2)K - 1$	b	a	p

S : For each $i \in \{1, 2, \ldots, n\}$, $(\alpha_1 + \alpha_2)k_i$

Scoring-Protocols$\setminus$\{Trivial, Plurality\}-CCWM: Reduction from Partition

We now show that:

$$(k_1, k_2, \ldots, k_n) \in \textsc{Partition} \quad \Longleftrightarrow \quad p \text{ can be made win } (C, V \cup S).$$

(⇒) If $(k_1, k_2, \ldots, k_n) \in \textsc{Partition}$, then there is a subset $A \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in A} k_i = \sum_{i \in \{1,2,\ldots,n\} \setminus A} k_i = K.$$

For $i \in A$, let $s_i \in S$ vote: $p\ a\ b$.

For $i \notin A$, let $s_i \in S$ vote: $p\ b\ a$.

Scoring-Protocols$\setminus\{$Trivial, Plurality$\}$-CCWM: Reduction from Partition

Then $score_{(C,V\cup S)}(p) = 2(\alpha_1+\alpha_2)\alpha_1 K$. But for $x \in \{a,b\}$, we have

$$
\begin{aligned}
score_{(C,V\cup S)}(x) &= (\alpha_1+\alpha_2)((2\alpha_1-\alpha_2)K-1)+\alpha_2 K(\alpha_1+\alpha_2)\\
&= (\alpha_1+\alpha_2)(2\alpha_1 K-1)\\
&< (\alpha_1+\alpha_2)2\alpha_1 K\\
&= score_{(C,V\cup S)}(p).
\end{aligned}
$$

Thus p is the unique winner.

Scoring-Protocols$\setminus${Trivial, Plurality}-CCWM: Reduction from Partition

($\Leftarrow$) Assume that p can be made win through the manipulators in S.

By monotonicity of scoring protocols, we can assume that all manipulators $s_i \in S$ put p in the first position.

Let

$$X = \sum_{s_i \in S \ : \ p \ \ a \ \ b} k_i \quad \text{and} \quad Y = \sum_{s_i \in S \ : \ p \ \ b \ \ a} k_i$$

We have $X + Y = 2K$. It follows that

$$\begin{aligned}
score_{(C,V \cup S)}(p) &= 2(\alpha_1 + \alpha_2)\alpha_1 K \\
score_{(C,V \cup S)}(a) &= (\alpha_1 + \alpha_2)((2\alpha_1 - \alpha_2)K - 1 + X\alpha_2) \\
score_{(C,V \cup S)}(b) &= (\alpha_1 + \alpha_2)((2\alpha_1 - \alpha_2)K - 1 + Y\alpha_2)
\end{aligned}$$

Scoring-Protocols$\setminus\{$Trivial, Plurality$\}$-CCWM: Reduction from PARTITION

Since p wins in $(C, V \cup S)$, we must have:

$$\begin{array}{rcll}
score_{(C,V\cup S)}(p) & \geq & score_{(C,V\cup S)}(a) & \\
2(\alpha_1+\alpha_2)\alpha_1 K & \geq & (\alpha_1+\alpha_2)((2\alpha_1-\alpha_2)K-1+X\alpha_2) & \\
2\alpha_1 K & \geq & 2\alpha_1 K - \alpha_2 K - 1 + X\alpha_2 & \\
\alpha_2 K + 1 & \geq & X\alpha_2 & \\
K + \dfrac{1}{\alpha_2} & \geq & X & \text{since } \alpha_2 > 0 \\
K & \geq & X & \text{since } \alpha_2 \geq 2
\end{array}$$

Analogously, $K \geq Y$. Since, $2K = X + Y$, it follows that $X = Y = K$.

Hence, $(k_1, k_2, \ldots, k_n) \in$ PARTITION. ❑

Remark: This result can be shown similarly in the unique-winner model.

Copeland-CCWM for four Candidates is Hard

Theorem (Conitzer, Sandholm, and Lang (2007))

Copeland-CONSTRUCTIVE COALITIONAL WEIGHTED MANIPULATION *for four candidates is* NP*-complete.*

Proof: Membership in NP is clear. To prove NP-hardness of Copeland-CCWM, we again reduce from PARTITION.

Given an instance $(k_1, k_2, \ldots, k_n)$ of PARTITION with $\sum_{i=1}^{n} k_i = 2K$ for some integer K, construct an election

$$(C, V \cup S)$$

with $C = \{a, b, c, p\}$, p distinguished, and the following votes in $V \cup S$.

Copeland-CCWM for four Candidates is Hard

V :

Vote Weight	Preference			
$2K+2$	p	a	b	c
$2K+2$	c	p	b	a
$K+1$	a	b	c	p
$K+1$	b	a	c	p

S : For each $i \in \{1, 2, \ldots, n\}$, k_i

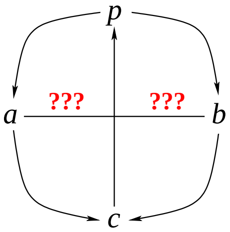
It remains to show that:

$$(k_1, k_2, \ldots, k_n) \in \textsc{Partition} \iff p \text{ can be made win } (C, V \cup S).$$

Copeland-CCWM for four Candidates is Hard

Pairwise comparisons and scores in (C, V):

	a	b	c	p	*CScore*
a	$\times$	0	$2K+2$	$-2K-2$	1.5
b	–	$\times$	$2K+2$	$-2K-2$	1.5
c	–	–	$\times$	$2K+2$	1
p	–	–	–	$\times$	2



Restricted to the sincere voters (i.e., to the election (C, V)), all but one of the pairwise comparisons are already decided, since the total weight of the manipulators, $2K$, is too low to flip the result of these comparisons.

Copeland-CCWM for four Candidates is Hard

The only as yet undecided comparison is the one between a and b.

If a or b wins this comparison in $(C, V \cup S)$, then this candidate has the same Copeland score as p.

However, the manipulators want to make p a *unique* winner, so they want to prevent that some of a and b outweighs the other.

Therefore, it is possible for them to make their favorite candidate p a unique winner of $(C, V \cup S)$ if and only if the pairwise comparison between a and b ends up in a tie.

They are tied already in the election (C, V) without the manipulators (this is the 0 entry in the above table).

Copeland-CCWM for four Candidates is Hard

This tie is preserved in $(C, V \cup S)$ exactly if

$$N_S(a,b) \ = \sum_{s \in S \ : \ a >_s b} w(s) \ = \sum_{s \in S \ : \ b >_s a} w(s) \ = \ N_S(b,a),$$

which in turn is equivalent to the equality $\sum_{i \in A} k_i = \sum_{i \in \{1,2,\ldots,n\} \setminus A} k_i$ for some subset $A \subseteq \{1, 2, \ldots, n\}$, where $i \in A$ if and only if the ith manipulator prefers a to b.

But that just says that $(k_1, k_2, \ldots, k_n)$ is a yes-instance of $\textsc{Partition}$.

Summing up, this shows that the manipulators can make p a unique winner of $(C, V \cup S)$ if and only if $(k_1, k_2, \ldots, k_n)$ is in $\textsc{Partition}$.

It follows that $\textsc{Partition} \leq_{\mathrm{m}}^{\mathrm{p}}$ Copeland-CCWM, which proves that Copeland-CCWM is NP-hard. ❑

Maximin-CCWM for four Candidates is Hard

Theorem (Conitzer, Sandholm, and Lang (2007))

*Maximin-*Constructive Coalitional Weighted Manipulation *for four candidates is* NP*-complete.*

Proof: Membership in NP is clear. To prove NP-hardness of Maximin-CCWM, we again reduce from Partition.

Given an instance $(k_1, k_2, \ldots, k_n)$ of Partition with $\sum_{i=1}^{n} k_i = 2K$ for some integer K, construct an election

$$(C, V \cup S)$$

with $C = \{a, b, c, p\}$, p distinguished, and the following votes in $V \cup S$.

Maximin-CCWM for four Candidates is Hard

	Vote Weight	Preference
V :	$7K-1$	$a \quad b \quad c \quad p$
	$7K-1$	$b \quad c \quad a \quad p$
	$4K-1$	$c \quad a \quad b \quad p$
	$5K$	$p \quad c \quad a \quad b$
S : For each $i \in \{1, 2, \ldots, n\}$,	$2k_i$	

It remains to show that:

$$(k_1, k_2, \ldots, k_n) \in \textsc{Partition} \iff p \text{ can be made win } (C, V \cup S).$$

Maximin-CCWM for four Candidates is Hard

(⇒) If $(k_1, k_2, \ldots, k_n) \in \textsc{Partition}$, then there is a subset $A \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in A} k_i = \sum_{i \in \{1,2,\ldots,n\} \setminus A} k_i = K.$$

For $i \in A$, let $s_i \in S$ with weight $2k_i$ vote: $p\ a\ b\ c$.

For $i \notin A$, let $s_i \in S$ with weight $2k_i$ vote: $p\ b\ c\ a$.

	p	a	b	c	$MScore$	
p	$\times$	$9K$	$9K$	$9K$	$\mathbf{9K}$	$\leftarrow$ max
a	$18K-3$	$\times$	$18K-2$	$9K-1$	$9K-1$	
b	$18K-3$	$9K-1$	$\times$	$18K-2$	$9K-1$	
c	$18K-3$	$18K-2$	$9K-1$	$\times$	$9K-1$	

$\Longrightarrow$ p is the unique maximin winner of $(C, V \cup S)$.

Maximin-CCWM for four Candidates is Hard

($\Leftarrow$) Assume p can be made win alone in $(C, V \cup S)$ via the votes in S.

Then p is also the unique winner if all $s_i \in S$ put p in the first position.

Hence, the worst pairwise comparison score of p is $MScore(p) = 9K$.

In (C, V), it is already decided who amongst a, b, c is worst off against whom, independently of the votes in S (having total weight $4K$):

	p	a	b	c
p	$\times$			
a	$18K - 3$	$\times$	$16K - 2$	$\boxed{7K - 1}$
b	$18K - 3$	$\boxed{7K - 1}$	$\times$	$14K - 2$
c	$18K - 3$	$16K - 2$	$\boxed{9K - 1}$	$\times$

1. $p\ a\ b\ c$
2. $p\ a\ c\ b$
3. $p\ b\ a\ c$
4. $p\ b\ c\ a$
5. $p\ c\ a\ b$
6. $p\ c\ b\ a$

Maximin-CCWM for four Candidates is Hard

	p	a	b	c
p	$\times$			
a	$18K-3$	$\times$	$16K-2$	$\boxed{7K-1}$
b	$18K-3$	$\boxed{7K-1}$	$\times$	$14K-2$
c	$18K-3$	$16K-2$	$\boxed{9K-1}$	$\times$

1. $p\ a\ b\ c$
2. $p\ a\ c\ b$
3. $p\ b\ a\ c$
4. $p\ b\ c\ a$
5. $p\ c\ a\ b$
6. $p\ c\ b\ a$

$$\text{Since}\quad MScore_{(C,V)}(c) = 9K-1 \quad \text{due to } b \text{ and}$$
$$MScore_{(C,V\cup S)}(p) = 9K,$$

no $s_i \in S$ can place c before b (or else p would not be the unique winner in $(C, V \cup S)$). Hence, preferences 2, 5, and 6 are excluded for $s_i \in S$.

Maximin-CCWM for four Candidates is Hard

	p	a	b	c
p	$\times$			
a	$18K-3$	$\times$	$16K-2$	$\boxed{7K-1}$
b	$18K-3$	$\boxed{7K-1}$	$\times$	$14K-2$
c	$18K-3$	$16K-2$	$\boxed{9K-1}$	$\times$

1. $p\ a\ b\ c$
2. ~~$p\ a\ c\ b$~~
3. $p\ b\ a\ c$
4. $p\ b\ c\ a$
5. ~~$p\ c\ a\ b$~~
6. ~~$p\ c\ b\ a$~~

Further, if any $s_i \in S$ puts a right before c, swapping their positions has no effect other than to decrease a's final score, so we may also assume this (preference 3) does not occur.

(Similarly, we can exclude that b is put directly before a: These are the already excluded preferences 3 and 6.)

Maximin-CCWM for four Candidates is Hard

	p	a	b	c
p	$\times$			
a	$18K-3$	$\times$	$16K-2$	$\boxed{7K-1}$
b	$18K-3$	$\boxed{7K-1}$	$\times$	$14K-2$
c	$18K-3$	$16K-2$	$\boxed{9K-1}$	$\times$

1. $p\ a\ b\ c$
2. ~~$p\ a\ c\ b$~~
3. ~~$p\ b\ a\ c$~~
4. $p\ b\ c\ a$
5. ~~$p\ c\ a\ b$~~
6. ~~$p\ c\ b\ a$~~

For $s_i \in S$, the following two preferences remain possible:

1. $p\ a\ b\ c$ with weight $\leq 2K$: $MScore_{(C,V)}(a) = 7K-1$ due to c;
4. $p\ b\ c\ a$ with weight $\leq 2K$: $MScore_{(C,V)}(b) = 7K-1$ due to a.

Hence, $\sum_{s_i \in S\ :\ p\ a\ b\ c} k_i = \sum_{s_i \in S\ :\ p\ b\ c\ a} k_i = K$, so $(k_1, k_2, \ldots, k_n) \in \textsc{Partition}$. ❑

STV-CCWM for three Candidates is Hard

Theorem (Conitzer, Sandholm, and Lang (2007))

STV-Constructive Coalitional Weighted Manipulation for three candidates is NP-complete.

Proof: Membership in NP is clear. To prove NP-hardness of STV-CCWM, we again reduce from Partition.

Given an instance $(k_1, k_2, \ldots, k_n)$ of Partition with $\sum_{i=1}^{n} k_i = 2K$ for some integer K, construct an election

$$(C, V \cup S)$$

with $C = \{a, b, p\}$, p distinguished, and the following votes in $V \cup S$.

STV-CCWM for three Candidates is Hard

		Vote Weight	Preference		
V :		$6K - 1$	b	p	a
		$4K$	a	b	p
		$4K$	p	a	b
S :	For each $i \in \{1, 2, \ldots, n\}$,	$2k_i$			

It remains to show that:

$$(k_1, k_2, \ldots, k_n) \in \textsc{Partition} \quad \Longleftrightarrow \quad p \text{ can be made win } (C, V \cup S).$$

STV-CCWM for three Candidates is Hard

(⇒) If $(k_1, k_2, \ldots, k_n) \in$ PARTITION, then there is a subset $A \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in A} k_i = \sum_{i \in \{1,2,\ldots,n\} \setminus A} k_i = K.$$

For $i \in A$, let $s_i \in S$ with weight $2k_i$ vote: $a\ p\ b$.

For $i \notin A$, let $s_i \in S$ with weight $2k_i$ vote: $p\ a\ b$.

Round 1:

	p	a	b
score	$6K$	$6K$	$6K - 1$

$\Longrightarrow$ b is eliminated and b's votes are transferred to p

Round 2:

	p	a
score	$12K - 1$	$6K$

$\Longrightarrow$ p is the only winner in $(C, V \cup S)$

STV-CCWM for three Candidates is Hard

(⇐) Suppose p were the only winner in $(C, V \cup S)$.

Certainly, p cannot be eliminated in the first round.

But also a cannot be eliminated in the first round; or else a's votes would be transferred to b (so $score(b) = 10K - 1$) and p would not win.

Hence, b must be eliminated in the first round.

Both a and p need weight at least $2K$ to defeat a in the first round.

STV-CCWM for three Candidates is Hard

Hence, there is a set $A \subseteq \{1, 2, \ldots, n\}$ such that (for weight function w):

$$\sum_{\substack{s_i \in S \,:\, p \cdots \\ i \in A}} w(s_i) \geq 2K \quad \text{and} \quad \sum_{\substack{s_i \in S \,:\, a \cdots \\ i \notin A}} w(s_i) \geq 2K.$$

Thus there is a set $A \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in A} k_i = \sum_{i \notin A} k_i = K.$$

It follows that $(k_1, k_2, \ldots, k_n) \in \textsc{Partition}$. ❑

Destructive Manipulation

Definition (Destructive Coalitional Weighted Manipulation)

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-DESTRUCTIVE COALITIONAL WEIGHTED MANIPULATION ($\mathcal{E}$-DCWM).

Given:
- A set C of candidates,
- a list V of nonmanipulative voters over C each having a nonnegative integer weight,
- a list of the weights of the manipulators in S (whose votes over C are still unspecified) with $V \cap S = \emptyset$, and
- a distinguished candidate $c \in C$.

Question: Can the preferences of the voters in S be set such that c is not a $\mathcal{E}$-winner of $(C, V \cup S)$?

When Is Destructive Manipulation Easy?

Theorem (Conitzer, Sandholm, and Lang (2007))

Let $\mathcal{E}$ be a voting system such that:

- *Each candidate gets a numerical score based on the votes, and all candidates with the highest score win.*
- *The score function is monotonic: If changing a vote v satisfies*

$$\begin{aligned} & \{b \mid v \text{ prefers } a \text{ to } b \text{ before the change}\} \\ \subseteq \quad & \{b \mid v \text{ prefers } a \text{ to } b \text{ after the change}\}, \end{aligned}$$

  then a's score does not decrease.
- *Winner determination in $\mathcal{E}$ can be done in polynomial time.*

Then $\mathcal{E}$-DCWM is in P.

When Is Destructive Manipulation Easy?

Proof: Consider the following algorithm for $\mathcal{E}$-DCWM.

Input: (C, V, S, d) with $C = \{c_1, \dots, c_{m-1}, d\}$

1. For each $c_i \in C$, $c_i \neq d$, consider the election E_i in which all $s \in S$ vote: $c_i \;\cdots\; d$, where the candidates from $C \setminus \{c_i, d\}$ between c_i and d come in arbitrary order (e.g., in lexicographic order).
2. Accept if and only if in some of the elections $E_1, \dots, E_{m-1}$, d does not win.

<u>**Correctness of the algorithm:**</u>

We show: There are strategic preferences for the voters in S such that d does not win in $(C, V \cup S)$. $\Longleftrightarrow$ In some of the elections $E_1, \dots, E_{m-1}$, d does not win.

When Is Destructive Manipulation Easy?

($\Leftarrow$) is obvious.

($\Rightarrow$) Suppose there are preferences for the voters in S such that $c_1 \neq d$ wins in $(C, V \cup S)$. (Argument for $c_i \neq c_1$ is analogous.)

Monotonicity implies:

$$score_{E_1}(c_1) \geq score_{(C,V \cup S)}(c_1)$$

because the set $\{b \mid s_i \text{ prefers } c_1 \text{ to } b\}$ is maximal for each $s_i \in S$, and

$$score_{(C,V \cup S)}(d) \geq score_{E_1}(d)$$

because the set $\{b \mid s_i \text{ prefers } d \text{ to } b\}$ is minimal for each $s_i \in S$.

When Is Destructive Manipulation Easy?

Since c_1 wins in $(C, V \cup S)$ (and d does not), it follows from

$$score_{(C,V\cup S)}(c_1) > score_{(C,V\cup S)}(d)$$

that

$$score_{E_1}(c_1) > score_{E_1}(d).$$

Hence, d does not win in E_1. ❑ Correctness

Runtime of the algorithm:

The algorithm runs in polynomial time because it calls the P algorithm for winner determination $(m-1)$ times. ❑

When Is Destructive Manipulation Easy?

Corollary (Conitzer, Sandholm, and Lang (2007))

For any number of candidates, DCWM *is in* P *for*

- *Borda,*
- *veto,*
- *Copeland, and*
- *maximin.*

When Is Destructive Manipulation Easy?

Remark: Destructive manipulation can be harder than constructive manipulation by at most a factor of $m-1$ (where m is the number of candidates). Indeed, suppose we can solve $\mathcal{E}$-CCWM in P.

Then $\mathcal{E}$-DCWM is in P as follows:

Given an instance (C, V, S, d) with $C = \{c_1, \ldots, c_{m-1}, d\}$,

- decide "$(C, V, S, c_i) \in \mathcal{E}$-$\mathrm{CCWM}$?" for each i, $1 \leq i \leq m-1$, and
- accept if at least one answer is "yes" (in the unique-winner model); otherwise, reject.

Corollary (Conitzer, Sandholm, and Lang (2007))

DCWM *is in* P *for plurality and regular cup for any number of candidates.*

STV-DCWM for three Candidates is Hard

Theorem (Conitzer, Sandholm, and Lang (2007))

STV-DESTRUCTIVE COALITIONAL WEIGHTED MANIPULATION *for three candidates is* NP*-complete.*

Proof: Membership in NP is clear. To prove NP-hardness of STV-DCWM, we again reduce from PARTITION.

Given an instance $(k_1, k_2, \ldots, k_n)$ of PARTITION with $\sum_{i=1}^{n} k_i = 2K$ for some integer K, construct an election

$$(C, V \cup S)$$

with $C = \{a, b, d\}$, d distinguished, and the following votes in $V \cup S$.

STV-DCWM for three Candidates is Hard

		Vote Weight	Preference		
V :		$6K$	a	d	b
		$6K$	b	d	a
		$8K - 1$	d	a	b
S :	For each $i \in \{1, 2, \ldots, n\}$,	$2k_i$			

It remains to show that:

$$(k_1, k_2, \ldots, k_n) \in \textsc{Partition} \iff d \text{ can be made to not win } (C, V \cup S).$$

STV-DCWM for three Candidates is Hard

Proposition: d does not win $\iff$ d is eliminated in the first round.

Proof of Proposition: **(⇐)** is obvious.

(⇒) We show the contrapositive: Assume d survives the first round.

Then either a or b is eliminated in the first round and transfers $6K$ votes to d.

But $score(d) = 14K - 1$ with a total weight of $24K - 1$ means: d wins. ❑ Proposition

Now we show that:

$$(k_1, k_2, \ldots, k_n) \in \textsc{Partition} \iff d \text{ can be made to not win } (C, V \cup S).$$

STV-DCWM for three Candidates is Hard

($\Rightarrow$) If $(k_1, k_2, \dots, k_n) \in \text{PARTITION}$, then there is a subset $A \subseteq \{1, 2, \dots, n\}$ such that

$$\sum_{i \in A} k_i = \sum_{i \in \{1,2,\dots,n\} \setminus A} k_i = K.$$

For $i \in A$, let $s_i \in S$ with weight $2k_i$ vote: $a\ b\ d$.

For $i \notin A$, let $s_i \in S$ with weight $2k_i$ vote: $b\ a\ d$.

Thus

$$score_{(C, V \cup S)}(a) = score_{(C, V \cup S)}(b) = 8K$$
$$score_{(C, V \cup S)}(d) = 8K - 1$$

Hence, d is eliminated in the first round and, by our proposition, does not win.

STV-DCWM for three Candidates is Hard

($\Leftarrow$) Suppose d does not win $(C, V \cup S)$.

By our proposition, d is eliminated in the first round. Thus

$$score_{(C,V\cup S)}(a) \geq 8K - 1 \qquad \text{and} \qquad score_{(C,V\cup S)}(b) \geq 8K - 1.$$

Hence,

$$score_{(C,S)}(a) \geq 2K - 1 \qquad \text{and} \qquad score_{(C,S)}(b) \geq 2K - 1.$$

Let

$$\begin{aligned} A &= \{i \mid s_i \in S \text{ has } a \text{ in the first position}\}, \\ B &= \{i \mid s_i \in S \text{ has } b \text{ in the first position}\}. \end{aligned}$$

STV-DCWM for three Candidates is Hard

Hence,

$$\sum_{i \in A} k_i \geq K - \frac{1}{2} \qquad \text{and} \qquad \sum_{i \in B} k_i \geq K - \frac{1}{2}.$$

Since all k_i, $1 \leq i \leq n$, are integers, it follows that

$$\sum_{i \in A} k_i \geq K \qquad \text{and} \qquad \sum_{i \in B} k_i \geq K.$$

Since A and B are disjoint, it follows that

$$\sum_{i \in A} k_i = \sum_{i \in B} k_i = K,$$

so $(k_1, k_2, \ldots, k_n) \in \textsc{Partition}$. ❑

Overview: Results for CCWM

# of Candidates	2	3	$\geq$ 4
Plurality	P	P	P
Regular Cup	P	P	P
Maximin	P	P	NP-complete
Copeland	P	NP-complete / P*	NP-complete
Veto	P	NP-complete	NP-complete
Borda	P	NP-complete	NP-complete
STV	P	NP-complete	NP-complete

Table: Results for Constructive Coalitional Weighted Manipulation.

NP-complete / P* for Copeland-CCWM means: "NP-complete" in the nonunique-winner model and "P" in the unique-winner model.

Overview: Results for DCWM

# of Candidates	2	≥ 3
Plurality	P	P
Regular Cup	P	P
Maximin	P	P
Copeland	P	P
Veto	P	P
Borda	P	P
STV	P	NP-complete

Table: Results for Destructive Coalitional Weighted Manipulation

Overview: Some More Results for CCWM

	CCWM			DCWM	
	Number of candidates				
	≤ 2	3	≥ 4	≤ 2	≥ 3
Plurality	P	P	P	P	P
Cup protocol	P	P	P	P	P
Fallback	P	P	P	P	P
Simpson	P	P	NP-complete	P	P
Nanson	P	P	NP-complete	P	?
Copeland	P	NP-complete / P*	NP-complete	P	P
Bucklin	P	NP-complete	NP-complete	P	P
Veto	P	NP-complete	NP-complete	P	P
Borda	P	NP-complete	NP-complete	P	P
STV	P	NP-complete	NP-complete	P	NP-complete