Basics

- We will use *cake* $X = [0, 1]$ to represent an infinitely divisible resource. Cake X is defined by the unit interval $[0, 1]$ of real numbers.
- Using bigger intervals such as $[0, 117]$ or even higher-dimensional intervals like $[0, 1]^2$ does not add anything to the description of the problem. Obviously, X does not necessarily need to be a cake.
- In fact, interval X represents *any* infinitely divisible resource that is to be divided among n *players* $p_1, p_2, \ldots, p_n$.
- Given n players, the aim is to perform cuts to divide cake X into n *portions* (or, *shares*) X_j, $1 \leq j \leq n$, where player p_j receives portion $X_j \subseteq X$ and it holds that $X = \bigcup_{j=1}^{n} X_j$ and $X_i \cap X_j = \emptyset$ for $1 \leq i < j \leq n$.

Basics

- Such a partition of cake X into n portions, which are assigned to the n players, is called a *division of X*.
- Note that a portion is not necessarily a single, contiguous piece of cake, but it can also be a collection of disjoint, possibly noncontiguous pieces of X.
- In other words, to obtain a division of X, more than $n-1$ cuts may be made and hence more than n pieces may be cut.
- For $1 \leq i \leq n$, player p_i's preferences over the pieces of X are measured using an individual, private *valuation function*

$$v_i : \{X' \mid X' \subseteq X\} \to [0,1]$$

that maps every piece of cake X (i.e., every subinterval $X' \subseteq X$) to some real number in $[0,1]$.

Basics

- We will not go into detail of the division of a *homogeneous* cake, as this is only a simplified version of the problem.
- In this case, due to the resource being homogeneous, a single player values all pieces of equal size the same.
- To fairly divide the cake, a simple and straightforward procedure would be to assign a piece of size $1/n$ to each of the n players.
- This trivially solves the problem, since every player feels that every other player received a piece of exactly the same value as she herself.

Basic Properties of Valuation Functions

Each valuation function v_i is assumed to satisfy the following properties:

1. **Normalization:** $v_i(X) = 1$ and $v_i(\emptyset) = 0$.
2. **Positivity:** For all subsets $A \subseteq X$, $A \neq \emptyset$ and A does not consist of "isolated points" (which, in terms of measure theory, have Lebesgue measure zero) only, we have $v_i(A) > 0$.
3. **Additivity:** For all subsets $A, B \subseteq X$, $A \cap B = \emptyset$, we have
$$v_i(A \cup B) = v_i(A) + v_i(B).$$
(Unless stated otherwise, we consider *finite* unions of intervals only.)
4. **Divisibility:** For all subsets $A \subseteq X$ and all real numbers α, $0 \leq \alpha \leq 1$, there exists a subset $B \subseteq A$, such that $v_i(B) = \alpha \cdot v_i(A)$.

Cake-cutting Protocols

- A *cake-cutting protocol* describes an interactive procedure recommended to be followed to divide cake X among n players.
- The protocol itself does not hold any information about the players' valuation functions.
- However, the protocol may ask some player to provide his valuation of a specific piece of cake.
- Based on the player's answer, the protocol may then recommend how to continue (e.g., for this player to either accept the piece, or to trim it if he valued it above a certain threshold).

Cake-cutting Protocols

- Robertson and Webb (1998) formalize this interaction between the protocol and the players by distinguishing two kinds of requests:

1. *evaluation requests* of the form $\texttt{eval}_i(S)$ for $S \subseteq X = [0,1]$ require player p_i to return the value $v_i(S)$, and
2. *cut requests* of the form $\texttt{cut}_i(S, \alpha)$ for $S \subseteq X = [0,1]$ and $0 \leq \alpha \leq 1$ require player p_i
    - either to return the value $x \in S = [s_1, s_2]$ such that $v_i([s_1, x]) = \alpha$ (that is, the protocol asks p_i to cut—or make a marking—at point x to produce (or mark) a subpiece of value α),
    - or to announce that this is impossible because no such x exists in S.

Cake-cutting Protocols: Rules versus Strategies

- Every cake-cutting protocol is characterized by a set of rules and a set of strategies.
- In game theory, a *strategy* is a complete and precise policy defining an action for every contingency (i.e., for every situation a player may encounter in the course of a game), and every player has a set of such (rule-consistent) strategies to choose from.
  Note that cake-cutting can be viewed as a game as well.
- However, in cake-cutting protocols, a strategy does not provide actions for the entirety of contingencies but only focuses on actions required to achieve a *fair* and/or *efficient* solution.

Cake-cutting Protocols: Rules versus Strategies

Rules	**Strategies**
• determine the course of action by giving general instructions to the players on what to do next • these instructions are independent of the players' valuation functions and their execution is always verifiable	• are recommendations for the players to make decisions that guarantee them a "fair" share • are not verifiable by a referee
"Cut the cake into two pieces!"	**"Choose your best piece!"**

Proportionality, Super-Proportionality, and Exactness

Definition (proportionality, super-proportionality, exactness)

Let $v_1, v_2, \ldots, v_n$ be the valuation functions of the n players.

The share X_i of player p_i is said to be

1. *proportional* (or, *simply fair*) if $v_i(X_i) \geq 1/n$;
2. *super-proportional* (or, *strongly fair*) if $v_i(X_i) > 1/n$;
3. *exact* if $v_i(X_i) = 1/n$.

A division of cake $X = \bigcup_{i=1}^{n} X_i$ is said to be *proportional* (or, *simply fair*), *super-proportional* (or, *strongly fair*), and *exact*, respectively, if every player receives, respectively, a proportional, super-proportional, and exact share.

Proportionality, Super-Proportionality, and Exactness

Definition

A cake-cutting protocol is said to be

- *proportional*,
- *super-proportional*, or
- *exact*

if every player who follows the rules and strategies of the protocol is guaranteed to receive a

- proportional,
- super-proportional, or
- exact

share of the cake—regardless of the valuation functions of the players and regardless of whether or not the other players follow their recommended strategies.

Remark

Note that all players will always have to follow the rules of the protocol.

Proportionality

- Proportionality is perhaps the most obvious criterion for evaluating the fairness of a division.
- All players ought to consider their portion to be at least a proportional share of the cake.
- That is,
  - when there are two players, both of them ought to feel to have received at least half of the cake;
  - with three players, everyone should value their share to be worth at least one third;
  - and so on.

Super-Proportionality

- Although super-proportionality is quite similar to proportionality, it is a more stringent criterion:

  Every super-proportional division is also proportional, but a proportional division does not need to be super-proportional.

- Compared to designing proportional cake-cutting protocols, designing a cake-cutting protocol that always achieves a super-proportional division is somewhat more challenging.

- In addition, a super-proportional division can only be obtained if the players' valuation functions are not identical.

- There is no need for the valuation functions to be completely different, but they do have to differ for at least the tiniest bit of cake.

Super-Proportionality

- In the latter case, every player could receive even more than a proportional share.
- However, this is merely a rather general statement on what is required for a super-proportional division to exist.
- When designing an algorithmic solution (i.e., a super-proportional cake-cutting protocol), it might become necessary to specify in what way the players' valuation functions need to differ in order to guarantee a super-proportional division.

Exactness

- Exactness, too, is a more stringent criterion than proportionality:

  Every exact division is also proportional, but a proportional division does not need to be exact. (Example: a super-proportional division.)

- However, for proportional divisions exactness and super-proportionality are not complementary, i.e., it does **not** hold that:

  a proportional division is exact if and only if it is not super-proportional.

- Consider, for example, a proportional division that is not super-proportional. This implies that *at least one* of the players receives a portion that she values to be exactly a proportional share.

- An exact division, in contrast, would require *every* player to value their portion to be exactly a proportional share. Hence, there are proportional divisions that are neither super-proportional nor exact.

Looking Beyond One's Own Plate

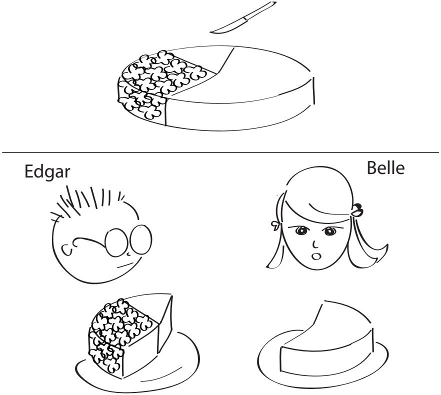

Figure: Belle and Edgar share a cake without envying each other

Looking Beyond One's Own Plate

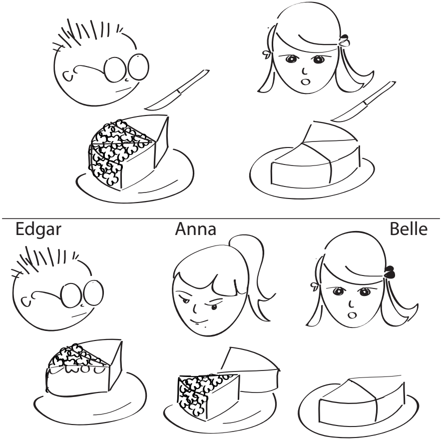

Figure: Anna, Belle, and Edgar share a cake, and envy raises its ugly head

Envy-Freeness, Super-Envy-Freeness, and Equitability

Definition (envy-freeness, super-envy-freeness, equitability)

Let $v_1, v_2, \ldots, v_n$ be the valuation functions of the n players.

The share X_i of player p_i is said to be

1. *envy-free* if $v_i(X_i) \geq v_i(X_j)$ for each j, $1 \leq j \leq n$;
2. *super-envy-free* if $v_i(X_j) < 1/n$ for each j, $1 \leq j \leq n$ and $i \neq j$;
3. *equitable* if $v_i(X_i) = v_j(X_j)$ for each j, $1 \leq j \leq n$.

A division of cake $X = \bigcup_{i=1}^{n} X_i$ is said to be *envy-free*, *super-envy-free*, and *equitable*, respectively, if every player receives, respectively, an envy-free, super-envy-free, and equitable share.

Envy-Freeness, Super-Envy-Freeness, and Equitability

Definition

A cake-cutting protocol is said to be

- *envy-free*
- *super-envy-free*, or
- *equitable*

if every player who follows the rules and strategies of the protocol is guaranteed to receive an

- envy-free,
- super-envy-free, or
- equitable

share of the cake—regardless of the valuation functions of the players and regardless of whether or not the other players follow their recommended strategies.

Envy-Freeness

Fact

Every envy-free division is also proportional, but the converse implication does not hold in general.

Proof: See blackboard. ❑

Theorem

Every proportional cake-cutting protocol for two players is envy-free.

Proof: See blackboard. ❑

Remark

It is quite a challenge to find a way to obtain an envy-free division for more than three players. Even for three players some clever ideas are needed.

Super-Envy-Freeness

- This criterion was introduced by Barbanel (1996) and is a stricter criterion than just envy-freeness.
- As with exactness, there are only very few results on this criterion.
- Barbanel (1996) showed that a super-envy-free division exists if and only if the valuation functions of all players are linearly independent.
- Webb (1999) developed the very first super-envy-free cake-cutting protocol.
- A super-envy-free division gives every player the feeling to have obtained more than every other player.

Super-Envy-Freeness

Fact

1. *For a super-envy-free division, it holds that* $v_i(X_i) > v_i(X_j)$ *for all* i *and* j*, where* $1 \leq i,j \leq n$ *and* $i \neq j$*.*
2. *Every super-envy-free division is also envy-free (and hence proportional), but the converse implication does not hold in general.*

Proof: See blackboard. ❑

Equitability

- Equitability means that all players are equally happy with their portion of the cake.
- For example, Cut & Choose is not equitable. Why not?
- In an equitable division, it is possible that everyone receives a very large value of the cake and it is also possible that everyone receives a very small value of the cake, from every player's own perspective.

Remark

For all the valuation criteria described and discussed above, it is necessary for the cake to be divided in a way such that there are no leftovers—only in this case will it be called a division: $X = \bigcup_{i=1}^{n} X_i$.

Relationships Between the Valuation Criteria

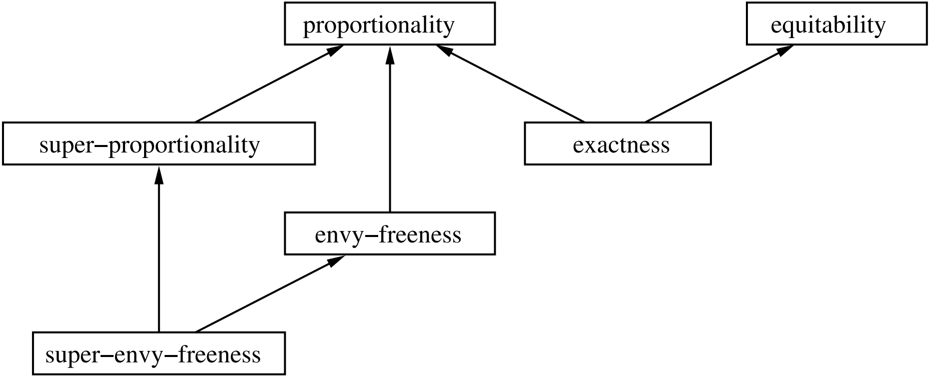

Figure: Implications between the valuation criteria for divisions

Relationships Between the Valuation Criteria

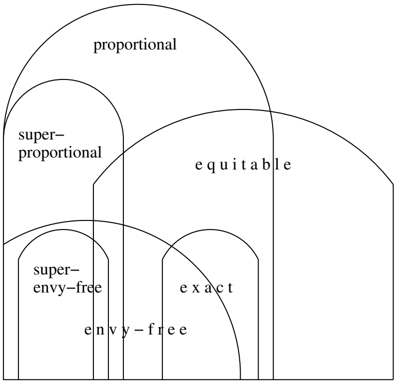

Figure: Relationships between the valuation criteria for divisions

Pareto Optimality, or Pareto Efficiency

Definition (Pareto optimality, or Pareto efficiency)

1. Let $v_1, v_2, \ldots, v_n$ be the valuation functions of the n players. A division of cake $X = \bigcup_{i=1}^{n} X_i$ (where X_i is the portion of player p_i) is said to be *Pareto-optimal* (or, *Pareto-efficient*) if there is no other division $Y = \bigcup_{i=1}^{n} Y_i$ (where Y_i is the portion of player p_i) such that
   1. $v_i(Y_i) \geq v_i(X_i)$ for all players p_i and
   2. there is at least one player p_j with $v_j(Y_j) > v_j(X_j)$.
2. A cake-cutting protocol is said to be *Pareto-optimal* (or, *Pareto-efficient*) if every division of cake X obtained using this protocol (independently of the players' valuation functions) is Pareto-optimal.

Pareto Optimality, or Pareto Efficiency

Fact

The cut-and-choose protocol is not Pareto-optimal.

Proof: Recall our previous example for the cut-and-choose protocol:

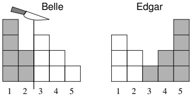

Figure: Cut-and-choose protocol: Belle cuts, Edgar chooses

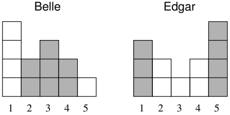

Figure: Alternative cake division assigns a more valuable portion to Belle without making Edgar worse off. ❑

Manipulability

- A player is *cheating* if she does not follow the strategies of the protocol in order to secure a more valuable portion than the one the protocol would guarantee her to receive.
- Even when a player is cheating, the Cut & Choose protocol guarantees an envy-free (and thus proportional and in that sense fair) share for all players playing truthfully.
- The following definition captures what it means for a cake-cutting protocol to be immune against such manipulation attempts.
- One could also consider *"strategy-proofness"* from a game-theoretic point of view, in the sense that playing truthfully is a "dominant strategy" for the players in the "cake-cutting game."

Manipulability

Definition (immunity to manipulation)

- Given a division of cake $X = \bigcup_{i=1}^{n} X_i$, where X_i is player p_i's portion, $v_i(X_i)$ is the *payoff of player p_i*.
- A player is said to be *risk-averse* if she seeks to maximize her payoff in the *worst case* (i.e., with respect to all possible valuation functions or actions of all players—alternatively: *. . . in all cases*).
- A cake-cutting protocol is said to be *immune to manipulation for risk-averse players* if all players maximize their payoff in the worst case only by playing truthfully.

Manipulability

Remark

- *Which portion a player gets (and thus her payoff) depends on her and all other players' actions during the execution of the protocol.*
- *The players' actions result from their valuation functions and, of course, they depend on the rules and the strategies of the protocol.*
- *Unlike strategies, a player's actions can usually be observed by the others (unless the protocol requests the player to do something in private).*
- *When some action of a player is observed, one can never be sure if he really has followed the proposed strategy.*

Manipulability

Remark

- *A risk-averse player is very conservative and always expects the worst to happen.*
- *His goal therefore is to maximize his payoff even in the worst case.*
- *He doesn't take chances.*
- *He just wants to be on the safe side, rather letting some potential gain slip through his fingers than to jeopardize his best worst-case payoff.*
- *His motto is:*

  "A bird in the hand is worth two in the bush!"

Manipulability

Remark

- *The point of the above definition is that only playing truthfully (i.e., playing according to the strategies proposed by the protocol) can give some player his maximum payoff in the worst case.*
- *Whenever a player tries to cheat by departing from the protocol's strategies, he is no longer guaranteed his maximum payoff, i.e., in the worst case he will get less than that.*
- *That does not exclude that he might be lucky in other cases than the worst case (i.e., for certain other, fortunate valuations/actions of the other players) in which he might increase his payoff by cheating.*

Manipulability

Definition (immunity to manipulation—continued)

A bit more formally, let

- $A_i = \sigma_i(v_1, \ldots, v_n)$ denote the sequence of actions of player p_i, $1 \leq i \leq n$, that result from the sequence of strategies σ_i that the procotocol proposes for p_i when applied to the valuation functions $v_1, \ldots, v_n$, and
- $A'_i = \sigma'_i(v_1, \ldots, v_n)$ denote the sequence of actions of player p_i, $1 \leq i \leq n$, that result from another sequence of strategies σ'_i when applied to the valuation functions $v_1, \ldots, v_n$.

Manipulability

Definition (immunity to manipulation—continued)

- *Immunity to manipulation for risk-averse players* means:

$$(\forall i)\,(\forall \sigma_i')\,(\exists v_{-i})\,[A_i' \neq A_i \implies v_i(X_i') < v_i(X_i)], \qquad (1)$$

  where

  - $v_{-i} = v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ denotes the valuation functions of all other players,
  - X_i' is the portion p_i receives when playing the actions A_i', and
  - X_i is the portion p_i receives when playing the (truthful) actions A_i.

- *Manipulability for risk-averse players* means:

$$(\exists i)\,(\exists \sigma_i')\,(\forall v_{-i})\,[A_i' \neq A_i \wedge v_i(X_i') \geq v_i(X_i)]. \qquad (2)$$

Manipulability

Immunity to manipulation for risk-averse players:

$$(\forall i)\,(\forall \sigma_i')\,(\exists v_{-i})\,[A_i' \neq A_i \implies v_i(X_i') < v_i(X_i)]$$

means:

- For all i and for all sequences of strategies σ_i',
- there exists some worst case (caused by the other players' valuation functions v_{-i})
- in which p_i is punished (because of $v_i(X_i') < v_i(X_i)$) for playing the actions $A_i' = \sigma_i'(v_{-i}, v_i)$,
- provided they differ from the truthful actions $A_i = \sigma_i(v_{-i}, v_i)$ proposed by the protocol.

Manipulability

Manipulability for risk-averse players:

$$(\exists i)\,(\exists \sigma_i')\,(\forall v_{-i})\,[A_i' \neq A_i \wedge v_i(X_i') \geq v_i(X_i)]$$

means:

- For some i and for some sequence of strategies σ_i',
- whatever valuation functions v_{-i} the other players have,
- p_i cannot be punished (because of $v_i(X_i') \geq v_i(X_i)$) for playing actions $A_i' = \sigma_i'(v_{-i}, v_i)$
- that are different from the truthful actions $A_i = \sigma_i(v_{-i}, v_i)$ proposed by the protocol.

Manipulability

Remark

- *Recall: A cake-cutting protocol is said to be immune to manipulation for risk-averse players if all players maximize their payoff in the worst case* **only** *by playing truthfully.*

- *Why* **only**? *this word would mean to replace (1) by*

$$(\forall i)\,(\forall \sigma_i')\,(\exists v_{-i})\,[A_i' \neq A_i \implies v_i(X_i') \leq v_i(X_i)] \tag{3}$$

i.e., all (risk-averse) players would then maximize their payoff in the worst case by playing truthfully, but playing untruthfully could also give them maximum payoff.

- *Our "immunity to manipulation" is the stronger of the two notions.*

Finite, Finite Bounded, and Continuous Protocols

Definition (finite vs continuous cake-cutting protocol)

1. A cake-cutting protocol is said to be *finite* if it always (i.e., independently of the players' valuation functions) terminates with a solution after only a finite number of decisions (i.e., evaluation requests, markings, and actual cuts) have been made.

2. A cake-cutting protocol is said to be *continuous* (or, a *moving-knife protocol*) if a solution can be achieved by the players continuously making decisions.

Finite, Finite Bounded, and Continuous Protocols

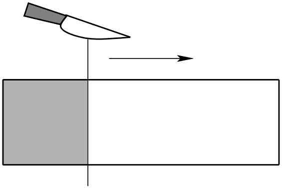

Figure: Schematic knife movement in a moving-knife protocol

Definition (finite bounded vs unbounded cake-cutting protocol)

A finite cake-cutting protocol is said to be *finite bounded* if the number of decisions (which may depend on the number of players) required to solve the problem is known in advance, irrespective of the players' valuation functions; otherwise, it is said to be *finite unbounded*.