Algorithmic Game Theory

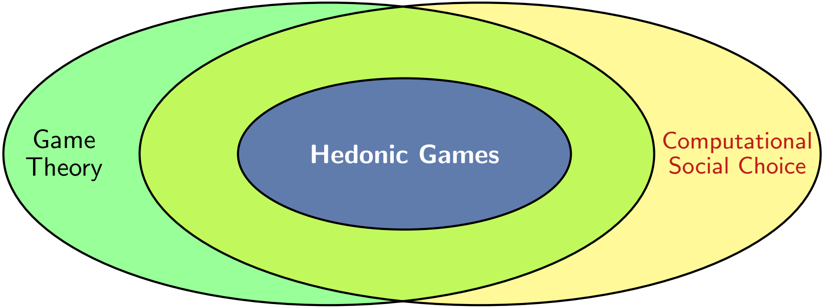
Algorithmische Spieltheorie

Hedonic Games

Wintersemester 2022/2023

Dozent: Prof. Dr. J. Rothe

Hedonic Games



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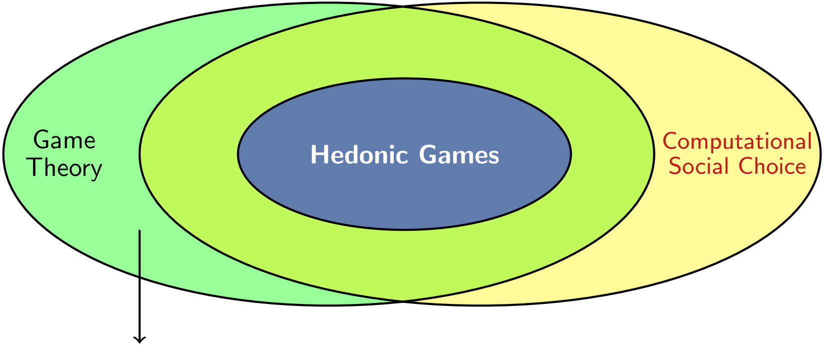

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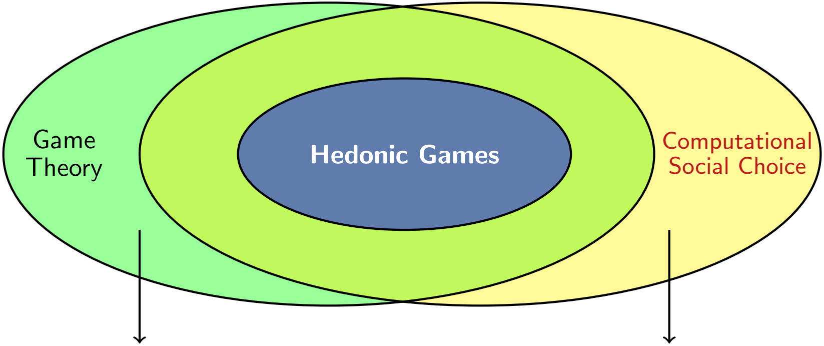

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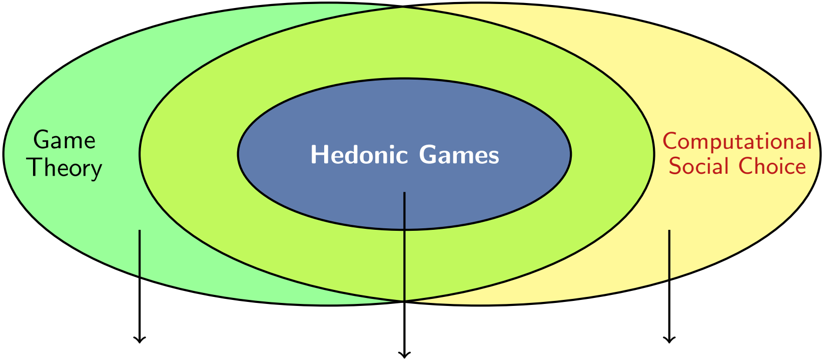

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Players, coalitions, and preferences. Every player wants to join her most preferred coalition.

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Definition (Drèze & Greenberg (1980))

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  - a finite *set of players* $N = \{1, \ldots, n\}$ and
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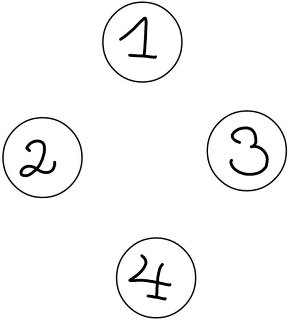
- $\Gamma(i)$ denotes the coalition of Γ that contains player $i \in N$.

Example of a Hedonic Game



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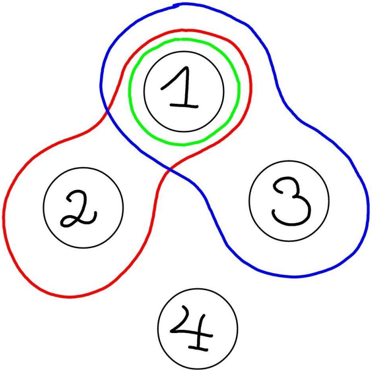

preferences: $\{1,3\} \succ_1 \{1,2\} \succ_1 \{1\} \succ_1 \cdots$

Compact Representations of Hedonic Games

- Individually rational hedonic games (Ballester, GEB 2004):

  Players list their individually rational coalitions only; those that they weakly prefer to being alone.

- Anonymous hedonic games (Ballester, GEB 2004):

  Players are indifferent about coalitions of equal size.

- Singleton encoding of hedonic games (Cechlárová & Romero-Medina, IJGT 2001):

  Every player ranks single players only rather than coalitions of players.

- Hedonic coalition nets (Elkind & Wooldridge, AAMAS 2009):

  a rule-based representation for hedonic games that is universally expressive.

Compact Representations of Hedonic Games

- **Additive hedonic games** (Aziz, Brandt, & Seedig, AIJ 2013):
  Each player i has a preference function $v_i : N \to \mathbb{R}$ such that for all coalitions $C, D \subseteq N$, we have $C \succeq_i D$ if and only if

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- Fractional hedonic games (Aziz, Brandt, & Harrenstein, AAMAS 2014):
  Every player assigns some value to each other player and 0 to herself; player i's utility of a coalition is her average value assigned to the members of this coalition; and for all coalitions $C, D \subseteq N$, we have $C \succeq_i D$ if and only if i's utility of C is at least as high as her utility of D.

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- **Friend-oriented** and **enemy-oriented** encoding (Dimitrov, Borm, Hendrickx, & Sung, SCW 2006).

Friends and Enemies

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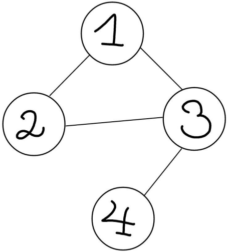

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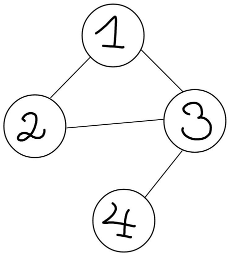

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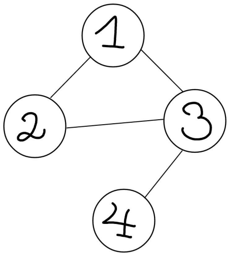
Here, only *symmetric* friendship relations matter.



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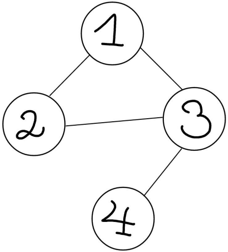

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Friend- and Enemy-Oriented Preferences Are Additive

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In particular, enemy-oriented preferences are additive:

- Set $v_i(j) = 1$ if i considers j a friend.
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Similarly, friend-oriented preferences are additive:

- Set $v_i(j) = \|N\|$ if i considers j a friend.
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Definition (Drèze & Greenberg (1980))

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A nonempty coalition $C \subseteq N$

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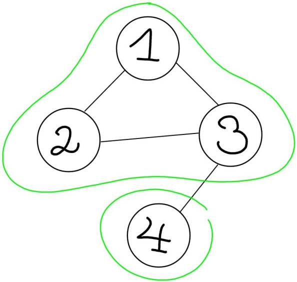

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Five players $0, 1, 2, 3, 4$ are sitting (in this order) around a round table.
Every player i (modulo 5 throughout) assigns

- a value $v_i(i+1) = 1$ to the player to his right,
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If a partition contains two single-player coalitions $\{i\}$ and $\{i+1\}$, then it would be blocked by $\{i, i+1\}$.
In the only remaining case for a potentially core stable partition, there is one single-player coalition $\{i\}$ and two two-player coalitions $\{i+1, i+2\}$ and $\{i+3, i+4\}$; this partition is blocked by $\{i, i+1\}$.

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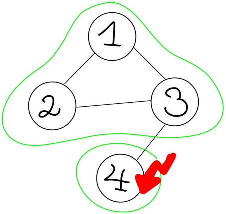

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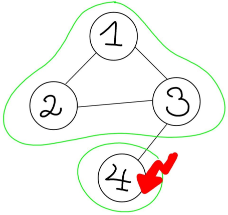

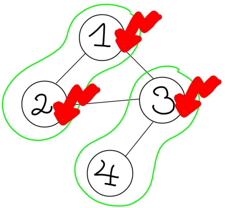

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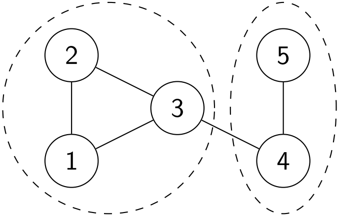

The partition Π into cliques indicated by the dashed lines is wonderfully stable since every vertex is in a clique of maximum size.

Wonderful Stability vs Strict Core Stability

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Lemma

Let $G = (V, E)$ be the graph representation of the enemy-oriented hedonic game $\mathscr{G} = (N, \succeq)$. Let Π be a partition of V and Γ the corresponding coalition structure in $\mathscr{G}$.

1. *If Π is a wonderfully stable partition for G, then Γ is a strictly core stable coalition structure for $\mathscr{G}$.*
2. *If there is an integer $c \in \mathbb{N}$ such that $\omega_G(v) = c$ for all vertices $v \in V$ and Γ is a strictly core stable coalition structure for $\mathscr{G}$, then Π is a wonderfully stable partition for G.*

Challenge: Wonderful Stability

Open Problem (Woeginger (SOFSEM 2013))

Pinpoint the computational complexity of deciding whether a given undirected graph has a wonderfully stable partition.

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By definition, $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{DP} \subseteq \mathrm{P}^{\mathrm{NP}[\log]} \subseteq \Sigma_2^p$.
$\mathrm{P}^{\mathrm{NP}[\log]}$-completeness is known, e.g., for the winner problems in

- Dodgson (Hemaspaandra, Hemaspaandra, & Rothe, JACM 1997),
- Young (Rothe, Spakowski, & Vogel, TOCS 2003), and
- Kemeny elections (Hemaspaandra, Spakowski, & Vogel, TCS 2005).

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Remark (Woeginger (SOFSEM 2013))

Suppose the preferences can be evaluated in polynomial time, i.e.,
$\{(i, C, D) \mid i \in N \text{ and } C, D \subseteq N \text{ and } C \succeq_i D\} \in \mathrm{P}$.

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Observation (Woeginger (SOFSEM 2013))

- $\mathrm{CSPV} \in \mathrm{P} \Rightarrow \mathrm{CSPE} \in \mathrm{NP}$.
- *However, hardness of* CSPV *does not necessarily imply hardness of* CSPE.

Core Stability under Enemy-Oriented Preferences:

Theorem (Dimitrov, Borm, Hendrickx, & Sung (SCW 2006))

Under enemy-oriented preferences, there always exists a core stable partition; hence $\mathrm{CSPE} \in \mathrm{P}$.

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Open Problem (Woeginger (SOFSEM 2013))

Pinpoint the computational complexity of deciding whether a given hedonic game with enemy-oriented preferences has a strictly core stable partition.

Core Stability under Additive Preferences

Corollary (Sung & Dimitrov (ORL 2007 and EJOR 2010))

For additive preferences, CSPV *is* NP*-complete and* CSPE *is* NP*-hard.*

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Theorem (Woeginger (MSS 2013))

In additive hedonic games, CSPE *is* Σ_2^p*-complete.*

Wonderfully Stable Partition Problems

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Again, WSPV and WSPE are closely related:

- $(G, \Pi) \in \text{WSPV} \iff (\exists \text{ clique } C)[C \text{ blocks } \Pi]$;
- $G \in \text{WSPE} \iff (\exists \Pi)(\forall \text{ cliques } C)[\neg(C \text{ blocks } \Pi)]$.

So $\text{WSPV} \in \text{NP}$ and $\text{WSPE} \in \Sigma_2^p$.

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NP-hardness is shown via a reduction from the NP-complete problem

Clique	
Given:	An undirected graph $G = (V, E)$ and a positive integer k.
Question:	Does G have a clique of size at least k?

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Given an instance $(G = (V, E), k)$ of Clique, we construct the following graph $G' = (V', E')$:

- The vertex set V' is obtained from V by adding, for each $v \in V$, $k-2$ vertices.
- We connect each of the $k-2$ new vertices and v to form a clique of size $k-1$, for each $v \in V$.
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This can obviously be achieved in polynomial time.

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Furthermore, $\|\Pi(v)\| = k-1$, for each $v \in V'$. Thus, there is no blocking clique for Π in G'. ❑

Strictly Core Stable Coalition Structures

STRICTLY CORE STABLE COALITION STRUCTURE (SCSCS)	
Given:	A hedonic game $(N, \succeq)$ with enemy-oriented preferences.
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Theorem

For $k \geq 3$, k-WSPE (and thus k-SCSCS) is NP-complete.

Challenge: Are WSPE and SCSCS Θ_2^p-Hard?

Open Problem (Woeginger (SOFSEM 2013))

1. **Pinpoint the computational complexity** *of deciding whether a given enemy-oriented hedonic game has a strictly core stable partition.*

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- It is also possible that both problems belong to DP (and so would be DP-complete) or are complete for another class.