Pure and Mixed Strategies

In all games so far, all players had to choose exactly one strategy:

- Smith and Wesson had to either confess or remain silent in the prisoners’ dilemma;
- George and Helena had to go either to the soccer match or the concert in the battle of the sexes;
- David and Edgar could only either swerve or go on driving in the chicken game;
- in the penalty game, the kicker and the goalkeeper had each to choose one side of the goal, left or right;
- in the paper-rock-scissors game, David and Edgar would form upon pon either paper, rock, or scissors with their hands; and
- each player had to choose exactly one number in the guessing numbers game.
Pure and Mixed Strategies

All players play *pure strategies* in these games.

However, if one such game is played several times in a row, the players might change their minds and choose different strategies.

It would be pretty dull in certain games to always decide for the same strategy. For example, a goalkeeper who always jumps to the left side will be very predictable; instead he should choose randomly where to jump, sometimes to the left, sometimes to the right.

If the players make their decisions on which strategy to choose randomly under some probability distribution, we say they use a *mixed strategy*.

In many games, especially so in those with mixed strategies, one does not win by intelligence only, one has also to be lucky.
Nash Equilibrium in Mixed Strategies

Definition (Nash equilibrium in mixed strategies)
Let $\mathcal{S} = S_1 \times S_2 \times \cdots \times S_n$ be the set of strategy profiles of the $n$ players in a noncooperative game in normal form and let $g_i$ be the gain function of player $i$, $1 \leq i \leq n$. For simplicity, let us assume that all sets $S_i$ are finite.

1. A mixed strategy for player $i$ is a probability distribution $\pi_i$ on $S_i$, where $\pi_i(s_j)$ is the probability of the event that $i$ chooses the strategy $s_j \in S_i$. Let $\Pi_i$ be the set of all probability distributions on $S_i$ (so $\pi_i \in \Pi_i$). Let $\Pi = \Pi_1 \times \Pi_2 \times \cdots \times \Pi_n$.

2. The expected utility of a mixed-strategy profile $\bar{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ for player $i$ is

$$G_i(\bar{\pi}) = \sum_{\bar{s} = (s_1, \ldots, s_n) \in \mathcal{S}} g_i(\bar{s}) \prod_{j=1}^{n} \pi_j(s_j).$$
**Nash Equilibrium in Mixed Strategies**

**Remark**

- Intuitively, to compute the expected utility of $G_i(\bar{\pi})$ for player $i$,
  - we first calculate the probability of reaching each outcome given $\bar{\pi}$, and
  - we then calculate the average of the gains of the outcomes weighted by the probabilities of each outcome.

- We assume players to be risk-neutral, i.e., they seek to maximize their expected utility.

- The support of a mixed strategy $\pi_i$ for player $i$ is the set of pure strategies $\{s_j \mid \pi_i(s_j) > 0\}$.
  - A pure strategy is the special case of a mixed strategy whose support is a singleton.
  - A strategy $\pi_i$ is fully mixed if it has full support, i.e., every pure strategy $s_j \in S_i$ occurs in it with nonzero probability.
Nash Equilibrium in Mixed Strategies

Definition (Nash equilibrium in mixed strategies—continued)

3 A mixed strategy $\pi_i \in \Pi_i$ is player $i$’s best response to the mixed-strategy profile $\vec{\pi}_{-i} = (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n) \in \Pi_{-i}$ of the other players if for all mixed strategies $\pi_i' \in \Pi_i$,

$$G_i(\pi_1, \ldots, \pi_{i-1}, \pi_i, \pi_{i+1}, \ldots, \pi_n) \geq G_i(\pi_1, \ldots, \pi_{i-1}, \pi_i', \pi_{i+1}, \ldots, \pi_n). \quad (1)$$

4 A profile $\vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ of mixed strategies is in a Nash equilibrium in mixed strategies if $\pi_i$ is a best response to $\vec{\pi}_{-i}$ for all players $i$. 
Remark

That is, a profile $\vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ of mixed strategies is in a Nash equilibrium in mixed strategies if and only if no player $i$ has a mixed strategy $\pi_i' \in \Pi_i$ that would give her a higher profit than her mixed strategy $\pi_i$ on $S_i$ in response to the mixed strategies she expects the other players to choose.

For each player, one-sided deviation from their mixed strategies would thus be not beneficial (and might even be punished), assuming that the other players stick to their mixed strategies of the Nash equilibrium.
Nash Equilibrium in Mixed Strategies

Remark (continued)

- Consequently, for a Nash equilibrium in mixed strategies, every player is indifferent to each strategy she chooses with positive probability in her mixed strategy (i.e., to each strategy in her support).

- Also, the players’ probability distributions in the profile of their mixed strategies are independent. (Compare: “correlated equilibrium.”)

Theorem

1. Let \( \vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_n) \) be a profile of mixed strategies in a noncooperative game in normal form. A mixed strategy \( \pi_i \) is a best response to the mixed-strategy profile \( \vec{\pi}_{-i} \) if and only if all pure strategies in its support are best responses.

2. Every pure Nash equilibrium is also a mixed Nash equilibrium.
Nash Equilibrium in Mixed Strategies

Why?

1. For a contradiction, suppose that a best response mixed strategy contains in its support a pure strategy that itself is not a best response.

   Then the player’s expected utility would be improved by decreasing the probability of the worst such pure strategy (increasing proportionally the remaining nonzero probabilities to fill the gap).

   This contradicts that the given mixed strategy was a best response.

   The converse is immediate.

Exercise.
Remark

- **As the following examples demonstrate, the converse is not necessarily true:** The existence of a Nash equilibrium in mixed strategies does not imply the existence of a Nash equilibrium in pure strategies.

- **That is, there can exist Nash equilibria in mixed strategies in addition to those in pure strategies.**

- **In particular, Nash equilibria in mixed strategies may exist in games that have no Nash equilibrium in pure strategies at all.**
Penalty Game: Mixed-Strategy Nash Equilibrium

There is no Nash equilibrium in pure strategies.

However, there is a Nash equilibrium in mixed strategies if the kicker $S$ and the goalkeeper $T$ both randomize uniformly:

$$\pi_S = (\pi_S(L), \pi_S(R)) = (1/2, 1/2) = (\pi_T(L), \pi_T(R)) = \pi_T.$$
Modified Penalty Game: Mixed-Strategy Nash Equilibrium

Table: The penalty game with a goalkeeper acting awkwardly on the left

<table>
<thead>
<tr>
<th>Kicker</th>
<th>Goalkeeper</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Left</td>
<td>(0,0)</td>
<td>(1,−1)</td>
<td></td>
</tr>
<tr>
<td>Right</td>
<td>(1,−1)</td>
<td>(−1,1)</td>
<td></td>
</tr>
</tbody>
</table>

Again, there is *no* Nash equilibrium in pure strategies.

However, there is a Nash equilibrium in mixed strategies:

\[(π_S, π_T) = ((2/3, 1/3), (2/3, 1/3)).\]

Table: The paper-rock-scissors game

<table>
<thead>
<tr>
<th></th>
<th>Edgar</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Rock</td>
<td>Scissors</td>
<td>Paper</td>
</tr>
<tr>
<td>David</td>
<td>Rock</td>
<td>(0, 0)</td>
<td>(1, −1)</td>
</tr>
<tr>
<td></td>
<td>Scissors</td>
<td>(−1, 1)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td></td>
<td>Paper</td>
<td>(1, −1)</td>
<td>(−1, 1)</td>
</tr>
</tbody>
</table>

Again, there is no Nash equilibrium in pure strategies.

However, there is a Nash equilibrium in mixed strategies:

\[ (\pi_D, \pi_E) = \left( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \right). \]
Battle of the Sexes: Mixed-Strategy Nash Equilibria

Table: The battle of the sexes

<table>
<thead>
<tr>
<th></th>
<th>Soccer</th>
<th>Concert</th>
</tr>
</thead>
<tbody>
<tr>
<td>George</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Soccer</td>
<td>(10, 1)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>Concert</td>
<td>(0, 0)</td>
<td>(1, 10)</td>
</tr>
</tbody>
</table>

• Nash equilibria in pure strategies:
  (Soccer, Soccer) and (Concert, Concert).

• In addition, there is also a third Nash equilibrium in mixed strategies:
  \((\pi_G, \pi_H) = ((10/11, 1/11), (1/11, 10/11))\).
Nash Equilibria in Mixed Strategies

**Examples**

### Chicken Game: Mixed-Strategy Nash Equilibria

**Table:** The chicken game

<table>
<thead>
<tr>
<th></th>
<th>Edgar</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Swerve</td>
</tr>
<tr>
<td>Swerve</td>
<td>(2, 2)</td>
</tr>
<tr>
<td>Drive on</td>
<td>(3, 1)</td>
</tr>
</tbody>
</table>

- Nash equilibria in pure strategies:
  - (Drive on, Swerve) and (Swerve, Drive on).

- Again, there is a third Nash equilibrium in mixed strategies:
  \[
  (\pi_D, \pi_E) = \left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right).
  \]
Chicken Game: Mixed-Strategy Nash Equilibria

Interpreting the three Nash equilibria in this game as recommendations for action, one could advice the players to do the following (and wish them good luck in evaluating their opponents well!):

1. If you expect your opponent to be a chicken, then you should definitely go all out and win heroically.
   
   This corresponds to one of the two Nash equilibria in pure strategies.

2. If you expect your opponent to be undaunted by death and risk it all, then you should be wise and swerve. You won’t win, but you’ll survive at least.
   
   This corresponds to the other one of the two pure Nash equilibria.

3. If you can’t judge your opponent well and just have no idea of what he is up to do, then you should toss a coin and go all out with heads, but cautiously swerve with tails. Maybe you win; if not, maybe you survive—good luck!
   
   This corresponds to the additional Nash equilibrium in mixed strategies.
Prisoners’ Dilemma: More Mixed-Strategy Nash Equilibria?

Table: The prisoners’ dilemma

<table>
<thead>
<tr>
<th></th>
<th>Wesson</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Confession</td>
</tr>
<tr>
<td>Smith</td>
<td>Confession</td>
</tr>
<tr>
<td></td>
<td>Silence</td>
</tr>
</tbody>
</table>

- Nash equilibrium in pure strategies: (Confession, Confession).
- There exists no additional Nash equilibrium in mixed strategies.
Different Interpretations of Mixed-Strategy Nash Equilibria

What does it mean to play a mixed strategy?

- Randomize to *confuse* your opponent:
  - Penalty game
  - Paper-Rock-Scissors game

- Randomize when you are *uncertain* about the other players' actions:
  - Battle of the sexes
  - Chicken game

- Mixed strategies describe what might happen in *repeated play*:
  - Number/frequency of pure strategies in the limit

- Mixed strategies describe *population dynamics*:
  - Some players chosen from a population of players, each with deterministic (i.e., pure) strategies
  - A mixed strategy is the probability of picking a player who will play one pure strategy or another
### Properties of Some Two-Player Games

**Table:** Properties of some two-player games

<table>
<thead>
<tr>
<th></th>
<th>Prisoners' dilemma</th>
<th>Battle of the sexes</th>
<th>Chicken game</th>
<th>Penalty game</th>
<th>Paper-Rock-Scissors game</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dominant strategies?</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Strictly dominant strategies?</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Number of NE in pure str.</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Number of NE in mixed str.</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Number of PO</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>PO = NE?</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>
Nash’s Theorem

Theorem (Nash (1950; 1951))

For each noncooperative game in normal form with a finite number of players each having a finite set of strategies, there exists a Nash equilibrium in mixed strategies.

• Nash provided two proofs of his celebrated result.

• We sketch the first and give a more detailed outline of the second.
Sketch of First Proof of Nash’s Theorem

- Let $\mathcal{S} = S_1 \times S_2 \times \cdots \times S_n$ be the set of strategy profiles of the $n$ players in a noncooperative game in normal form. All sets $S_i$ are here assumed to be finite.

- How can the abstract notion of “strategy” (in pure and in mixed form) be made accessible to mathematical or, specifically, topological arguments?

  How can, for example, the very concrete strategy *Drive on* in the chicken game be compared with another concrete strategy from a different game, such as *Confession* in the prisoners’ dilemma or *Left* in the penalty game?

- Nash views pure strategies as the unit vectors in an appropriate real vector space; every strategy from $S_i$ is thus in $\mathbb{R}^{m_i}$. 
Sketch of First Proof of Nash’s Theorem

- Strategies can then be mixed using the common operations in vector spaces:
  - Every mixed strategy is the linear combination of pure strategies, each weighted by a certain probability, and
  - since a mixed strategy corresponds to a probability distribution on $S_i$, these probabilities sum up to 1.

- Mathematically speaking, mixed strategies over $S_i$ are the points of a *simplex*, which can be viewed as a convex subset of $\mathbb{R}^{m_i}$.

- Such a subset is said to be *convex* if the direct connection between any two points of this subset completely lies within this subset.
Sketch of First Proof of Nash’s Theorem

Figure: A convex and a nonconvex set
Sketch of First Proof of Nash’s Theorem

- In addition, the strategy sets are required to be *compact* (which is defined using the mathematical terms of closure and boundedness).

- Also the gain functions $g_i$, $1 \leq i \leq n$, mapping each strategy profile $\bar{s} = (s_1, s_2, \ldots, s_n) \in \mathcal{S}$ to a real number, must satisfy certain conditions so that known fixed point theorems from topology can be applied to them.

- To wit, it is required that the (multilinear) extensions of the functions $g_i$ to the set of mixed strategies over $\mathcal{S}$ be *continuous* and *quasi-concave in $s_j$* for all $j$, $1 \leq j \leq n$.

  Continuity means that if there are only very small changes in the profiles of mixed strategies, then also the corresponding gains change only very little, i.e., there are no “jumps” (technically speaking, no points of discontinuity) in these gain functions.
Sketch of First Proof of Nash’s Theorem

- A function $g : \mathbb{R} \to \mathbb{R}$ is said to be **quasi-convex** if all sets of the form

  $$M_c = \{ x \in \mathbb{R} \mid g(x) \leq c \}$$

  are convex, and

- A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **quasi-concave** if its negation, $-f$, is quasi-convex. For example,
  - every monotonic function is both quasi-convex and quasi-concave, and
  - every function monotonically increasing up to a certain point and then monotonically decreasing is quasi-concave.
Sketch of First Proof of Nash’s Theorem

For concreteness, suppose that Anna and Belle play a two-player noncooperative game in normal form with
- Anna having the pure strategies $a_1$, $a_2$, and $a_3$ and
- Belle having the pure strategies $b_1$ and $b_2$.

<table>
<thead>
<tr>
<th>Anna</th>
<th>Belle Strategy $b_1$</th>
<th>Belle Strategy $b_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy $a_1$</td>
<td>$(1, 2)$</td>
<td>$(1, 2)$</td>
</tr>
<tr>
<td>Strategy $a_2$</td>
<td>$(4, 1)$</td>
<td>$(2, 0)$</td>
</tr>
<tr>
<td>Strategy $a_3$</td>
<td>$(3, 0)$</td>
<td>$(4, 3)$</td>
</tr>
</tbody>
</table>
Sketch of First Proof of Nash’s Theorem

(a) Anna’s gains

(b) Belle’s gains

Figure: Convex gain sets for pure and mixed strategy sets
Sketch of First Proof of Nash’s Theorem

- Since finite sets cannot be convex, the existence of a Nash equilibrium in *pure* strategies cannot be guaranteed by the proof of Nash’s Theorem.

- The set of *mixed* strategies over $\mathcal{S}$ (including the pure strategies as special cases), however, is compact and convex and the extensions of the gain functions on these sets satisfy all required conditions, which makes certain fixed point theorems of topology applicable.

- It is then possible to define suitable transformations whose fixed points correspond to the Nash equilibria in mixed strategies.
Sketch of First Proof of Nash’s Theorem

- For \( \vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_n) \in \Pi \) and every player \( i \), a best response correspondence \( b_i(\vec{\pi}_{-i}) \) is defined as a relation from the set of probability distributions \( \Pi_{-i} \) over the other players’ strategies.

- Setting

\[
b(\vec{\pi}) = b_1(\vec{\pi}_{-1}) \times b_2(\vec{\pi}_{-2}) \times \cdots \times b_n(\vec{\pi}_{-n})
\]

and using the fixed point theorem of Kakutani, one can prove that \( b \) must have a fixed point under the hypotheses mentioned.

- That is, there exists a strategy profile \( \vec{\pi}^* \) with \( \vec{\pi}^* \in b(\vec{\pi}^*) \).
Sketch of First Proof of Nash’s Theorem

- However, since \( b(\vec{π}) \) contains the best response strategies of all players to \( \vec{π} \) by definition, this fixed point

  \[
  \vec{π}^* \in b(\vec{π}^*)
  \]

shows that the mixed strategies of all players in \( \vec{π}^* \) are simultaneously in a Nash equilibrium in mixed strategies.

- No player has an incentive to deviate from her mixed strategy in \( \vec{π}^* \), assuming that all other players stick to their strategies in \( \vec{π}^* \) as well.

- This is the idea of the original proof of Nash’s Theorem.
Nash’s Second Proof: Some Basic Definitions

Definition

1. A set $X \subseteq \mathbb{R}^m$ is **convex** if for all $\vec{x}, \vec{y} \in X$ and for all real numbers $\lambda \in [0, 1]$,
   $$\lambda \cdot \vec{x} + (1 - \lambda) \cdot \vec{y} \in X.$$  

2. For vectors $\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^m$ and nonnegative scalars $\lambda_0, \lambda_1, \ldots, \lambda_n$ satisfying $\sum_{i=0}^n \lambda_i = 1$, the vector
   $$\sum_{i=0}^n \lambda_i \cdot \vec{x}_i$$
   is a **convex combination of** $\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_n$.

3. A finite set $\{\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_n\}$ of vectors in $\mathbb{R}^m$ is said to be **affinely independent** if
   $$\left( \sum_{i=0}^n \lambda_i \cdot \vec{x}_i = \vec{0} \text{ and } \sum_{i=0}^n \lambda_i = 0 \right) \Rightarrow \lambda_0 = \lambda_1 = \cdots = \lambda_n = 0.$$
Nash’s Second Proof: Simplex

Definition

1. An \( n \)-simplex is the set of all convex combinations of the affinely independent set \( \{ \vec{x}_0, \vec{x}_1, \ldots, \vec{x}_n \} \) of vectors:

\[
\vec{x}_0 \cdots \vec{x}_n = \left\{ \sum_{i=0}^{n} \lambda_i \cdot \vec{x}_i \mid \lambda_i \geq 0 \text{ for each } i, 0 \leq i \leq n, \text{ and } \sum_{i=0}^{n} \lambda_i = 1 \right\}.
\]

(a) Every \( \vec{x}_i \) is a \textit{vertex of the n-simplex} \( \vec{x}_0 \cdots \vec{x}_n \).

(b) Every \( k \)-simplex \( \vec{x}_{i_0} \cdots \vec{x}_{i_k}, i_0, \ldots i_k \in \{0,1,\ldots,n\} \), is a \textit{k-face of} \( \vec{x}_0 \cdots \vec{x}_n \).

2. The \textit{standard n-simplex} \( \Delta_n \) is defined as

\[
\Delta_n = \left\{ \vec{y} = (y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1} \mid y_i \geq 0, 0 \leq i \leq n, \text{ and } \sum_{i=0}^{n} y_i = 1 \right\}.
\]

That is, \( \Delta_n = \vec{u}_0 \cdots \vec{u}_n \), where \( \vec{u}_i \) denotes the \( i \)-th unit vector in \( \mathbb{R}^{n+1} \).
Nash’s Second Proof: Simplex

(a) 0-simplex

(b) 1-simplex

(c) 2-simplex

(d) 3-simplex

Figure: \( n \)-simplexes for \( 0 \leq n \leq 3 \)
Nash’s Second Proof: Simplicial Subdivision & Labeling

Definition

1. A simplicial subdivision of an $n$-simplex $T$ is a finite set of simplexes $\{T_i \mid 1 \leq i \leq k\}$ such that
   (a) $\bigcup_{T_i \in T} T_i = T$ and
   (b) for each $T_i, T_j \in T$, $T_i \cap T_j$ is either empty or equal to a common face.

2. Let $T = \vec{x}_0 \cdots \vec{x}_n$ be a simplicial subdivided $n$-simplex, and let $V$ denote the set of all distinct vertices of all the subsimplexes.
   For a point $\vec{y} \in T$, $\vec{y} = \sum_{i=0}^{n} \lambda_i \cdot \vec{x}_i$, let $\sigma(\vec{y}) = \{i \mid \lambda_i > 0\}$ be the set of vertices “involved” in $\vec{y}$.

   A function $L : V \rightarrow \{0, 1, \ldots, n\}$ is a proper labeling of a subdivision of $T$ if $L(\vec{v}) \in \sigma(\vec{v})$.

3. A subsimplex of $T$ is completely labeled by $L$ if $L$ takes on all the values $0, 1, \ldots, n$ on its set of vertices.
Nash’s Second Proof: Simplicial Subdivision & Labeling

Figure: A properly labeled simplicial subdivision of a 2-simplex
Nash’s Second Proof: Sperner’s Lemma

Lemma (Sperner’s Lemma)

Let $T = \vec{x}_0 \cdots \vec{x}_n$ be a simplicially subdivided $n$-simplex and let $\mathcal{L}$ be a proper labeling of the subdivision of $T$. There are an odd number of subsimplexes that are completely labeled by $\mathcal{L}$ in this subdivision of $T$.

Proof: See blackboard.
Nash’s Second Proof: Illustrating Sperner’s Lemma

Figure: Walking through the 2-simplex $T_2 = \vec{x}_0 \vec{x}_1 \vec{x}_2$
Nash Equilibria in Mixed Strategies

Nash’s Second Proof: Illustrating Sperner’s Lemma

Figure: All walks through the 2-simplex \( T_2 = \vec{x}_0 \vec{x}_1 \vec{x}_2 \)
Nash’s Second Proof: Compact Set & Centroid

Definition (compact set)
A subset of $\mathbb{R}^m$ is *compact* if it is closed and bounded.

Remark
- $\Delta_m$ is compact.
- A compact set has the property that every infinite sequence has a convergent subsequence.

Definition (centroid)
The *centroid of an $n$-simplex* $\vec{x}_0 \cdots \vec{x}_n$ is the “average” of its vertices:

$$\frac{1}{n+1} \sum_{i=0}^{n} \vec{x}_i.$$
Nash’s Second Proof: Brouwer’s Fixed Point Theorem

Theorem (Brouwer’s Fixed Point Theorem)

Every continuous function \( f : \Delta_m \rightarrow \Delta_m \) has a fixed point, i.e., there exists some \( \vec{z} \in \Delta_m \) such that

\[
    f(\vec{z}) = \vec{z}.
\]

Proof: The proof proceeds in two parts:

1. **We construct a simplicial subdivision with a proper labeling function \( \mathcal{L} \) for \( \Delta_m \) so that Sperner’s lemma can be applied, yielding at least one completely labeled \( m \)-subsimplex in this subdivision.**

2. **Making such subdivisions finer and finer, we show that this \( m \)-subsimplex contracts to a fixed point of \( f \).**
Nash’s Second Proof: Brouwer’s Fixed Point Theorem

For the first part:

- Fix an $\varepsilon > 0$.

- Subdivide $\Delta_m$ simplicially such that the Euclidean distance between any two points $\vec{x} = (x_0, \ldots, x_m)$ and $\vec{y} = (y_0, \ldots, y_m)$ in $\mathbb{R}^{m+1}$ in the same $m$-subsimplex of this subdivision is at most $\varepsilon$:

$$\sqrt{(x_0 - y_0)^2 + \cdots + (x_m - y_m)^2} \leq \varepsilon.$$ 

- We here assume that it is always possible to find such a simplicial subdivision of $\Delta_m$, regardless of the dimension $m$, which is true, but not trivial to show.
Nash’s Second Proof: Brouwer’s Fixed Point Theorem

Now define a labeling function \( \mathcal{L} : V \rightarrow \{0, 1, \ldots, m\} \) as follows.

For each vertex \( \vec{v} \in V \) of the \( m \)-subsimplices in this subdivision, we choose a label \( \mathcal{L}(\vec{v}) \) from the set

\[
\sigma(\vec{v}) \cap \{i \mid f_i(\vec{v}) \leq v_i\},
\]

where

- \( \vec{v} = (v_0, v_1, \ldots, v_m) \) and \( f(\vec{v}) = (f_0(\vec{v}), f_1(\vec{v}), \ldots, f_m(\vec{v})) \) are points in \( \Delta_m \),
- \( \sigma(\vec{v}) = \{i \mid v_i > 0\} \) for \( \vec{v} = \sum_{i=0}^{m} v_i \cdot \vec{u}_i \), since \( \vec{u}_i \) is the \( i \)th unit vector in \( \mathbb{R}^{m+1} \).

That is, \( \mathcal{L}(\vec{v}) = i \) means that \( v_i > 0 \) and \( f_i(\vec{v}) \leq v_i \).
Nash’s Second Proof: Brouwer’s Fixed Point Theorem

- We have to show that this labeling function is well-defined, i.e., that
  \[ \sigma(\vec{v}) \cap \{ i \mid f_i(\vec{v}) \leq v_i \} \neq \emptyset. \]

- Intuitively, this is true because
  - \( \vec{v} \) and \( f(\vec{v}) \) are points in \( \Delta_m \), so their components each add up to one by definition of \( \Delta_m \).
  - Thus there exists an \( i \) such that \( f_i(\vec{v}) \leq v_i \), and this holds true even when restricted to \( \sigma(\vec{v}) \), so \( v_i > 0 \).
Nash’s Second Proof: Brouwer’s Fixed Point Theorem

1. Formally, for a contradiction suppose that $\sigma(\vec{v}) \cap \{i \mid f_i(\vec{v}) \leq v_i\} = \emptyset$.
2. Since $\vec{v}$ is a point in $\Delta_m$ (i.e., $\sum_{i=0}^{m} v_i = 1$) and $v_j > 0$ exactly if $j \in \sigma(\vec{v})$, we have
   \[ \sum_{j \in \sigma(\vec{v})} v_j = \sum_{i=0}^{m} v_i = 1. \]
3. From our assumption we know that $f_j(\vec{v}) > v_j$ for each $j \in \sigma(\vec{v})$, which implies
   \[ \sum_{j \in \sigma(\vec{v})} f_j(\vec{v}) > \sum_{j \in \sigma(\vec{v})} v_j = 1. \]  
   (2)
4. However, since $f(\vec{v})$ is a point in $\Delta_m$ as well, we have
   \[ \sum_{j \in \sigma(\vec{v})} f_j(\vec{v}) \leq \sum_{i=0}^{m} f_i(\vec{v}) = 1, \]
   contradicting (2). Thus $\mathcal{L}$ is well-defined.
Nash’s Second Proof: Brouwer’s Fixed Point Theorem

- By construction,

\[ L(\bar{\nu}) \in \sigma(\bar{\nu}) \]

for each \( \bar{\nu} \in V \).

- Thus \( L \) is also proper.

- By Sperner’s lemma, in this simplicial subdivision of \( \Delta_m \) there exists at least one \( m \)-subsimplex \( T^\epsilon_m \) that depends on \( \epsilon \) and is completely labeled by \( L \).
Nash’s Second Proof: Brouwer’s Fixed Point Theorem

In the second part of the proof:

- We will show that when $\varepsilon$ goes to zero, the resulting $m$-subs simplex
  \[ T_m^\varepsilon = \vec{t}_0 \cdots \vec{t}_m \]
  contracts to a fixed point of $f$.

- $T_m^\varepsilon$ is completely labeled; without loss of generality, we may assume that $\mathcal{L}(\vec{t}_i) = i$. (Otherwise, we simply rename the labels accordingly.)

- Furthermore, by construction of $\mathcal{L}$, we have
  \[ f_i(\vec{t}_i) \leq (\vec{t}_i)_i \quad (3) \]
  for each $i$, $0 \leq i \leq m$, where $(\vec{t}_i)_i$ denotes the $i$th component of $\vec{t}_i$. 
Nash’s Second Proof: Brouwer’s Fixed Point Theorem

- For $\varepsilon$ going to zero, we consider the infinite sequence of centroids in these completely labeled $m$-subsimplexes $T^\varepsilon_m$.

- Since $\Delta_m$ is compact, there exists a convergent subsequence with limit $\bar{z}$.

- The vertices of these $m$-subsimplexes $T^\varepsilon_m$ then move toward $\bar{z}$ with $\varepsilon$ going to zero, that is, $\vec{t}_i \xrightarrow[\varepsilon \to 0]{} \bar{z}$ for each $i$, $0 \leq i \leq m$. 

Nash’s Second Proof: Brouwer’s Fixed Point Theorem

- Since $f$ is continuous, it follows from (3) that
  \[ f_i(\vec{z}) \leq \vec{z}_i \]
  for each $i$, $0 \leq i \leq m$.

- This implies that $f(\vec{z}) = \vec{z}$, as desired, since otherwise, by the same argument as used in the first part of this proof, we would have
  \[ 1 = \sum_{i=0}^{m} f_i(\vec{z}) < \sum_{i=0}^{m} \vec{z}_i = 1, \]
  a contradiction. □
Nash’s Second Proof: Brouwer’s Fixed Point Theorem

**Reminder:** What we have shown is

**Theorem (Brouwer’s Fixed Point Theorem)**

*Every continuous function* \( f : \Delta_m \to \Delta_m \) *has a fixed point, i.e., there exists some* \( \vec{z} \in \Delta_m \) *such that*

\[
f(\vec{z}) = \vec{z}.
\]

**Corollary (Brouwer’s Fixed Point Theorem, applied to simplices)**

Let \( K = \prod_{j=1}^{k} \Delta_{m_j} \) be a simploptope (i.e., a Cartesian product of simplexes).

*Every continuous function* \( f : K \to K \) *has a fixed point.* **without proof**
Nash’s Second Proof

Theorem (Nash (1950; 1951))

For each noncooperative game in normal form with a finite number of players each having a finite set of strategies, there exists a Nash equilibrium in mixed strategies.

“A proof of this existence theorem based on Kakutani’s generalized fixed point theorem was published in Proc. Nat. Acad. Sci. U.S.A., 36, pp. 48–49. The proof given here is a considerable improvement over that earlier version and is based directly on the Brouwer theorem.”

John F. Nash (1951)
Nash’s Second Proof

Proof:

- Let $\vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_n) \in \Pi$ be a profile of mixed strategies with the expected gain functions $G_i(\vec{\pi})$.

- Let $\mathcal{S} = S_1 \times S_2 \times \cdots \times S_n$ be the underlying set of pure strategy profiles, where each $S_i$ is finite.

- For each pure strategy $s_j$ of each player $i$, let $G_i(\vec{\pi}_{-i}, s_j)$ be $i$’s gain when switching one-sidedly from $\pi_i$ to $s_j$.

- Define the functions

$$\varphi_{ij}(\vec{\pi}) = \max(0, G_i(\vec{\pi}_{-i}, s_j) - G_i(\vec{\pi}))$$

for each $i$ and $j$ with $1 \leq i \leq n$ and $1 \leq j \leq \|S_i\|$.
Nash’s Second Proof

Since the expected gain functions are continuous, so is each function $\varphi_{ij}$.

Now, define the function $f : \Pi \rightarrow \Pi$ by $f(\vec{\pi}) = \vec{\pi}' = (\pi'_1, \pi'_2, \ldots, \pi'_n)$, where the modifications $\pi'_i$ of $\pi_i$ are defined by

$$\pi'_i(s_j) = \frac{\pi_i(s_j) + \varphi_{ij}(\vec{\pi})}{\sum_{s_k \in S_i} (\pi_i(s_k) + \varphi_{ik}(\vec{\pi}))} = \frac{\pi_i(s_j) + \varphi_{ij}(\vec{\pi})}{1 + \sum_{s_k \in S_i} \varphi_{ik}(\vec{\pi})}. \hspace{1cm} (4)$$

Intuitively, $\vec{\pi}'$ puts more probability weight $\pi'_i$ on those pure strategies of each player $i$ that are “better” responses to the other players’ mixed strategies $\vec{\pi} _{\neg i}$.
Nash’s Second Proof

- Since every function $\varphi_{ij}$ is continuous, so is $f$.

- Since $\Pi$, as a simplotope, is convex and compact and since $f : \Pi \to \Pi$ is continuous, $f$ has at least one fixed point by Brouwer’s fixed point theorem for simplotope.

- It remains to show that $\bar{\pi}$ is a Nash equilibrium in mixed strategies if and only if $f(\bar{\pi}) = \bar{\pi}$.

- From left to right, if $\bar{\pi}$ is a Nash equilibrium in mixed strategies, we have $\varphi_{ij}(\bar{\pi}) = 0$ for all $i$ and $j$.

- Hence, $f(\bar{\pi}) = \bar{\pi}' = \bar{\pi}$, so $\bar{\pi}$ is a fixed point.
Nash’s Second Proof

- From right to left, suppose $f(\vec{\pi}) = \vec{\pi}$.

- Consider player $i$.

- Since $G_i$ is linear in its $i$th component, there exists at least one pure strategy $s_j$ in the support of $\pi_i$ (i.e., $\pi_i(s_j) > 0$) such that

$$G_i(\vec{\pi}_{-i}, s_j) \leq G_i(\vec{\pi}).$$

In other words, by linearity of $G_i$ in its $i$th component, we see that the situation where for each pure strategy $s_k$ (in the support of $\pi_i$) it holds that $G_i(\vec{\pi}_{-i}, s_k) > G_i(\vec{\pi})$ is impossible.
Nash’s Second Proof

- By definition of $\varphi_{ij}$, it follows that $\varphi_{ij}(\vec{\pi}_{-i}, s_j) = 0$.

- Since $f(\vec{\pi}) = \vec{\pi}$, this enforces that $\pi'_i(s_j) = \pi_i(s_j)$.

That is, the enumerator in (4) simplifies to $\pi_i(s_j)$ (due to $\varphi_{ij}(\vec{\pi}_{-i}, s_j) = 0$) and it is positive because $s_j$ is in the support of $\pi_i$.

- This implies, by simple arithmetic, that the denominator in (4) must be one. Consequently,

$$\sum_{s_k \in S_i} \varphi_{ik}(\vec{\pi}) = 0.$$
Nash’s Second Proof

- This holds true for each player $i$ and, in effect, for all $i$ and $k$, we have $\varphi_{ik}(\vec{\pi}) = 0$.

- That is, no player $i$ can increase her gain by moving one-sidedly from her mixed strategy $\pi_i$ to some pure strategy.

- However, since we know that
  \[
  \max_{\pi_i' \in \Pi_i} G_i(\vec{\pi}_{-1}, \pi'_i) = \max_{s_j \in \Pi_i} G_i(\vec{\pi}_{-1}, s_j)
  \]
  from the theorem on slide 7, this means that $\vec{\pi}$ is a Nash equilibrium in mixed strategies.
Nash’s Theorem

Nash has won numerous prizes and awards and has been loaded with the highest academic honors for his superb insights and pathbreaking ideas, such as

- the 1978 *John von Neumann Theory Prize* for inventing the equilibria in noncooperative games named after him and

- the 1994 *Nobel Prize in Economics* (jointly with the game theoreticians Reinhard Selten and John Harsanyi).

“That’s trivial, you know. That’s just a fixed point theorem.”

*John von Neumann (1950)*