On Computing the Smallest Four-Coloring of Planar Graphs and Non-Self-Reducible Sets in P ⋆

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Abstract

We show that computing the lexicographically first four-coloring for planar graphs is Δ_2^p-hard. This result optimally improves upon a result of Khuller and Vazirani who prove this problem NP-hard, and conclude that it is not self-reducible in the sense of Schnorr, assuming $\mathrm{P} \neq \mathrm{NP}$. We discuss this application to non-self-reducibility and provide a general related result. We also discuss when raising a problem's NP-hardness lower bound to Δ_2^p-hardness can be valuable.

Key words: computational complexity, graph colorability, self-reducibility

1 Introduction

Khuller and Vazirani [13] proved that computing the lexicographically smallest solutions of `Pl-4-Color` instances is NP-hard, where `Pl-4-Color` denotes the pla-

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nar graph four-colorability problem. They conclude that, unless $\mathrm{P} = \mathrm{NP}$, the polynomial-time decidable problem `Pl-4-Color` is not self-reducible in the sense of Schnorr [22,23]. Noting that their result appears to be the first such non-self-reducibility result for problems in P, they proposed as an interesting task to find other problems in P that are not self-reducible under some plausible assumption.

We raise Khuller and Vazirani's NP-hardness lower bound for computing the lexicographically smallest four-coloring of a planar graph to Δ_2^p-hardness. Our result is optimal, since this problem belongs to (the function analog of) the class Δ_2^p.

The class $\Delta_2^p = \mathrm{P}^{\mathrm{NP}}$, which belongs to the second level of the polynomial hierarchy [17,26], contains exactly the problems solvable in deterministic polynomial time with an NP oracle. Papadimitriou [18] proved that `Unique-Optimal-Traveling-Salesperson` is Δ_2^p-complete, and Krentel [15] and Wagner [28] established many more Δ_2^p-completeness results, including the result that the problem `Odd-Max-SAT` is Δ_2^p-complete. The complexity of colorability problems has been studied in a number of papers, see, e.g., [1,2,25,6,28,13,20].

As mentioned above, if for some problem in P computing the lexicographically smallest solution is hard, then the problem itself cannot be self-reducible in the sense of Schnorr [22,23], unless $\mathrm{P} = \mathrm{NP}$. We discuss this application to non-self-reducibility and provide a general related result. In particular, it follows from this result that even a set as simple as Σ^* has representations in which it is not self-reducible in Schnorr's sense, unless $\mathrm{P} = \mathrm{NP}$. Finally, we conclude this paper with a discussion of when raising a problem's NP-hardness lower bound to Δ_2^p-hardness can be valuable, and pose some open questions.

2 Computing the Smallest Four-Coloring of a Planar Graph

Appel and Haken [1,2] showed that every planar graph can be colored with no more than four colors, thus solving the famous Four Color Conjecture in the affirmative. In contrast, for each $k \geq 4$, computing the lexicographically first k-coloring of a planar graph is hard: Khuller and Vazirani [13] established an NP-hardness lower bound for this problem. We raise their lower bound to Δ_2^p-hardness. Since the lexicographically smallest k-coloring of a planar graph can be computed in (the function analog of) Δ_2^p, this improved lower bound is optimal.

Definition 2.1 *Let $k > 1$, and let $0, 1, \ldots, k-1$ represent k colors.*

- *A* k-coloring *of an undirected graph $G = (V, E)$ is a mapping $\psi_G : V \rightarrow \{0, 1, \ldots, k-1\}$.*
- *A k-coloring ψ_G is said to be* legal *if and only if for each edge $\{u, v\} \in E$, $\psi_G(u) \neq \psi_G(v)$.*

- *A graph G is said to be* k-colorable *if and only if there exists a legal k-coloring of G.*
- *Let* `Pl`-k-`Color` *denote the planar graph k-colorability problem.*

Stockmeyer [25] proved that `Pl-3-Color` is NP-complete, see also Garey et al. [6]. By Appel and Haken's above-mentioned result, every planar graph is four-colorable. Thus, `Pl`-k-`Color` is in P for each $k \geq 4$.

Definition 2.2 (Khuller and Vazirani [13]) *Let $k > 1$, and let the vertex set of a given undirected graph $G = (V, E)$ with n vertices be ordered as $V = \{v_1, v_2, \ldots, v_n\}$. Then, every k-coloring ψ_G of G can be represented by a string ψ_G in $\{0, 1, \ldots, k-1\}^n$, which is defined by $\psi_G = \psi_G(v_1)\psi_G(v_2)\cdots\psi_G(v_n)$.*

Define the lexicographically smallest (legal) k-coloring *by*

$$\mathrm{LF}_{\texttt{Pl-}k\texttt{-Color}}(G) = \min\{\psi_G \mid \psi_G \textit{ is a legal } k\textit{-coloring of } G\},$$

if $G \in$ `Pl`-k-`Color`, where the minimum is taken with respect to the lexicographic ordering of strings, and define $\mathrm{LF}_{\texttt{Pl-}k\texttt{-Color}}(G) = 10^n$ if $G \notin$ `Pl`-k-`Color`.

We now prove our main result.

Theorem 2.3 *Computing the lexicographically smallest k-coloring for planar graphs is Δ_2^p-hard for any $k \geq 4$.*

Proof. For simplicity, we show this claim only for $k = 4$. Let ρ_4 be the reduction of Khuller and Vazirani [13, Theorem 3.1]. Recall that ρ_4 maps a given planar graph $G = (V, E)$, whose vertices are ordered as $V = \{v_1, v_2, \ldots, v_m\}$, to the planar graph $H = (U, F)$ defined as follows:

- The vertex set of H is ordered as $U = \{u_1, u_2, \ldots, u_{2m}\}$, where u_i is a new vertex and $u_{m+i} = v_i$ is an old vertex for each i, $1 \leq i \leq m$.
- The edge set of H is defined by $F = E \cup \{\{u_i, u_{m+i}\} \mid 1 \leq i \leq m\}$.

It follows immediately from this construction that

(1) $\quad G \in \texttt{Pl-3-Color} \iff \mathrm{LF}_{\texttt{Pl-4-Color}}(\rho_4(G)) \in \{0^m w \mid w \in \{1, 2, 3\}^m\}$,

that is, "$G \in$ `Pl-3-Color`?" can be decided by looking at the first m bits of $\mathrm{LF}_{\texttt{Pl-4-Color}}(H)$.

We give a reduction from the problem `Odd-Min-SAT`, which is defined to be the set of all boolean formulas $F = F(x_1, x_2, \ldots, x_n)$ in conjunctive normal form for which, assuming F *is* satisfiable, the lexicographically smallest satisfying assignment $\alpha : \{x_1, x_2, \ldots, x_n\} \to \{1, 2\}$ is "odd," i.e., for which $\alpha(x_n) = 1$. Here, "1" represents "true," and "2" represents "false."

It is well known that Odd-Min-SAT is Δ_2^p-complete; Krentel [15] and also Wagner [28] proved the corresponding claim for the dual problem Odd-Max-SAT.

Let $F = F(x_1, x_2, \ldots, x_n)$ be any given boolean formula. Without loss of generality, we may assume that F is in conjunctive normal form with exactly three literals per clause. Assume that F has z clauses. Let σ be the Stockmeyer reduction from 3-SAT to Pl-3-Color, see Stockmeyer [25] and also Garey et al. [6]. This reduction σ, on input F, yields a graph $G = (V, E)$ with $m > n$ vertices, where $m = m(F)$ depends on the number n of variables, the number z of clauses, and the structure of F. Note that F's structure induces a certain number of "crossovers" of edges to guarantee the planarity of G; see [6,25] for details.

Order the vertex set of G as $V = \{v_1, v_2, \ldots, v_m\}$ such that

(a) for each i, $1 \leq i \leq n$, v_i represents the variable x_i, and
(b) for each i, $n < i \leq m$, v_i represents some other vertex of G.

Note that G is a planar graph satisfying the following properties:

(i) F is satisfiable if and only if G is 3-colorable, using the colors 1, 2, and 3.
(ii) Every satisfying assignment α of F corresponds to a 3-coloring ψ_α of G such that for each i, $1 \leq i \leq n$, $\psi_\alpha(v_i) = \alpha(v_i) \in \{1, 2\}$. The color 3 is used for the other vertices of G.

Now apply the reduction ρ_4 of Khuller and Vazirani to G and obtain a planar graph $H = \rho_4(G) = \rho_4(\sigma(F))$ that satisfies Equation (1) as described above. It follows immediately from this construction and from Equation (1) that

$$\begin{aligned}&F \in \texttt{Odd-Min-SAT} \Longleftrightarrow \\ &\mathrm{LF}_{\texttt{Pl-4-Color}}(\rho_4(\sigma(F))) \in \{0^m w 1 y \mid w \in \{1,2\}^{n-1} \text{ and } y \in \{1,2,3\}^{m-n}\},\end{aligned}$$

that is, "$F \in$ Odd-Min-SAT?" can be decided by looking at the first m bits and at the $(m+n)$th bit of $\mathrm{LF}_{\texttt{Pl-4-Color}}(H)$.

For $k > 4$, the claim of the theorem follows from an analogous argument that employs in place of ρ_4 the appropriate reduction ρ_k from [13, Thm. 3.2]. ❑

3 Non-Self-Reducible Sets in P

From their NP-hardness lower bound for computing the lexicographically first four-coloring of planar graphs, Khuller and Vazirani [13] conclude that, unless $\mathrm{P} = \mathrm{NP}$, the polynomial-time decidable problem Pl-k-Color is not self-reducible for $k \geq 4$.

The type of (functional) self-reducibility used by Khuller and Vazirani is due to Schnorr [22,23], see also [5]. For more background on self-reducibility, see, e.g., [24,12,21].

Definition 3.1 (Schnorr [22,23])

- *Let Σ and Γ be alphabets with at least two symbols each. Instances of problems are encoded over Σ, and solutions of problems are encoded over Γ. For any set $B \subseteq \Sigma^* \times \Gamma^*$ and any polynomial p, the* p-projection of B *is defined to be the set*

$$\mathrm{proj}_p(B) = \{x \in \Sigma^* \mid (\exists y \in \Gamma^*)\, [|y| \leq p(|x|) \textit{ and } (x, y) \in B]\}.$$

  If $A = \mathrm{proj}_p(B)$, we say A has the representation (B, p).
- *A partial order $\leq$ on Σ^* is* polynomially well-founded and length-bounded *if and only if there exists a polynomial q such that*
  (a) *every $\leq$-decreasing chain with maximum element x has at most $q(|x|)$ elements, and*
  (b) *for all strings $x, y \in \Sigma^*$, $x < y$ implies $|x| \leq q(|y|)$.*
- *Let $A = \mathrm{proj}_p(B)$ for some set $B \subseteq \Sigma^* \times \Gamma^*$ and some polynomial p. The projection A is said to be* self-reducible with respect to its representation (B, p) *if and only if there exist a polynomial-time computable function g mapping from $\Sigma^* \times \Gamma$ to Σ^* and a polynomially well-founded and length-bounded partial order $\leq$ such that for all strings $x \in \Sigma^*$, for all strings $y \in \Gamma^*$, and for all symbols $\gamma \in \Gamma$,*
  (i) $g(x, \gamma) < x$, *and*
  (ii) $(x, \gamma y) \in B \iff (g(x, \gamma), y) \in B$.
  If the representation (B, p) of $A = \mathrm{proj}_p(B)$ is clear from the context, we omit the phrase "with respect to its representation (B, p)."

We mention in passing that various other important types of self-reducibility have been studied, such as the self-reducibility defined by Meyer and Paterson [16] and the disjunctive self-reducibility studied by Selman [24], Ko [14], and many others. We refer the reader to the excellent survey by Joseph and Young [12] for an overview and for pointers to the literature. Note that, in sharp contrast with Schnorr's self-reducibility, every set in P is self-reducible in the sense of Meyer and Paterson [16], Ko [14], and Selman [24].

Definition 3.2 *Let $\Sigma = \{0, 1\}$. Given any set A in NP with $A \subseteq \Sigma^*$, there is an associated set $B_A \subseteq \Sigma^* \times \Sigma^*$ and an associated polynomial p_A such that B_A is in P and $A = \mathrm{proj}_{p_A}(B_A)$.*

- *For any $x \in \Sigma^*$, define the* set of solutions for x with respect to B_A and p_A *by*

$$\mathrm{Sol}_{(B_A, p_A)}(x) = \{y \in \Sigma^* \mid |y| \leq p_A(|x|) \textit{ and } (x, y) \in B_A\}.$$

  Note that $x \in A$ if and only if $\mathrm{Sol}_{(B_A, p_A)}(x) \neq \emptyset$.

- *For any $x \in \Sigma^*$, define the* lexicographically first solution with respect to B_A and p_A *by*

$$\mathrm{LF}_{(B_A,p_A)}(x) = \begin{cases} \min \mathrm{Sol}_{(B_A,p_A)}(x) & \textit{if } x \in A \\ \mathrm{bin}(2^{p(|x|)}) & \textit{otherwise,} \end{cases}$$

  where the minimum is taken with respect to the lexicographic ordering of Σ^*, and $\mathrm{bin}(n)$ *denotes the binary representation of the integer n without leading zeros.*

  If the representation (B_A, p_A) of $A = \mathrm{proj}_{p_A}(B_A)$ is clear from the context, we use $\mathrm{Sol}_A(x)$ *and* $\mathrm{LF}_A(x)$ *as shorthands for, respectively,* $\mathrm{Sol}_{(B_A,p_A)}(x)$ *and* $\mathrm{LF}_{(B_A,p_A)}(x)$.

It is well known that if A is self-reducible then LF_A can be computed in polynomial time by prefix search, via suitable queries to the oracle A. Moreover, if A is in P then LF_A can even be computed in polynomial time without any oracle queries. It follows that if A is in P yet computing LF_A is NP-hard then A cannot be self-reducible, assuming $\mathrm{P} \neq \mathrm{NP}$.

Khuller and Vazirani [13] propose to find polynomial-time decidable problems other than `Pl-4-Color` that are non-self-reducible, under the assumption $\mathrm{P} \neq \mathrm{NP}$. Theorem 3.5 below provides a general result showing that it is almost trivial to find such problems: For any NP problem A for which LF_A is hard to compute, one can define a P-decidable version D of A such that LF_D is still hard to compute; hence, D is not self-reducible, assuming $\mathrm{P} \neq \mathrm{NP}$.

To formulate this result, we now define the functional many-one reducibility that was introduced by Vollmer [27] as a potentially stricter reducibility notion than Krentel's metric reducibility [15]. We also define the function class $\min \cdot \mathrm{P}$ that was introduced by Hempel and Wechsung [11].

Definition 3.3 (Vollmer [27]) *Let f and h be functions from Σ^* to Σ^*.*

- *We say that f is* polynomial-time functionally many-one reducible *to h (in symbols, $f \leq_{\mathrm{m}}^{\mathrm{FP}} h$) if and only if there exists a polynomial-time computable function g such that for all $x \in \Sigma^*$, $f(x) = h(g(x))$.*
- *We say that h is* $\leq_{\mathrm{m}}^{\mathrm{FP}}$-hard *for a function class $\mathcal{C}$ if and only if for every $f \in \mathcal{C}$, $f \leq_{\mathrm{m}}^{\mathrm{FP}} h$.*
- *We say that h is* $\leq_{\mathrm{m}}^{\mathrm{FP}}$-complete *for $\mathcal{C}$ if and only if $h \in \mathcal{C}$ and h is $\leq_{\mathrm{m}}^{\mathrm{FP}}$-hard.*

Definition 3.4 (Hempel and Wechsung [11]) *Define the class $\min \cdot \mathrm{P}$ to consist of all functions f for which there exist a set $A \in \mathrm{P}$ and a polynomial p such that for all $x \in \Sigma^*$,*

$$f(x) = \min\{y \in \{0,1\}^* \mid |y| \leq p(|x|) \textit{ and } (x,y) \in A\},$$

where $(\cdot,\cdot) : \Sigma^* \times \Sigma^* \to \Sigma^*$ is a standard pairing function. If the set over which the minimum is taken is empty, define by convention $f(x) = \mathrm{bin}(2^{p(|x|)})$.

Note that $\mathrm{LF}_A = \mathrm{LF}_{(B,p)}$ is in $\min\cdot\mathrm{P}$ for every NP set A and for every representation of A as a p-projection $A = \mathrm{proj}_p(B)$ of some suitable set $B \in \mathrm{P}$ and polynomial p.

Theorem 3.5 *Let $\Sigma = \{0,1\}$, let $A \subseteq \Sigma^*$ be any set in NP, let $B \subseteq \Sigma^* \times \Sigma^*$ and $D \subseteq \Sigma^*$ be sets in P, and let p be a polynomial such that $A = \mathrm{proj}_p(B) \subseteq D$ and LF_A is $\leq_{\mathrm{m}}^{\mathrm{FP}}$-complete for $\min\cdot\mathrm{P}$. Then, there exist a set $C \subseteq \Sigma^* \times \Sigma^*$ in P and a polynomial q such that $D = \mathrm{proj}_q(C)$ and computing LF_D is Δ_2^p-hard.*

Hence, D is not self-reducible with respect to (C,q), assuming $\mathrm{P} \neq \mathrm{NP}$.

Proof. Define the set

$$C = (B \cap \{(x,y) \mid |y| \leq p(|x|)\}) \cup \{(x, \mathrm{bin}(2^{p(|x|)})) \mid x \in D\},$$

and let $q(n) = p(n)+1$ for all n. Note that $C \in \mathrm{P}$ and $D = \mathrm{proj}_q(C)$. It also follows that $\mathrm{LF}_A(x) \equiv \mathrm{LF}_D(x) \bmod 2$ if $x \in D$, and $\mathrm{LF}_A(x) \equiv \mathrm{LF}_D(x) \equiv 0 \bmod 2$ if $x \notin D$. Thus, for all x, $\mathrm{LF}_A(x) \equiv \mathrm{LF}_D(x) \bmod 2$.

We now show that computing LF_D is as hard as deciding the Δ_2^p-complete problem `Odd-Min-SAT`, which was defined in Section 2. Since LF_A is $\leq_{\mathrm{m}}^{\mathrm{FP}}$-complete for $\min\cdot\mathrm{P}$, we have $\mathrm{LF}_{\mathtt{SAT}}(F) = \mathrm{LF}_A(t(F))$ for some polynomial-time computable function t. Hence,

$$\begin{aligned} F \in \texttt{Odd-Min-SAT} &\iff \mathrm{LF}_{\mathtt{SAT}}(F) \equiv 1 \bmod 2 \\ &\iff \mathrm{LF}_A(t(F)) \equiv 1 \bmod 2 \\ &\iff \mathrm{LF}_D(t(F)) \equiv 1 \bmod 2. \end{aligned}$$

Thus, one can decide whether or not F belongs to `Odd-Min-SAT` by looking at the last bit of $\mathrm{LF}_D(t(F))$. ❑

Corollary 3.6 *If $\mathrm{P} \neq \mathrm{NP}$ then Σ^* has representations in which it is not self-reducible.*

Proof. Replacing the set D of Theorem 3.5 by Σ^*, it is clear that the hypothesis of the theorem can be satisfied by suitably choosing A, B, and p.[3] It follows that Σ^*, unconditionally, has representations in which it is not self-reducible in the sense of Schnorr, unless $\mathrm{P} = \mathrm{NP}$. ❑

[3] Concrete examples of A, B, and p are given in Section 4, where we assume that the problem A (e.g., $A =$ `P-SAT`) as well as the set of solutions for instances of A are suitably encoded over Σ. Thus $A \subseteq \Sigma^* = D$ and $B \subseteq \Sigma^* \times \Sigma^*$.

4 Conclusions and Open Questions

In Theorem 2.3, we strengthened Khuller and Vazirani's [13] lower bound for computing the lexicographically first four-coloring for planar graphs from NP-hardness to Δ_2^p-hardness. The non-self-reducibility of the `Pl-4-Color` problem follows immediately from these lower bounds.

Since $\mathrm{P} \neq \mathrm{NP}$ is equivalent to $\mathrm{P} \neq \Delta_2^p$, our strengthened lower bound for computing $\mathrm{LF}_{\texttt{Pl-}k\texttt{-Color}}(G)$ from Theorem 2.3 does not give strengthened evidence regarding the non-self-reducibility of Pl-k-Color. However, raising a problem's lower bound so as to match its upper bound is important in its own right.

In addition, we now give another reason of why this improved lower bound may be valuable, by re-iterating a point that has first been made by Hemaspaandra et al. [10], who discuss the issue of why and when it may be valuable to raise a problem's NP-hardness lower bound to Θ_2^p-hardness, with regard to other computational models such as one-sided error randomized polynomial time or unambiguous polynomial time. Just as Δ_2^p, the class $\Theta_2^p = \mathrm{P}^{\mathrm{NP}[\log]}$ belongs to the second level of the polynomial hierarchy; note that $\mathrm{NP} \subseteq \Theta_2^p \subseteq \Delta_2^p \subseteq \mathrm{NP}^{\mathrm{NP}}$. Rephrasing for the class Δ_2^p a question that Hemaspaandra et al. [10] studied for Θ_2^p, we ask: Given a complexity class $\mathcal{C}$, is it currently known that $\mathrm{NP} \subseteq \mathcal{C}$ if and only if $\Delta_2^p \subseteq \mathcal{C}$? The answer to this question is the key to the issue of whether or not raising an NP-hardness lower bound to Δ_2^p-hardness indeed may have some value: If the answer is yes, then the raised lower bound is worthless with respect to the computational model captured by $\mathcal{C}$; if the answer is no, then the raised lower bound may potentially be valuable for $\mathcal{C}$.

Table 1 provides some answers to the above question for various classes $\mathcal{C}$ with respect to Δ_2^p-hardness. [4] In most cases (namely, for $\mathcal{C}$ being one of P, BPP, RP, ZPP, and UP), the answer for Δ_2^p is the same as for Θ_2^p, by essentially the argument given in [10]. However, if $\mathcal{C}$ is either PP or $\mathrm{C_{=}P}$, the answer for Δ_2^p differs from that for Θ_2^p. In particular, since $\mathrm{NP} \subseteq \mathrm{PP}$ and $\Theta_2^p \subseteq \mathrm{PP}$ (see [4]), raising NP-hardess to Θ_2^p-hardness is worthless for PP. In contrast, raising NP-hardess to Δ_2^p-hardness may potentially be valuable for PP, since it is not known whether $\Delta_2^p \subseteq \mathrm{PP}$; there is even a relativized counterexample for $\Delta_2^p \subseteq \mathrm{PP}$ (see [3]). And for $\mathcal{C} = \mathrm{C_{=}P}$, the closure properties of $\mathrm{C_{=}P}$ imply that raising NP-hardess to Θ_2^p-hardness is worthless (see [10]), but do not seem to suffice in any obvious way to yield the same claim for Δ_2^p-hardness. Again, there are relativized counterexamples for the inclusion $\Delta_2^p \subseteq \mathrm{C_{=}P}$—and even relativizations that separate the entire polynomial hierarchy from $\mathrm{C_{=}P}$ with immunity, see [19]. However, unlike for PP, these relativized separations do not give us any more insight regarding the value of raising NP-hardness to Δ_2^p-hardness for $\mathrm{C_{=}P}$, since $\mathrm{NP} \subseteq \mathrm{C_{=}P}$ is not known to hold either.

[4] For the definitions of the classes $\mathcal{C}$ discussed in Table 1, the reader is referred to [10] and the original literature cited therein.

Computational Model	$\mathcal{C}$	$\mathrm{NP} \subseteq \mathcal{C} \iff \Delta_2^p \subseteq \mathcal{C}$?	Reference
Deterministic Polynomial Time	P	yes	[17,26]
Probabilistic Polynomial Time	PP	not known	but see [3]
Bounded-Error Probabilistic Polynomial Time	BPP	yes	[7,29]
Zero-Error Probabilistic Polynomial Time	ZPP	yes	[7]
Random Polynomial Time	RP	not known	but see [10]
Exact Counting	$\mathrm{C}_=\mathrm{P}$	not known	but see [10]
Unambiguous Polynomial Time	UP	not known	but see [10]

Table 1
When can it be useful to raise NP-hardness to Δ_2^p-hardness?

Khuller and Vazirani [13] asked whether similar non-self-reducibility results can be proven for problems in P other than `Pl-4-Color`, under some plausible assumption such as $\mathrm{P} \neq \mathrm{NP}$. We established as Theorem 3.5 a general result showing that it is almost trivial to find such problems.

This general result subsumes a number of results [8] providing concrete—although somewhat artificial—problems in P that are not self-reducible in Schnorr's sense, unless $\mathrm{P} = \mathrm{NP}$. Why are these problems artificial? The reason is that they are P versions of standard NP-complete problems—such as the satisfiability problem, the clique problem, and the knapsack problem—that are defined by

(a) encoding directly into each solvable problem instance a trivial solution to this instance, and simultaneously
(b) ensuring that computing the smallest solution remains a hard problem by fixing a suitable ordering of the solutions to a given problem instance.

Here are some examples of such problems:

(1) (a) `P-SAT` is the set of pairs (F, x_i) such that F is a boolean formula in conjunctive normal form and x_i is a variable occurring in each clause of F in positive form.
(b) Let the variables of a given formula F be ordered as $F = F(x_1, x_2, \ldots, x_n)$. Just as for the satisfiability problem, a *solution to a* `P-SAT` *instance* $I = (F, x_i)$ is any satisfying assignment ψ_I of F. A solution ψ_I of I is represented by the string ψ_I in $\{0,1\}^n$ that is defined by $\psi_I = \psi_I(x_1)\psi_I(x_2)\cdots\psi_I(x_n)$, where "1" represents "true" and "0" represents "false."
(2) (a) `P-Clique` is the set of pairs (G, C) such that $G = (V, E)$ is a graph and

$C \subseteq V$ is a clique in G.

(b) Let the vertex set of a given graph $G = (V, E)$ be ordered as $V = \{v_1, v_2, \ldots, v_n\}$. Just as for the clique problem, a *solution to a* `P-Clique` *instance* $I = (G, C)$ is any clique $\hat{C} \subseteq V$ that is of size at least $\|C\|$. A solution $\hat{C}$ of I is represented by the string ψ_I in $\{0,1\}^n$ that is defined by $\psi_I = \chi_{\hat{C}}(v_1)\chi_{\hat{C}}(v_2)\cdots\chi_{\hat{C}}(v_n)$, where $\chi_{\hat{C}}$ denotes the characteristic function of $\hat{C}$, i.e., $\chi_{\hat{C}}(v) = 1$ if $v \in \hat{C}$, and $\chi_{\hat{C}}(v) = 0$ if $v \notin \hat{C}$.

(3) (a) `P-Knapsack` is the set of tuples (U, s, v, k, b) such that U is a finite set, s and v are functions mapping from U to the positive integers, and there exists an element $u \in U$ satisfying $s(u) \leq b$ and $v(u) \geq k$.

(b) Let the set U of a given `P-Knapsack` instance $I = (U, s, v, k, b)$ be ordered as $U = \{u_1, u_2, \ldots, u_n\}$. Just as for the knapsack problem, a *solution to* I is any subset $\hat{U} \subseteq U$ that satisfies the "knapsack property," i.e., that satisfies the conditions

$$\sum_{u \in \hat{U}} s(u) \leq b \text{ and } \sum_{u \in \hat{U}} v(u) \geq k.$$

A solution $\hat{U}$ of I is represented by the string ψ_I in $\{0,1\}^n$ that is defined by $\psi_I = \chi_{\hat{U}}(v_1)\chi_{\hat{U}}(v_2)\cdots\chi_{\hat{U}}(v_n)$.

Note that the lexicographic ordering of strings induces a suitable ordering of the solutions to a given problem instance. For each of the P problems Π defined above, computing LF_{Π} can be shown to be NP-hard [8], which implies that Π is non-self-reducible unless $\mathrm{P} = \mathrm{NP}$.

Analogously, every standard NP-complete problem yields such an artificial, non-self-reducible problem in P. In contrast, the `Pl-4-Color` problem is a quite natural problem. Is it possible to prove, under a plausible assumption such as $\mathrm{P} \neq \mathrm{NP}$, the non-self-reducibility of other *natural* problems in P?

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