Multivariate Complexity of Swap Bribery

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joint work with

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COMSOC 2010, Düsseldorf
Bribery in elections

spending money to influence the voters’ preferences

- pay money to voters/to chair
- campaigning

⇒ bad/good phenomenon

both hardness and tractability results interesting!
Bribery as a computational problem

**Bribery**

**Input:** \( \mathcal{E} \)-Election \( E = (C, V) \), preferred candidate \( p \in C \), cost function, budget \( \beta \).

**Question:** Is it possible to bribe voters such that \( p \) wins, respecting the budget?

In the following: \( m = |C| = \# \) candidates
\( n = |V| = \# \) votes.
Bribery as a computational problem

Special model considered here:

**Swap Bribery** [Elkind, Faliszewski, Slinko, SAGT 2009]

cost function: every voter assigns certain price for swapping the positions of two *consecutive* candidates in his preference list.

Example: \(v: a > b > p\)

\(v\)'s list of costs of swaps:

- \(c(a \bowtie b) = 2\)
- \(c(a \bowtie p) = 3\)
- \(c(b \bowtie p) = 1\)

briber wants \(\tilde{v}: p > b > a\)

cost of a set of swaps:

- \(v: a > b > p. \) swap \(a \bowtie b\) at cost 2
- \(\tilde{v}: b > a > p. \) swap \(a \bowtie p\) at cost 3
- \(b > p > a. \) swap \(b \bowtie p\) at cost 1
- \(p > b > a. \) total cost: 6
Bribery as a computational problem

**Swap Bribery**

**Input:** \( \mathcal{E} \)-Election \( E = (C, V) \), preferred candidate \( p \in C \), cost functions, budget \( \beta \).

**Question:** Is there a set of swaps with total cost \( \leq \beta \), such that \( p \) wins the bribed election?

for costs in \( \{0, \delta > 0\} \), budget \( \beta = 0 \): Possible Winner.
Some known results for **Swap Bribery**

[**Elkind, Faliszewski, Slinko, SAGT 2009**]

- hardness results for Borda: **NP-c**
  - (from **Possible Winner** [Xia, Conitzer, AAAI, 2008]),
  - Copeland$^\alpha$: **NP-c**, Maximin: **NP-c**

- case study for $k$-approval $(1,1,\ldots,1,0,\ldots,0)$
  - $k = 1$ (Plurality): $P$ $k$-approval
  - $k = m - 1$ (Veto): $P$
  - $1 \leq k \leq m$, $m$ or $n$ constant: $P$
  - $k = 2$: **NP-c**
    - (from **Possible Winner**, [Betzler, Dorn, J.Comput.Syst.Sci., 2010])
  - $3 \leq k \leq m - 2$, $k$ fixed, costs in $\{0,1,2\}$: **NP-c**

- $k$ part of the input: **NP-c** even for 1 voter!
Multivariate complexity analysis of Swap Bribery

so far: complexity measured in size of the input (1-dimensional)

now: complexity measured in size of the input
and certain ‘parameters’ (multi-dimensional)

e.g.: # candidates
    # votes
    # candidates with special property
    cost
    budget
    ...

Which parameters have a significant influence on the hardness of the problem?
Multivariate complexity analysis of **Swap Bribery**

- **$t$ - parameter**

NP-hard problems: presumably cannot avoid exp. running times.

**But:** Maybe we can restrict exponential part of running time to a certain parameter! E.g. $2^t \cdot |x|^2$

⇒ If value of $t$ is small in certain settings: efficient algorithm!

**fixed-parameter tractability**

A problem is **fixed-parameter tractable** if it can be solved in

$$f(t) \cdot \text{poly}(|x|)$$

(time)

($|x|$ - size of the input)

**corresponding complexity class:** **FPT**

What about running time $|x|^t$? **not in FPT!**
Multivariate complexity analysis of Swap Bribery

Intractability results

Hardness classes
First level of fixed-parameter intractability: class $W[1]$

hardness/completeness via parameterized reduction.
Goal: Analyze complexity of Swap Bribery from a parameterized/multivariate point of view.

Special focus on \( k \)-approval.

Our investigations

Complexity depending on

\( (1) \) cost function, budget
\( (2) \) combined parameter \((n = \# \text{ votes}, \beta = \text{budget})\)
\( (3) \) \( m = \# \text{ candidates} \)
1. Complexity depending on cost function

\( k \)-approval

**Theorem 1**

Costs uniform (every swap has the same cost):
\[ \text{SWAP BRIBERY for } k \text{-approval is in } \mathbf{P} \]

\[ \rightarrow \text{ network flow problem} \]

**Theorem 2**

As soon as there are two different costs:
\[ \text{SWAP BRIBERY for } k \text{-approval is } \mathbf{NP} \text{-c.} \]
\[ \text{SWAP BRIBERY for } k \text{-approval is } \mathbf{W[1]} \text{-hard with respect to } \beta \]

\[ \rightarrow (\text{parameterized}) \text{ reduction from } \text{MULTICOLORED CLIQUE} \]
2. Complexity depending on combined parameter \((n, \beta)\)

\(k\)-approval

**Theorem 3**

If minimum cost of a swap is 1:

**Swap Bribery** for \(k\)-approval is in \textbf{FPT} with respect to \((n, \beta)\)
2. Complexity depending on combined parameter \((n, \beta)\)

\[
\begin{array}{c}
\underbrace{111\ldots111}_{k} \underbrace{000\ldots0}_{\beta} \\
\beta & \beta
\end{array}
\]

votes

minimum cost of a swap = 1: only candidates that can be swapped within budget \(\beta\) from 1- to 0-position or vice versa are interesting.

\(\Rightarrow\) cut votes (such that only relevant candidates stay)

\[
\begin{array}{c}
\underbrace{11\ldots100\ldots0}_{\beta} \\
\beta & \beta
\end{array}
\]

cut votes

\[\text{+}
\]

some more votes that take into account points of ‘lost’ positions
2. Complexity depending on combined parameter \((n, \beta)\)

\[
\begin{array}{c}
\beta \\
\begin{array}{cccc}
1 & 1 & \ldots & 1 \\
0 & 0 & \ldots & 0 \\
\end{array}
\end{array}
\]

- cut votes

+ some more votes that take into account points of ‘lost’ positions

remaining profile is much smaller:
- only \(O(n^2 \beta^2)\) candidates left
- new votes, but only \(O(n^2 \beta)\) many of them

→ brute force on the smaller instance (‘problem kernel’), leads to an \textbf{FPT} running time
3. Complexity depending on $m =$ number of candidates

Any voting system that can be described by *linear inequalities*, e.g. scoring rules, Maximin, Copeland$^\alpha$, Bucklin, Ranked Pairs, . . .

**Theorem 4**

For all voting rules that can be described by linear inequalities:

\textbf{Swap Bribery} is in \textbf{FPT} with respect to $m$.

→ \textbf{ILP formulation}

In a similar way:

Many other problems are in \textbf{FPT} with respect to $m$ as well, e.g.

- \textbf{Possible Winner}
- \textbf{Manipulation}
- \textbf{Control}
- \textbf{Lobbying}
Summary

Results

Complexity depending on

(1) cost function, budget: $\mathsf{P}/\mathsf{NP}$-c, $\mathsf{W}[1]$-hard ($\beta$)

(2) ($n = \# \text{ votes}, \beta = \text{ budget}$): $\mathsf{FPT}$

(3) $m = \# \text{ candidates}: \mathsf{FPT}$

$k$-approval
What else is interesting?

- different parameters
- different voting systems
- destructive case
- different models?