# Algorithmic Game Theory

## Algorithmische Spieltheorie

Power Indices in Simple Games

Wintersemester 2022/2023

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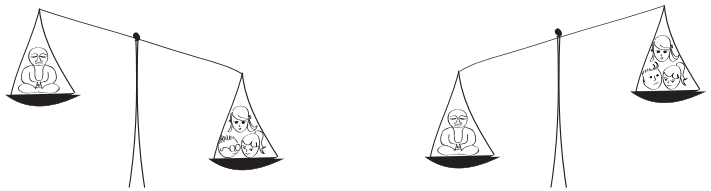
# Power Indices in Simple Games

- How can one measure the influence/power a player has in a weighted voting game or, more generally, in a simple or any cooperative game?
- A veto player $i$ has much power (if any exists), as $i$ is necessary for a winning coalition to form: $v(C) = 0$ for each coalition $C \subseteq P \setminus \{i\}$.

## Example

- Consider again the weighted voting game $G = (108, 106, 62, 66; 277)$:



in which Anna, Belle, David, and Edgar want to jointly lift a treasure.

- There are three veto players in this game: Anna, Belle, and Edgar.

# Power Indices in Simple Games

## Example (continued)

- On the other hand, David is too light to be useful for *any* coalition and thus has *absolutely no* influence.

  In other words, David is a dummy player!

## Definition (dummy player)

In a cooperative game $G = (P, v)$, a player $i \in P$ is a *dummy player* if $i$ is useless for all coalitions, i.e., if $v(C \cup \{i\}) = v(C)$ for each $C \subseteq P$.

For $i$ to be not a dummy player in a simple game, we have $v(C) = 0$ and $v(C \cup \{i\}) = 1$ for at least one coalition $C$.

# The Shapley Value and the Shapley–Shubik Index

- In games with an empty core, it is unclear how to divide the gains. Moreover, core imputations can be "unreasonable" or "unfair."
- An alternative way would be to pay the players according to their influence in this game.
- To this end, Shapley (1953) introduced the following method.
- Restricted to *simple* cooperative games, Shapley and Shubik (1954) proposed to use the Shapley value as a measure of the influence or power in such games.
- The Shapley value and the Shapley–Shubik index have been thoroughly investigated since, e.g., by Dubey and Shapley (1979) and Roth (1988).

# The Shapley Value: Definition

## Definition (Shapley value)

- Let $G = (P, v)$ be a cooperative game with $n = \|P\|$ players.
- Let $\Pi_P$ be the set of all permutations of $P$.
- For $\pi \in \Pi_P$ and $i \in P$, let $S_\pi(i)$ be the set of all predecessors of $i$ in $\pi$:

$$S_\pi(i) = \{j \in P \mid \pi(j) < \pi(i)\}.$$

- The *marginal contribution of player $i$ with respect to $\pi \in \Pi_P$ in game $G = (P, v)$* is defined by $\Delta_\pi^G(i) = v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))$.
- The *Shapley value of a player $i$ in $G$* is defined by

$$\varphi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi_P} \Delta_\pi^G(i).$$

# The Shapley Value: Properties

**Theorem**

*Let $G = (P, v)$ be a cooperative game with $n = \|P\|$ players. Then the payoff vector*

$$(\varphi_1(G), \varphi_2(G), \ldots, \varphi_n(G))$$

*satisfies the following properties:*

1. *Dummy player: If $i \in P$ is a dummy player then $\varphi_i(G) = 0$.*
2. *Efficiency: $\sum_{i=1}^{n} \varphi_i(G) = v(P)$.*
3. *Symmetry: $\varphi_i(G) = \varphi_j(G)$ for symmetric players $i$ and $j$, i.e., for players $i$ and $j$ satisfying $v(C \cup \{i\}) = v(C \cup \{j\})$ for all coalitions $C \subseteq P \setminus \{i, j\}$.*

Proof: For the first two properties, see blackboard. ❑

# The Shapley Value: Properties

**Symmetry:** For $\pi \in \Pi_P$, let $\pi' \in \Pi_P$ be the permutation resulting from $\pi$ by swapping $i$ and $j$:

$$\pi'(i) = \pi(j), \quad \pi'(j) = \pi(i), \quad \text{and} \quad \pi'(k) = \pi(k) \quad \text{for } i \neq k \neq j.$$

We show

$$\Delta_\pi^G(i) = \Delta_{\pi'}^G(j).$$

**Case 1:** $i$ precedes $j$ in $\pi$.

Then $S_\pi(i) = S_{\pi'}(j)$. Let $C = S_\pi(i) = S_{\pi'}(j)$. We have

$$\begin{aligned} \Delta_\pi^G(i) &= v(C \cup \{i\}) - v(C) \quad \text{and} \\ \Delta_{\pi'}^G(j) &= v(C \cup \{j\}) - v(C). \end{aligned}$$

# The Shapley Value: Properties

Since $i$ and $j$ are symmetric players, we have $v(C \cup \{i\}) = v(C \cup \{j\})$, so

$$\Delta_{\pi}^{G}(i) = \Delta_{\pi'}^{G}(j).$$

**Case 2:** $j$ precedes $i$ in $\pi$.

Let $C = S_{\pi}(i) \setminus \{j\}$. We have

$$\begin{aligned} \Delta_{\pi}^{G}(i) &= v(C \cup \{j\} \cup \{i\}) - v(C \cup \{j\}) \quad \text{and} \\ \Delta_{\pi'}^{G}(j) &= v(C \cup \{j\} \cup \{i\}) - v(C \cup \{i\}). \end{aligned}$$

Since $C \subseteq P$ contains neither $i$ nor $j$, it follows from the symmetry of $i$ and $j$ that $v(C \cup \{i\}) = v(C \cup \{j\})$, so

$$\Delta_{\pi}^{G}(i) = \Delta_{\pi'}^{G}(j).$$

# The Shapley Value: Properties

Since $\Delta_\pi^G(i) = \Delta_{\pi'}^G(j)$ for all $\pi \in \Pi_P$ and the mapping $\pi \mapsto \pi'$ is bijctive, we have

$$\Pi_p = \{\pi' \mid \pi \in \Pi_P\}.$$

It follows that

$$\begin{aligned} \varphi_i(G) &= \frac{1}{n!} \sum_{\pi \in \Pi_P} \Delta_\pi^G(i) \quad \text{by definition} \\ &= \frac{1}{n!} \sum_{\pi' \in \Pi_P} \Delta_{\pi'}^G(j) \\ &= \varphi_j(G), \end{aligned}$$

which completes the proof. ❑

# The Shapley Value: Properties

## Remark

1. *In superadditive games, the Shapley value thus*
   1. *defines a valid payoff vector (i.e., an imputation),*
   2. *assigns no payoff to dummies, and*
   3. *guarantees equal treatment to players who contribute to all coalitions in the same way.*

2. *In weighted voting games two players of the same weight are symmetric, but the converse is not necessarily true:*

   *For instance, in the weighted voting game* $G = (108, 106, 62, 66; 277)$*, Anna, Belle and Edgar are pairwise symmetric despite having very different weights. This illustrates that a player's Shapley value is a better measure of his contribution than his weight.*

# The Shapley Value: Properties

**Definition**

The *sum of two games* $G_1 = (P, v_1)$ and $G_2 = (P, v_2)$ with the same set of players is defined to be the game

$$G_1 + G_2 = (P, v),$$

whose characteristic function is given by $v(C) = v_1(C) + v_2(C)$ for all coalitions $C \subseteq P$.

**Theorem**

*The Shapley value also satisfies*

4. *Additivity: For each $i \in P$, $\varphi_i(G_1 + G_2) = \varphi_i(G_1) + \varphi_i(G_2)$.*

Proof: See blackboard. ❑

# The Shapley Value: Properties

## Remark

1. *The converse of the dummy player property:*

$$\text{“If } \varphi_i(G) = 0 \text{ then } i \text{ is a dummy player.”}$$

   - *is true if the game is monotonic,*
   - *but is not always true in nonmonotonic games.*

2. *The Shapley value is in fact the only payoff division scheme that satisfies these four properties simultaneously. These four axioms thus provide a unique characterization of the Shapley value.*

3. *They can be used to simplify the computation of the Shapley value.*

# The Shapley Value: Properties

**Remark**

- *The argument in the proof of*

**Theorem**

*A cooperative game $G = (P, v)$ has a nonempty core if and only if its superadditive cover $G^* = (P, v^*)$ has a nonempty core.*

*does not necessarily extend to other solution concepts, in particular not to the Shapley value.*

- *That is, paying the players in a game $G$ according to their Shapley value may require cross-coalitional transfers in the superadditive cover $G^*$.*

# The Shapley–Shubik Index

## Definition (Shapley–Shubik index)

- Let $G = (P, v)$ be a simple cooperative game with $n = \|P\|$ players.
- A player $i \in P$ is said to be *pivotal for a coalition* $C \subseteq P \setminus \{i\}$ if $C \cup \{i\}$ wins, but $C$ does not.
- Setting the *marginal contribution of player $i$ to the gains of coalition $C$ in $G$* to

$$d_G(C, i) = v(C \cup \{i\}) - v(C), \tag{1}$$

we have $d_G(C, i) = 1$ if $i$ is a pivotal player for $C$, and $d_G(C, i) = 0$ otherwise.

# The Shapley–Shubik Index

## Remark

- *Note that a dummy player is not pivotal for any coalition.*
- *Given a permutation $\pi$ of $P$, a player $i$ is pivotal for $\pi$ if $i$ is pivotal for the coalition $S_\pi(i) = \{j \in P \mid \pi(j) < \pi(i)\}$ of his predecessors in $\pi$.*
- *Player $i$'s marginal contribution with respect to $\pi$:*

$$\Delta_\pi^G(i) = v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))$$

  *equals 1 if $i$ is pivotal for $\pi$, and 0 otherwise.*
- *Observe that for each permutation $\pi$ there exists exactly one player who is pivotal for it.*

# The Shapley–Shubik Index

**Definition (Shapley–Shubik index—continued)**

The *raw Shapley–Shubik index of a player $i$ in $G$* is defined by

$$\mathrm{SSI}^*(G,i) = \sum_{\pi \in \Pi_P} \Delta_\pi^G(i) \tag{2}$$

and simply counts how many permutations $i$ is pivotal for.

# The Shapley–Shubik Index

- By summing expression (2) over all players $i \in P$, we obtain

$$\sum_{i \in P} \sum_{\pi \in \Pi_P} \Delta_\pi^G(i) = \sum_{\pi \in \Pi_P} \sum_{i \in P} \Delta_\pi^G(i) = \sum_{\pi \in \Pi_P} 1 = n! \ . \tag{3}$$

- The first equality is obtained by changing the order of summation, and the second equality follows from our observation that for each permutation there exists exactly one player that is pivotal for it.
- Thus the sum of the players' raw Shapley–Shubik indices in a simple $n$-player game is always $n!$.

# The Shapley–Shubik Index

- It is convenient and computationally advantageous to express the raw Shapley–Shubik index $\mathrm{SSI}^*(G,i)$ via $d_G(C,i)$ rather than $\Delta_\pi^G(i)$.
- To this end, observe that one coalition of size $k$ that $i$ is pivotal for corresponds to $k!(n-k-1)!$ permutations that $i$ is pivotal for:
- If $i$ is pivotal for $C$, it is pivotal for
  - all permutations obtained by placing the elements of $C$ in the first $\|C\|$ positions (in an arbitrary order),
  - followed by $i$,
  - followed by the elements of $P \setminus (C \cup \{i\})$ (in an arbitrary order).

# The Shapley–Shubik Index

- Thus we can rewrite expression (2) as follows:

$$\mathrm{SSI}^*(G,i) \quad = \quad \sum_{C \subseteq P \setminus \{i\}} \|C\|! \cdot (n - \|C\| - 1)! \cdot d_G(C,i) \qquad (4)$$

  which gives the number of coalitions for which $i$ is a pivotal player, where the order matters in which players join coalitions.

- Observe that expression (2) has $n!$ terms, while expression (4) has only $2^{n-1}$ terms.

- The factorial function grows much faster than the exponential function: For instance, for $n = 6$ we have $6! = 720$, while $2^5 = 32$.

- So if we are to compute a player's raw Shapley–Shubik index directly from the definition, it is more practical to use expression (4).

# The Shapley–Shubik Index

**Definition (Shapley–Shubik index—continued)**

The raw Shapley–Shubik index is then normalized by

$$\mathrm{SSI}(G,i) \;=\; \frac{1}{n!}\cdot \mathrm{SSI}^*(G,i),$$

which is the *Shapley–Shubik index of $i$ in $G$*.

$\mathrm{SSI}(G,i)$ is just the Shapley value $\varphi_i(G)$ restricted to simple games.

# The Shapley–Shubik Index

## Remark

1. *For a weighted voting game, this works as follows for* $\mathrm{SSI}^*(G,i)$*:*
   - *The players join a coalition in a fixed order until the total weight of the coalition meets or exceeds the quota and the coalition wins.*
   - *This change from losing to winning is attributed to the last player joining, as this player has been pivotal for the victory of the coalition.*
   - *This is done for all possible orders (or permutations) of the players, and one counts how often player* $i$ *is pivotal for a coalition.*
2. *By normalizing the raw* $\mathrm{SSI}^*(G,i)$*, it is ensured that* $\mathrm{SSI}(G,i)$
   - *is always between* $0$ *and* $1$*,*
   - *is* $0$ *if and only if* $i$ *is a dummy player (due to monotonicity), and*
   - $\sum_{i=1}^{n} \mathrm{SSI}(G,i) = 1$ *(see next slide).*

# The Shapley–Shubik Index

## Remark (continued)

3. *When is $i$'s Shapley–Shubik index equal to 1?*
   - *It may be tempting to conjecture that this is the case whenever $i$ is a veto player.*
   - *However, this is not necessarily true: A veto player is necessary, but not sufficient to form a winning coalition, so it may be the case that $v(C) = v(C \cup \{i\}) = 0$ for some coalition $C$ even if $i$ is a veto player.*
   - *In fact, it follows from the observations above (in particular, from (3)) that the Shapley–Shubik indices of all players in the game add up to 1:*

$$\sum_{i \in P} \mathrm{SSI}(G,i) \;=\; \sum_{i \in P} \frac{1}{n!} \cdot \mathrm{SSI}^*(G,i) \;=\; \frac{n!}{n!} \;=\; 1.$$

   - *Consequently, a player's Shapley–Shubik index is 1 if and only if all other players are dummies.*

# The Shapley–Shubik Index

## Example (Shapley–Shubik index)

- Let us determine the Shapley–Shubik index of, say, Anna in the weighted voting game $G = (108, 106, 62, 66; 277)$.
- Only two coalitions win:
  - $E = \{\text{Anna, Belle, Edgar}\}$ and
  - $P = \{\text{Anna, Belle, David, Edgar}\}$, the grand coalition.
- Anna is pivotal only for
  - $E \setminus \{\text{Anna}\} = \{\text{Belle, Edgar}\}$ and
  - $P \setminus \{\text{Anna}\} = \{\text{Belle, David, Edgar}\}$, i.e., we have

$$d_G(E \setminus \{\text{Anna}\}, \text{Anna}) = 1 \quad \text{and} \quad d_G(P \setminus \{\text{Anna}\}, \text{Anna}) = 1.$$

Only these terms contribute to the sum in (4); all other terms vanish because $d_G(C, \text{Anna}) = 0$ for all other coalitions $C \subseteq P \setminus \{\text{Anna}\}$.

# The Shapley–Shubik Index

## Example (Shapley–Shubik index—continued)

- It follows that:

$$
\begin{aligned}
&\mathrm{SSI}^*(G, \mathrm{Anna}) \\
&\quad = \|E \setminus \{\mathrm{Anna}\}\|! \cdot (4 - \|E \setminus \{\mathrm{Anna}\}\| - 1)! \cdot d_G(E \setminus \{\mathrm{Anna}\}, \mathrm{Anna}) + \\
&\qquad \|P \setminus \{\mathrm{Anna}\}\|! \cdot (4 - \|P \setminus \{\mathrm{Anna}\}\| - 1)! \cdot d_G(P \setminus \{\mathrm{Anna}\}, \mathrm{Anna}) \\
&\quad = 2! \cdot (4 - 2 - 1)! \cdot 1 + 3! \cdot (4 - 3 - 1)! \cdot 1 \\
&\quad = 2 + 6 \;=\; 8.
\end{aligned}
$$

- Normalizing Anna's raw Shapley–Shubik index, we obtain her Shapley–Shubik index in $G$:

$$
\mathrm{SSI}(G, \mathrm{Anna}) \;=\; \frac{1}{4!} \cdot \mathrm{SSI}^*(G, \mathrm{Anna}) \;=\; \frac{1}{24} \cdot 8 \;=\; \frac{1}{3}.
$$

# The Shapley–Shubik Index

**Example (Shapley–Shubik index—continued)**

- Analogously, one can show that Edgar and Belle have the same Shapley–Shubik index in $G$, only dummy player David has no influence in $G$ at all:

$$\begin{aligned} \mathrm{SSI}(G, \mathrm{Edgar}) &= \mathrm{SSI}(G, \mathrm{Belle}) = \frac{1}{3}, \\ \mathrm{SSI}(G, \mathrm{David}) &= 0. \end{aligned}$$

# The Shapley–Shubik Index

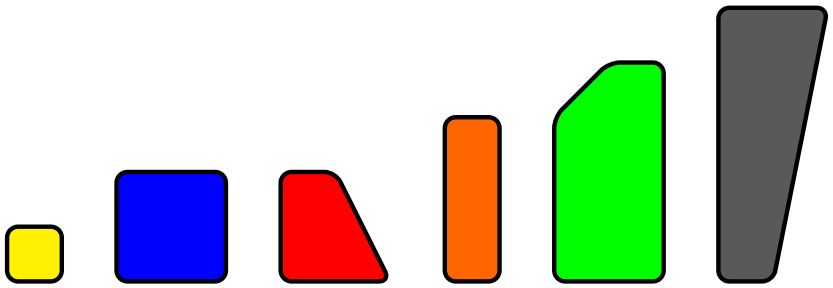
**Definition (coalitional game)**

players $P = \{1, \ldots, n\}$,

coalitional function $v : 2^P \to \mathbb{R}^+$

**Example (Shapley–Shubik index)**

players



$v(\emptyset) = 0$,

$v(\{$  $\}) = 2$,

$v(\{$  ,  $\}) = 5$,

$\vdots$

exponentially many

# The Shapley–Shubik Index

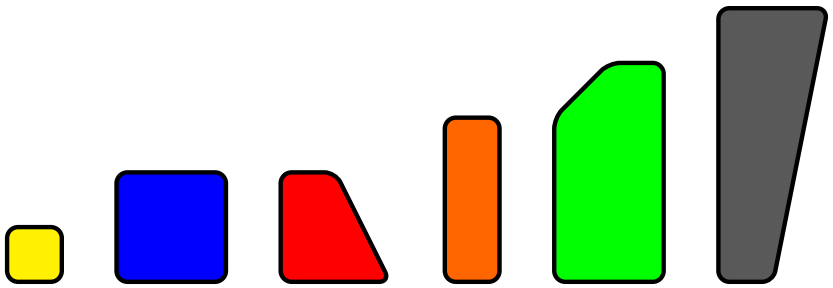
**Definition (simple game)**

players $P = \{1, \ldots, n\}$,

coalitional function $v : 2^P \to \{0, 1\}$, monotonicity

**Example (Shapley–Shubik index)**

players



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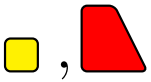
$v(\{$  $\}) = 1$,

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# The Shapley–Shubik Index

**Definition (weighted voting game)**

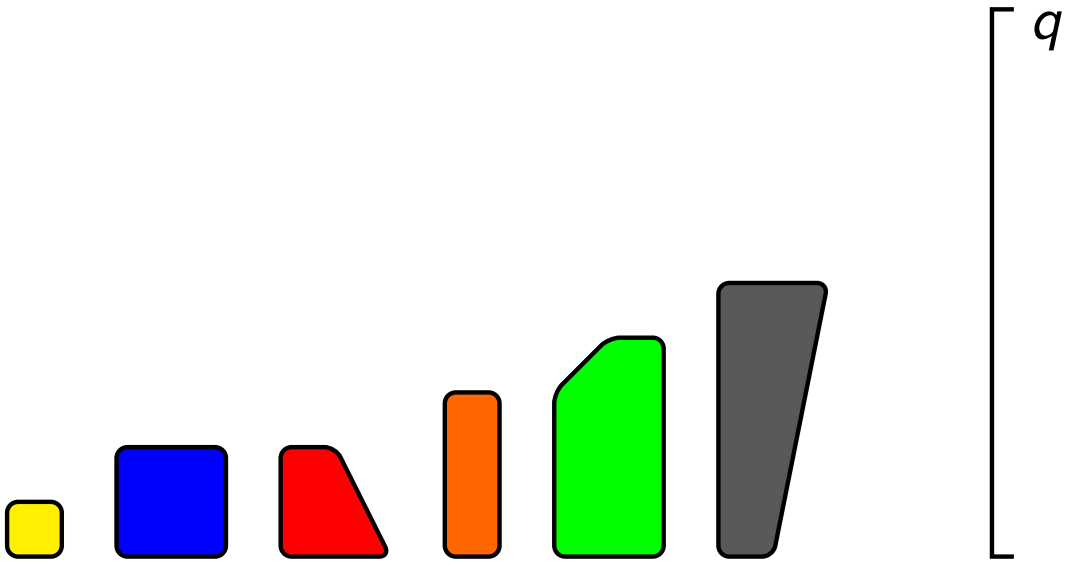
players $P = \{1, \ldots, n\}$,

*compact representation* $G = (w_1, \ldots, w_n; q)$

coalitional function $v(C) = 1 \iff \sum_{i \in C} w_i \geq q$

**Example (Shapley–Shubik index)**

players



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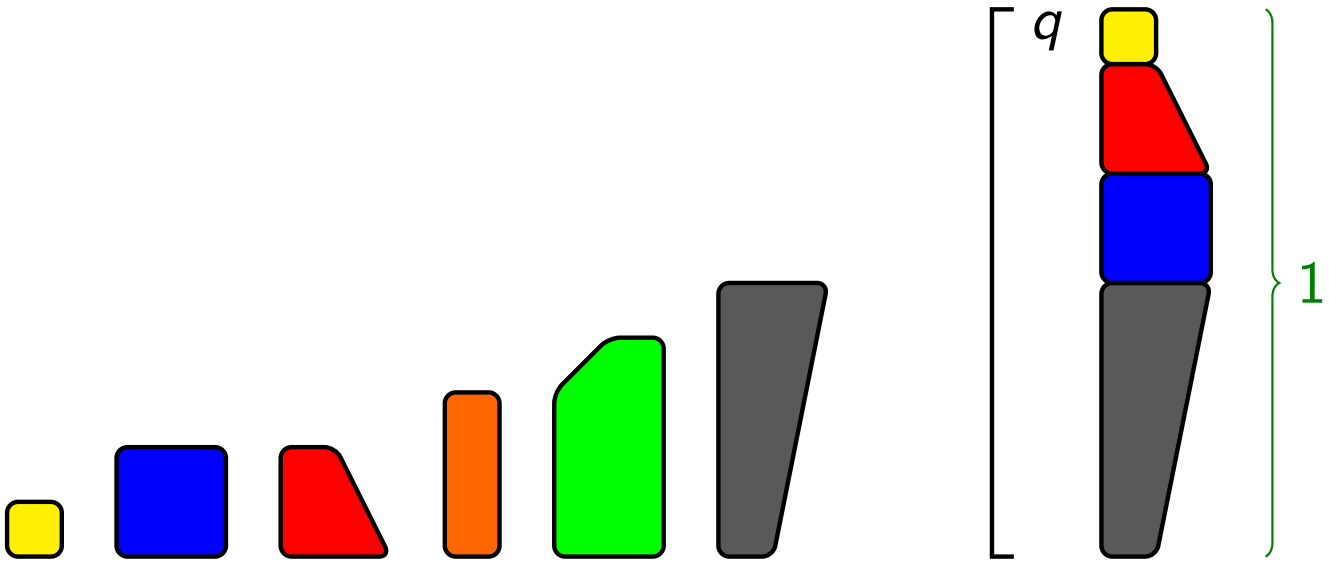
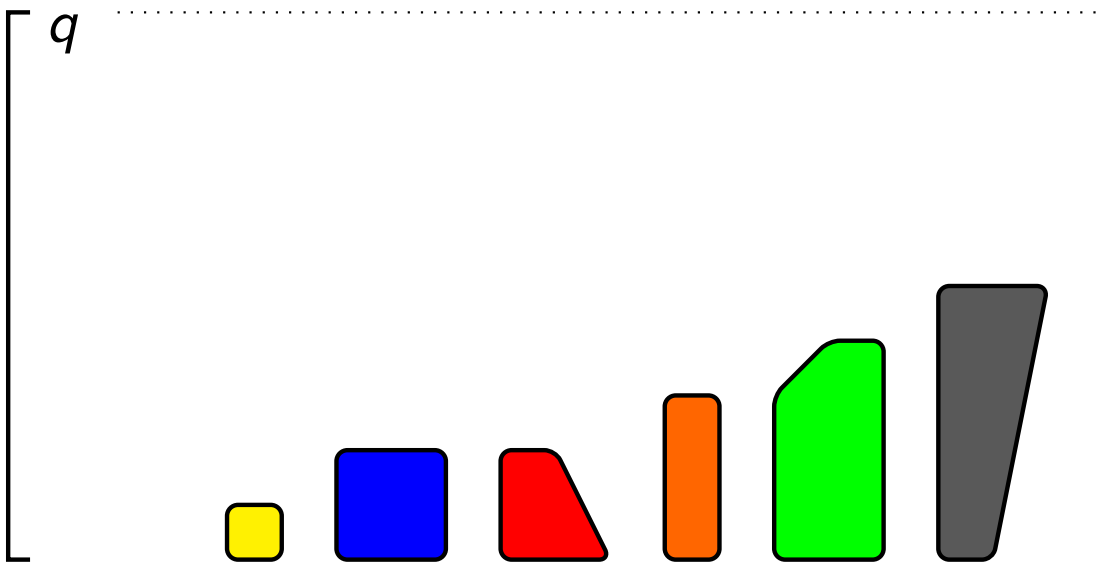


# The Shapley–Shubik Index

$$\mathrm{SSI}^*(G,i) \quad = \quad \sum_{C \subseteq P \setminus \{i\}} \|C\|! \cdot (n-1-\|C\|)! \cdot (v(C \cup \{i\}) - v(C))$$



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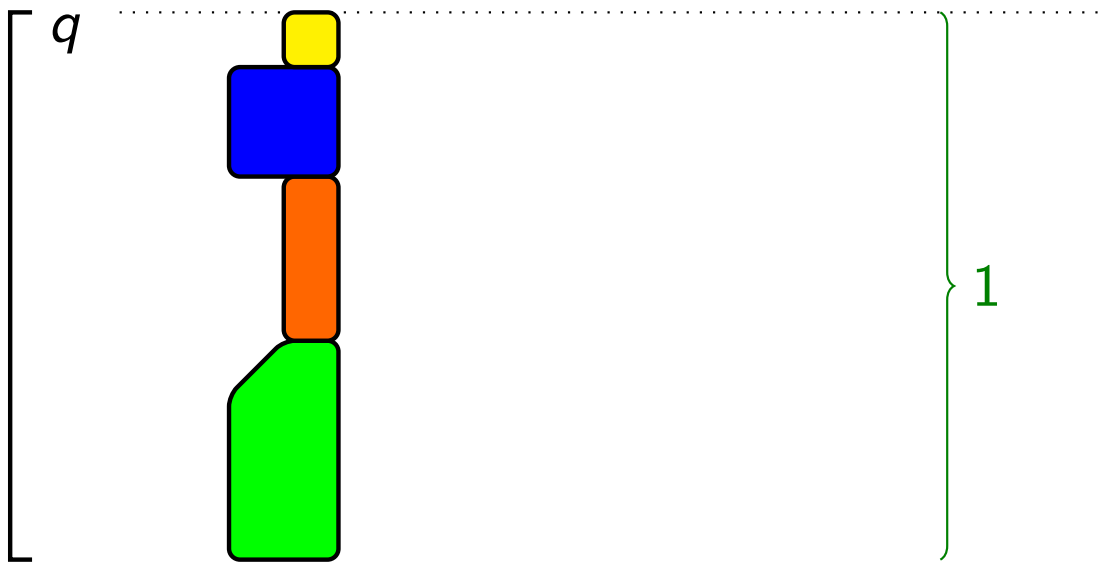

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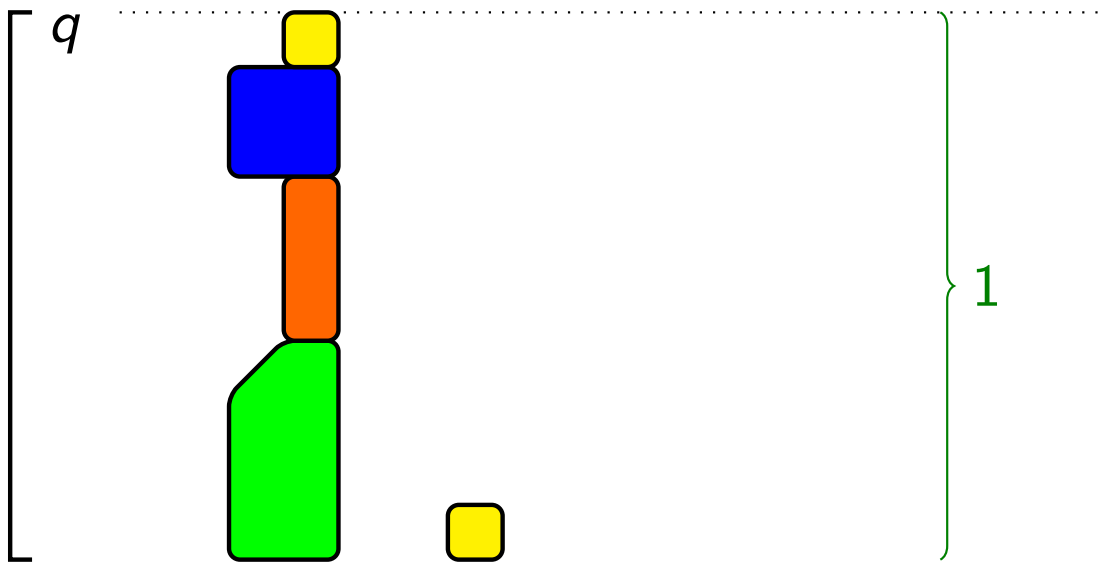

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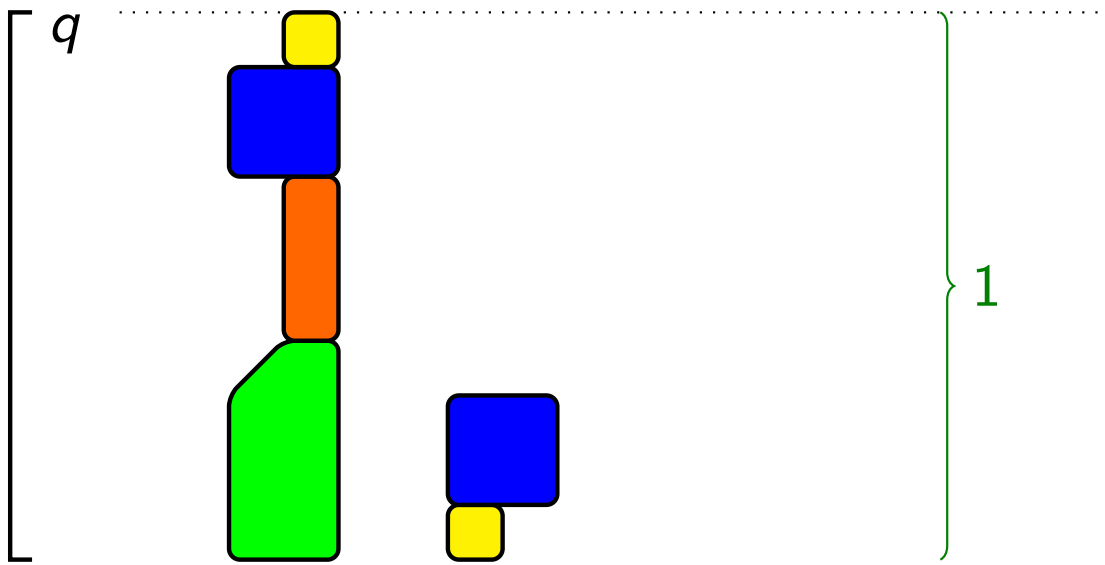

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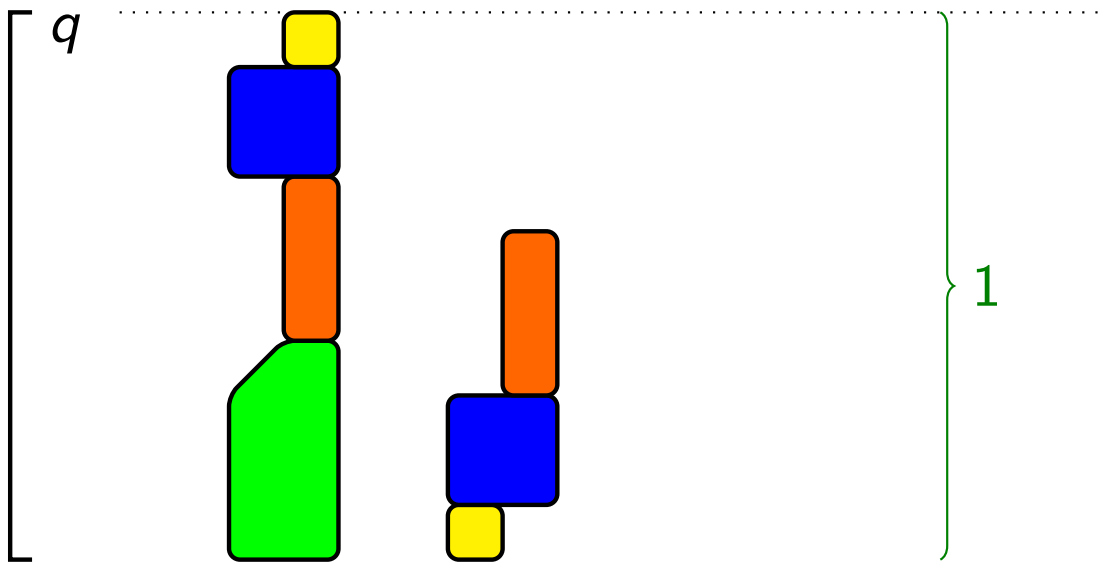

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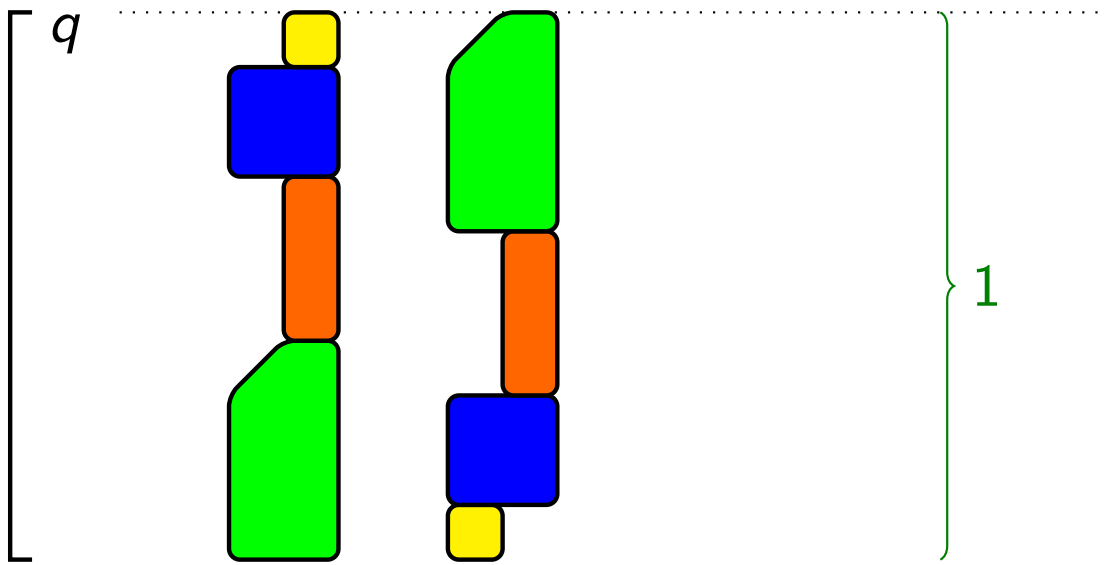

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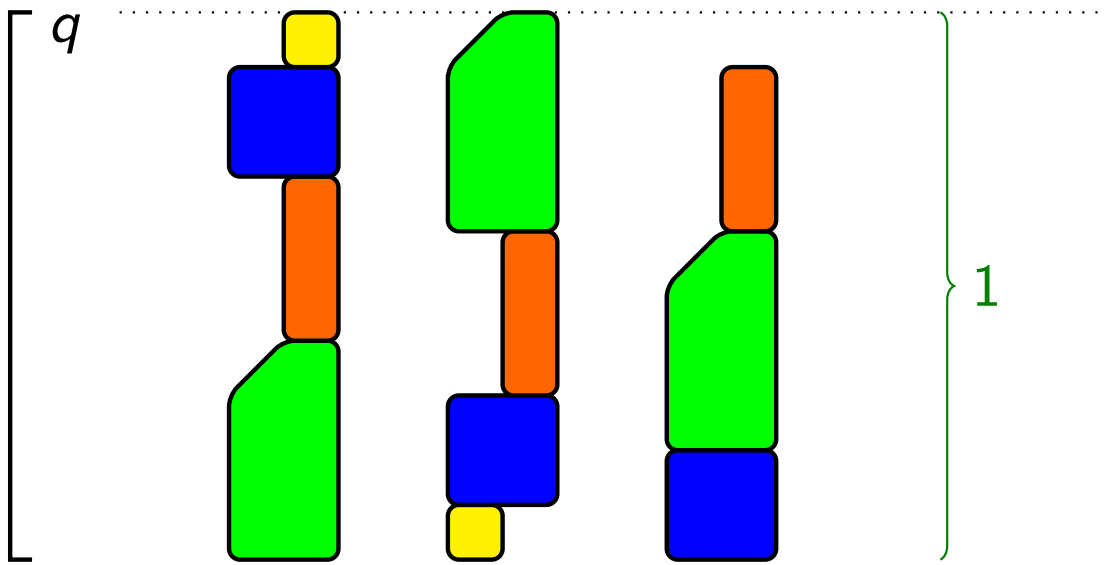

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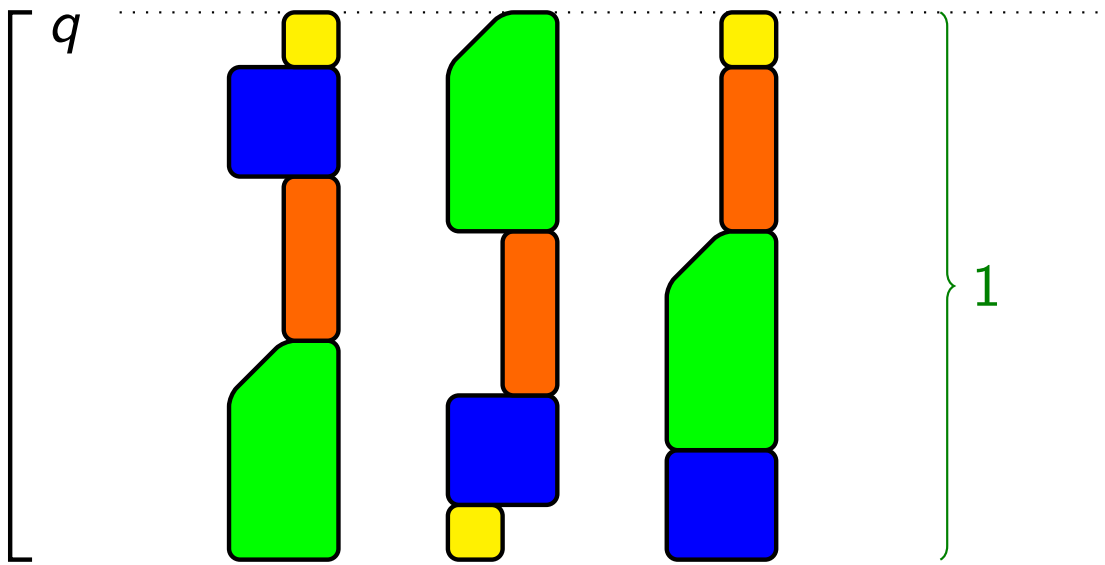

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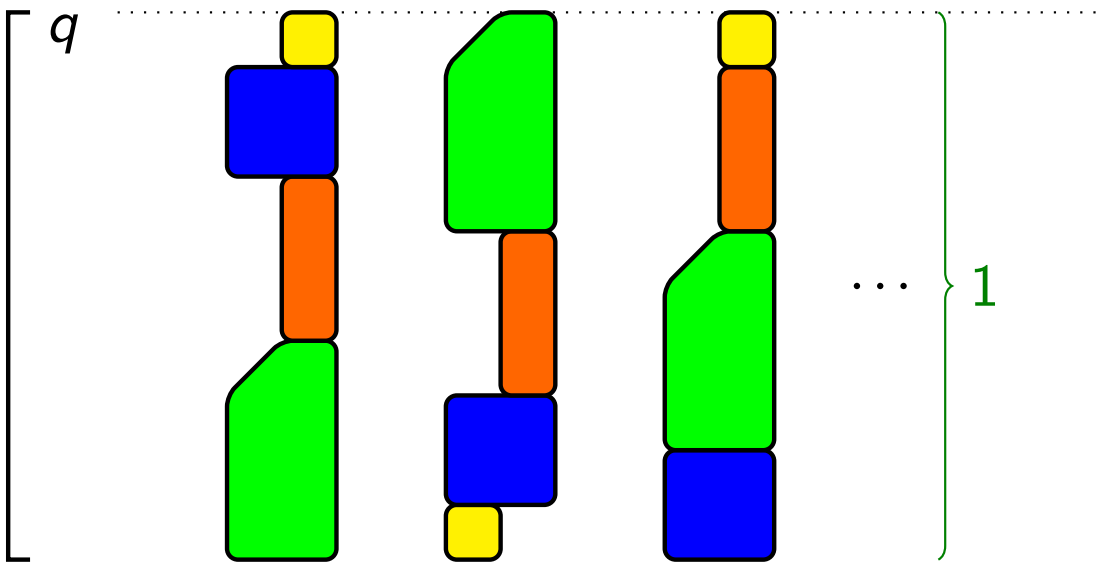
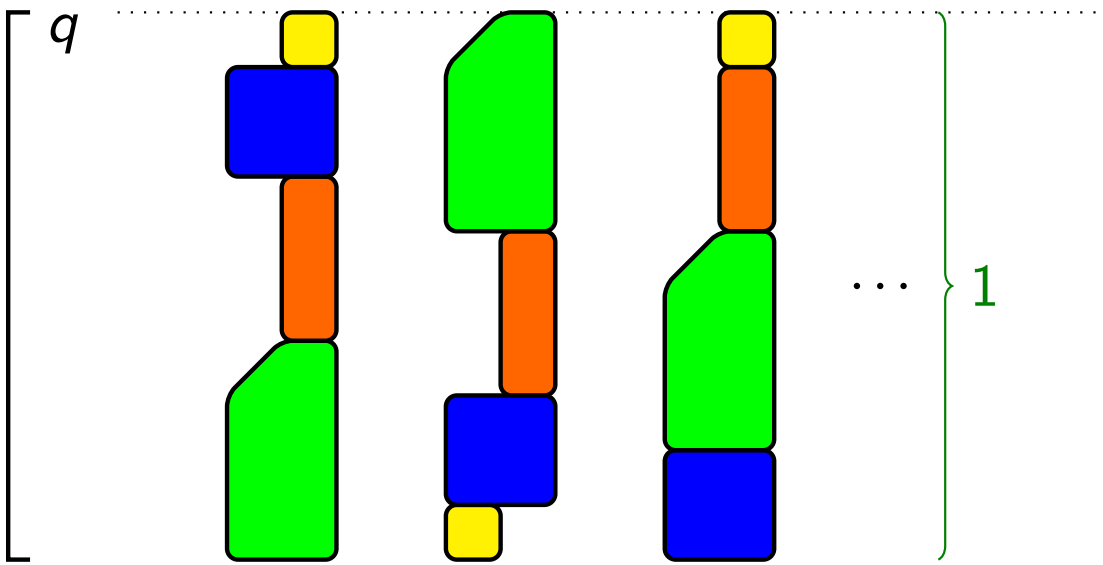



$\mathrm{SSI}^*(G, \square) = 48$

# The Shapley–Shubik Index

$$\mathrm{SSI}^*(G,i) = \sum_{C \subseteq P \setminus \{i\}} \|C\|! \cdot (n-1-\|C\|)! \cdot (v(C \cup \{i\}) - v(C))$$

$$\mathrm{SSI}(G,i) = \frac{\mathrm{SSI}^*(G,i)}{n!}$$




$\mathrm{SSI}^*(G, \square) = 48$

$\mathrm{SSI}(G, \square) = 1/15$

# Shapley Value versus Core

**What is the relationship between the Shapley value and the core?**

- One may expect that the payoff vector

$$(\varphi_1(G), \ldots, \varphi_n(G))$$

  is in the core of $G$ whenever $Core(G)$ is nonempty.
- However, in general this is not the case.
- The easiest way to see this is to focus on simple superadditive games:
  - A player receives a positive payoff in a core imputation only if he is a veto player,
  - whereas his Shapley value is positive as long as he is not a dummy.

# Shapley Value versus Core

- Thus any simple superadditive game that has some veto players (so its core is nonempty) as well as some players that are neither veto players nor dummies provides a counterexample to our conjecture.
- For concreteness, consider the 3-player simple game $G^1$ discussed earlier, in which a coalition wins if and only if it contains player 1 and at least one of the other two players.
- Clearly, player 1 is the only veto player in this game (so the core is given by $\{(1,0,0)\}$), but players 2 and 3 are not dummies, so their Shapley value is strictly positive.
- One can verify that $\varphi_1(G^1) = 2/3$ and $\varphi_2(G^1) = \varphi_3(G^1) = 1/6$.

# Shapley Value versus Core

- This example suggests that (at least in simple games) the Shapley value provides a more nuanced measure of a player's contribution than the core.
- However, for convex games the Shapley value is always in the core.
- Indeed, there is an obvious similarity between the definition of the Shapley value and the method we used to construct an outcome in the core of a convex game: The only difference is that
  - the latter pays each player according to his marginal contribution with respect to a *specific* permutation of the players, while
  - the former averages over *all* player permutations (Equation (2)).

# Shapley Value versus Core

- This observation, coupled with the fact that the core of any cooperative game is a convex set,[1] implies that for every convex game $G$ it holds that

$$(\varphi_1(G), \ldots, \varphi_n(G))$$

is in the core of $G$.

[1] For every $\alpha \in [0,1]$ it holds that if $\vec{x}$ and $\vec{y}$ are payoff vectors in the core of a game $G$, then their *convex combination* $\alpha\vec{x} + (1-\alpha)\vec{y}$ is also in the core of $G$.

# The Banzhaf Indices

- Banzhaf (1965) introduced a different power index in simple games.
- The same power index has actually been proposed even earlier by Penrose (1946) and was then rediscovered by Banzhaf, which is why it is sometimes referred to as the *Penrose–Banzhaf index*.
- Here the order in which players join the coalitions does not matter, all that matters is the number of coalitions for which they are pivotal.

# The Banzhaf Indices

## Definition (raw Banzhaf index)

- Formally, for a simple game $G = (P, v)$, the *raw Banzhaf index of a player $i$ in $G$* is defined by

$$\mathrm{BI}^*(G, i) = \sum_{C \subseteq P \setminus \{i\}} d_G(C, i), \tag{5}$$

  where the marginal contribution $d_G(C, i)$ of player $i$ to coalition $C$ is defined as in (1), i.e., $d_G(C, i) = 1$ if $i$ is pivotal for $C$, and $d_G(C, i) = 0$ otherwise.

- The value $\mathrm{BI}^*(G, i)$ thus gives the number of coalitions for which $i$ is pivotal.

# The Banzhaf Indices

**Definition (normalized and probabilistic Banzhaf index)**

- Banzhaf (1965) normalizes the raw Banzhaf index by

$$\overline{\mathrm{BI}}(G,i) \;=\; \frac{\mathrm{BI}^*(G,i)}{\sum_{j=1}^{n} \mathrm{BI}^*(G,j)},$$

  and thus obtains the *(normalized) Banzhaf index of $i$ in $G$*.

- Dubey and Shapley (1979) defined the *probabilistic Banzhaf index of $i$ in $G$* by

$$\mathrm{BI}(G,i) \;=\; \frac{\mathrm{BI}^*(G,i)}{2^{n-1}}.$$

# The Banzhaf Indices

## Remark

- *Unlike* $\mathrm{BI}(G,i)$, *the original Banzhaf index* $\overline{\mathrm{BI}}(G,i)$ *satisfies*
  - *neither additivity*
  - *nor the valuation property (due to Dubey and Shapley (1979)):*

    *If* $G_1 = (P, v_1)$ *and* $G_2 = (P, v_2)$ *are simple games and* $\mathbb{PI}$ *is a power index, then*

    $$\mathbb{PI}(G_\vee, i) + \mathbb{PI}(G_\wedge, i) = \mathbb{PI}(G_1, i) + \mathbb{PI}(G_2, i),$$

    *where* $G_\vee = (P, v_1 \vee v_2)$ *and* $G_\wedge = (P, v_1 \wedge v_2)$ *are defined by*

    $$\begin{aligned} (v_1 \vee v_2)(C) &= \max(v_1(C), v_2(C)) \text{ and} \\ (v_1 \wedge v_2)(C) &= \min(v_1(C), v_2(C)), \end{aligned}$$

    *respectively, for all coalitions* $C \subseteq P$.

# The Banzhaf Indices

## Remark

- *They conclude that* $\mathrm{BI}(G,i)$ *is "in many respects more natural" than the original Banzhaf index* $\overline{\mathrm{BI}}(G,i)$*:*

  *"This may be taken as an initial sign of trouble with the normalization [of the normalized Banzhaf index]."*

- *Both Banzhaf indices satisfy both*
  - *the dummy player property and*
  - *symmetry.*

- *On the other hand, unlike* $\overline{\mathrm{BI}}(G,i)$, $\mathrm{BI}(G,i)$ *is not efficient.*

# The Banzhaf Indices

## Example (Banzhaf indices)

- Consider again the weighted voting game $G = (108, 106, 62, 66; 277)$.
- As we have seen,
  - Anna, Belle, and Edgar are pivotal for two coalitions each, but
  - David is not pivotal for any coalition.
- Hence, the raw Banzhaf index of these four players is:

$$\begin{aligned} \mathrm{BI}^*(G, \mathrm{Anna}) &= \mathrm{BI}^*(G, \mathrm{Belle}) = \mathrm{BI}^*(G, \mathrm{Edgar}) = 2, \\ \mathrm{BI}^*(G, \mathrm{David}) &= 0. \end{aligned}$$

# The Banzhaf Indices

## Example (Banzhaf indices—continued)

- Thus, their normalized Banzhaf index is the same as their Shapley–Shubik index:

$$\begin{aligned} \overline{\mathrm{BI}}(G, \mathrm{Anna}) &= \overline{\mathrm{BI}}(G, \mathrm{Belle}) = \overline{\mathrm{BI}}(G, \mathrm{Edgar}) = 1/3, \\ \overline{\mathrm{BI}}(G, \mathrm{David}) &= 0, \end{aligned}$$

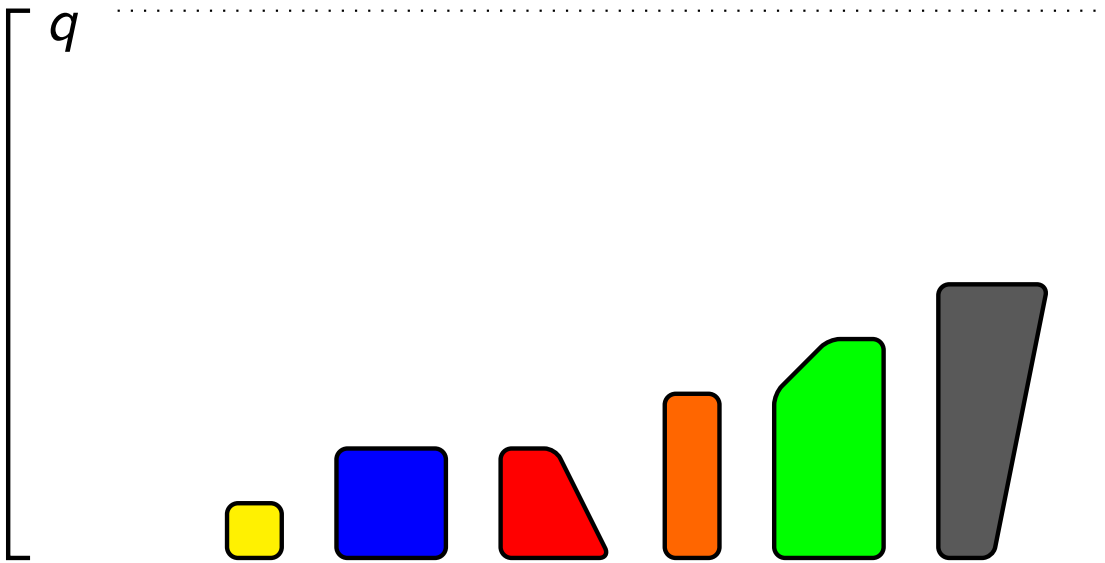
- but their probabilistic Banzhaf index is different:

$$\begin{aligned} \mathrm{BI}(G, \mathrm{Anna}) &= \mathrm{BI}(G, \mathrm{Belle}) = \mathrm{BI}(G, \mathrm{Edgar}) = 1/4, \\ \mathrm{BI}(G, \mathrm{David}) &= 0. \end{aligned}$$

# The Banzhaf Indices

raw Banzhaf index:

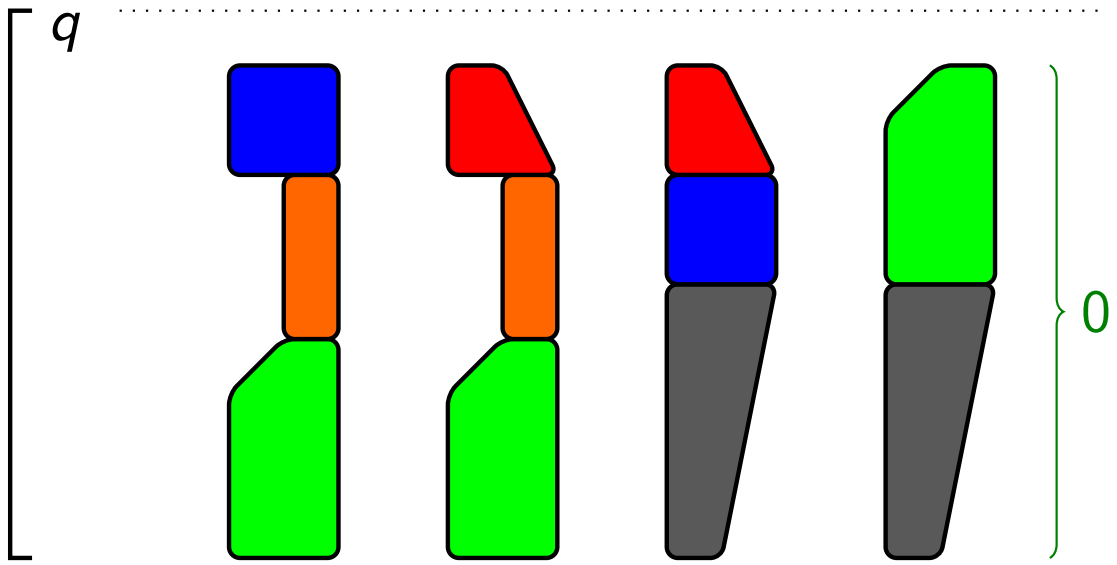
$$\mathrm{BI}^*(G,i) \;=\; \sum_{C \subseteq P \setminus \{i\}} (v(C \cup \{i\}) - v(C))$$



# The Banzhaf Indices

raw Banzhaf index:

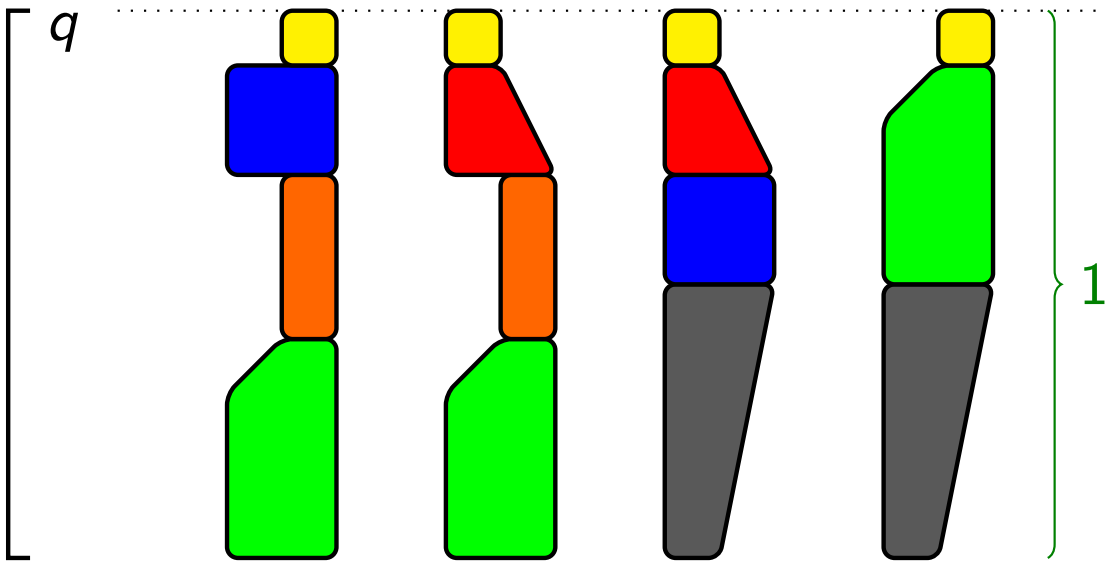
$$\mathrm{BI}^*(G,i) \;=\; \sum_{C \subseteq P \setminus \{i\}} (v(C \cup \{i\}) - v(C))$$



# The Banzhaf Indices

raw Banzhaf index:

$$\mathrm{BI}^*(G,i) \;=\; \sum_{C \subseteq P \setminus \{i\}} \big(v(C \cup \{i\}) - v(C)\big)$$




$\mathrm{BI}^*(G, \square) = 4$

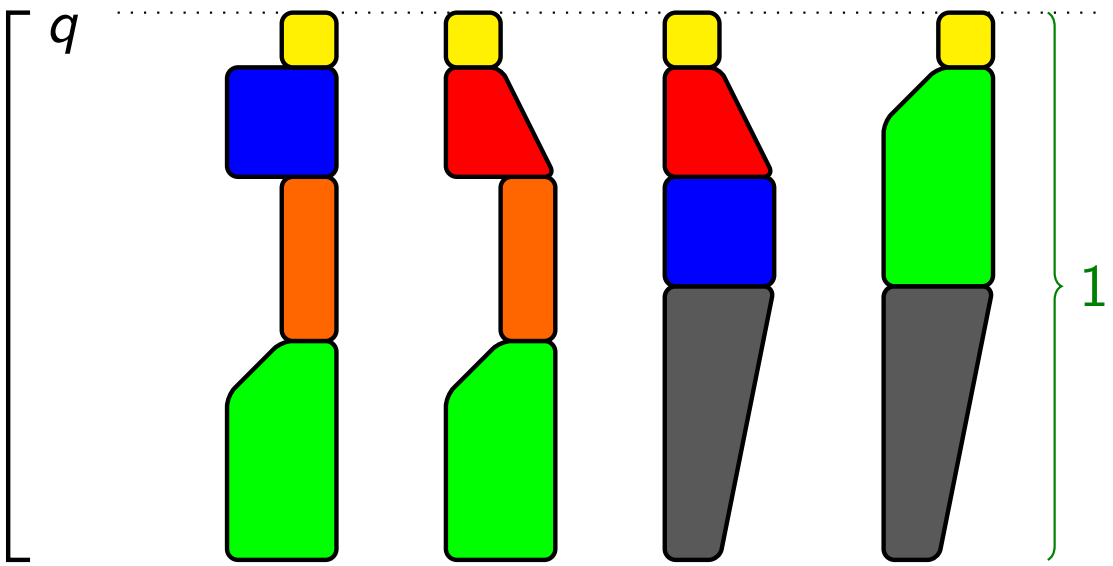
# The Banzhaf Indices

raw Banzhaf index:

$$\mathrm{BI}^*(G,i) \;=\; \sum_{C \subseteq P \setminus \{i\}} \left(v(C \cup \{i\}) - v(C)\right)$$

normalized Banzhaf index:

$$\overline{\mathrm{BI}}(G,i) \;=\; \frac{\mathrm{BI}^*(G,i)}{\sum_{j=1}^{n} \mathrm{BI}^*(G,j)}$$




$\mathrm{BI}^*(G, \square) = 4$

$\overline{\mathrm{BI}}(G, \square) = 1/14$

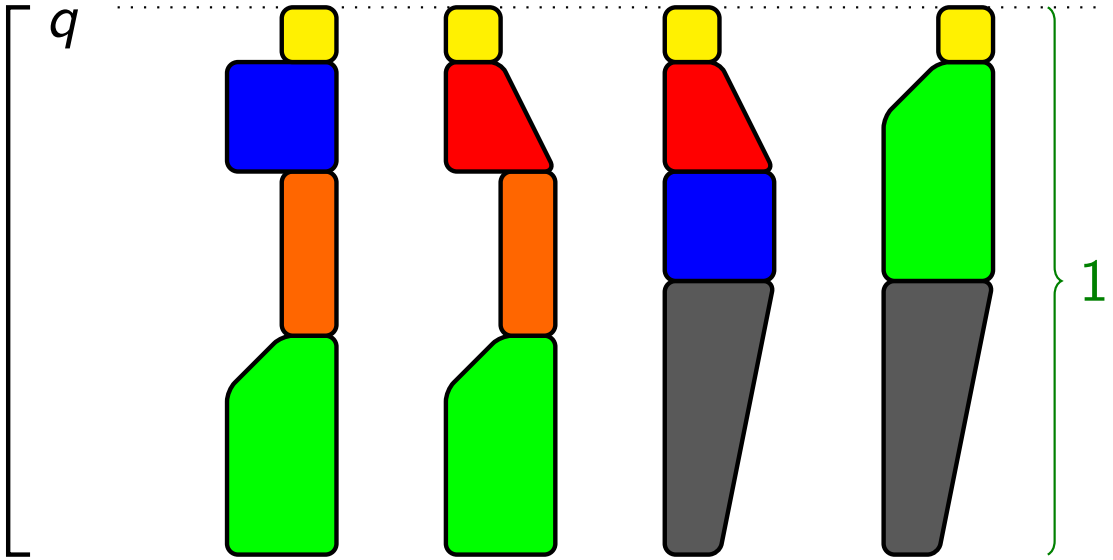
# The Banzhaf Indices

raw Banzhaf index:

$$\mathrm{BI}^*(G,i) \;=\; \sum_{C \subseteq P \setminus \{i\}} \big(v(C \cup \{i\}) - v(C)\big)$$

probabilistic Banzhaf index:

$$\mathrm{BI}(G,i) \;=\; \frac{\mathrm{BI}^*(G,i)}{2^{n-1}}$$

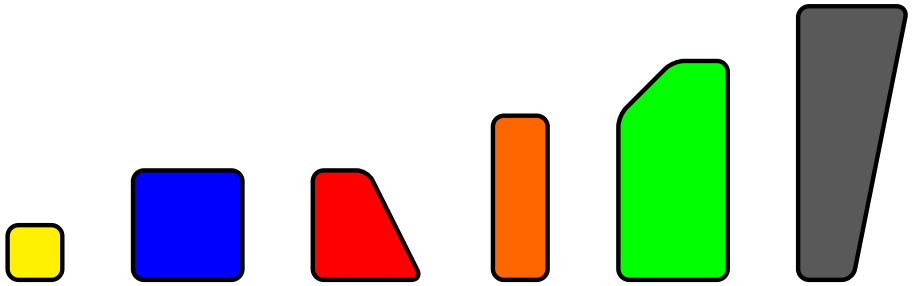



$\mathrm{BI}^*(G, \square) = 4$

$\mathrm{BI}(G, \square) = 1/8$

# The Banzhaf Index and the Shapley–Shubik Index

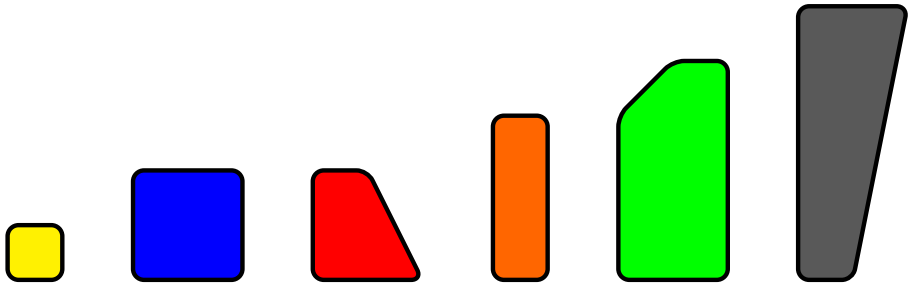
## Example



| | | | | | | |
|---|---|---|---|---|---|---|
| Banzhaf | $\frac{4}{32}$ | $\frac{6}{32}$ | $\frac{6}{32}$ | $\frac{10}{32}$ | $\frac{12}{32}$ | $\frac{18}{32}$ |

# The Banzhaf Index and the Shapley–Shubik Index

## Example



| | | | | | | |
|---|---|---|---|---|---|---|
| Banzhaf | $\frac{4}{32}$ | $\frac{6}{32}$ | $\frac{6}{32}$ | $\frac{10}{32}$ | $\frac{12}{32}$ | $\frac{18}{32}$ |
| Shapley–Shubik | $\frac{4}{60}$ | $\frac{6}{60}$ | $\frac{6}{60}$ | $\frac{11}{60}$ | $\frac{13}{60}$ | $\frac{20}{60}$ |