Fair Division under Ordinal Preferences: Computing Envy-Free Allocations of Indivisible Goods

Sylvain Bouveret
Onera Toulouse

Ulle Endriss
University of Amsterdam

Jérôme Lang
Université Paris Dauphine

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The fair division problem

Given

- a set of indivisible objects $O = \{o_1, \ldots, o_m\}$
- and a set of agents $A = \{1, \ldots, n\}$
- such that each agent has some preferences on the subsets of objects she may receive

Find

- an allocation $\pi : A \to 2^O$
- such that $\pi(i) \cap \pi(j)$ for every $i \neq j$
- satisfying some fairness and efficiency criteria

Separable ordinal preferences

- We assume that the preferences are ordinal.
- **Restriction**: each agent specifies a linear order $\rhd$ on O (**single** objects)

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Problem: How to compare **subsets** of objects ?

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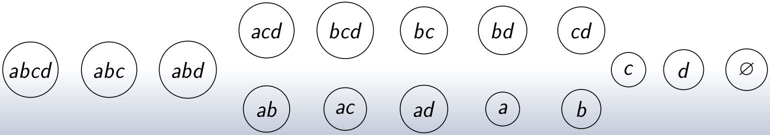
1. Assume monotonicity $\rightsquigarrow$ *e.g* $abc \succ ab$.
2. Assume separability: if $(X \cup Y) \cap Z = \varnothing$ then $X \succ Y$ iff $X \cup Z \succ Y \cup Z$.
   $\rightsquigarrow$ *e.g* $ab \succ ac$.

Example

- $\mathcal{N} : a \rhd b \rhd c \rhd d$
- Separability
- Monotonicity



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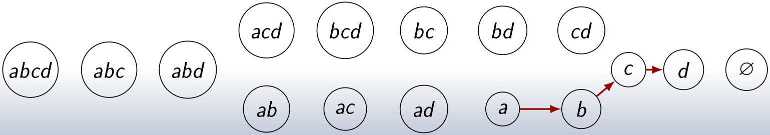

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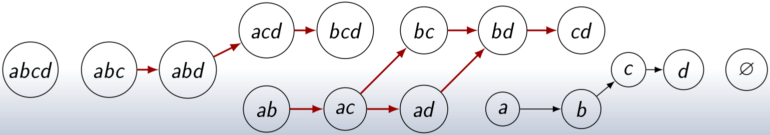

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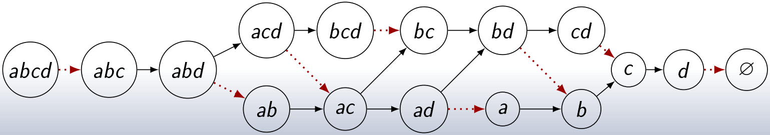

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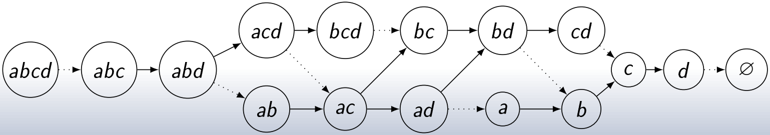

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$X \succ_{\mathcal{N}} Y \Leftrightarrow \exists$ an **injective mapping of improvements** $Y \mapsto X$.

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- $\{a, c, d\} \succ_{\mathcal{N}} \{b, c, e\}$
- $\{a, d, e\}$ and $\{b, c, f\}$ are incomparable.
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Brams, S. J., Edelman, P. H., and Fishburn, P. C. (2004).
Fair division of indivisible items.
Theory and Decision, 5(2):147–180.

Brams, S. J. and King, D. (2005).
Efficient fair division—help the worst off or avoid envy?
Rationality and Society, 17(4):387–421.

Envy-freeness

Fairness. . .

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Fairness...

Envy-freeness: $\langle \succ_1, \ldots, \succ_n \rangle$ **total** strict orders, allocation π.

$$\pi \text{ envy-free } \Leftrightarrow \forall i, j, \pi(i) \succ_i \pi(j)$$

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When $\langle \succ_1, \ldots, \succ_n \rangle$ *are partial orders ?*

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$\leadsto$ Envy-freeness becomes a **modal** notion

Possible and necessary Envy-freeness

- π is Possibly Envy-Free *iff* for all i, j, we have $\pi(j) \not\succ_i \pi(i)$;
- π is Necessary Envy-Free *iff* for all i, j, we have $\pi(i) \succ_i \pi(j)$.

Pareto-efficiency

Efficiency. . .

Pareto-efficiency

Efficiency...

- Complete allocation.
- Pareto-efficiency

Pareto-efficiency

Efficiency...

Classical Pareto dominance

π' **dominates** π if for all i, $\pi'(i) \succeq_i \pi(i)$ and for some j, $\pi'(j) \succ_j \pi(j)$

Extended to possible and necessary Pareto dominance.

- π is *possibly Pareto-efficient* (PPE) if there exists no allocation π' such that π' necessarily dominates π.
- π' is *necessarily Pareto-efficient* (NPE) if there exists no allocation π' such that π' possibly dominates π.

Envy-freeness and efficiency

complete PPE NPE ← Efficiency

Envy-freeness and efficiency

complete PPE NPE ← Efficiency

PEF
NEF
↑
Fairness

Envy-freeness and efficiency

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

← Efficiency

↑ Fairness

Envy-freeness and efficiency

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PEF	X	X	X
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Envy-freeness and efficiency cannot always be satisfied simultaneously

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Efficiency

Fairness

Envy-freeness and efficiency cannot always be satisfied simultaneously

Questions:

- under which conditions is it guaranteed that there exists a allocation that satisfies Fairness and Efficiency ?
- how hard it is to determine whether such an allocation exists?

Complete possibly envy-free allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

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Result

n agents, m objects, k distinct goods are top-ranked by some agent.

$$\exists \text{ complete PEF allocation } \Leftrightarrow m \geq 2n - k.$$

Constructive proof (algorithm/protocol)

Example

$$\mathcal{N}_1\colon a \rhd b \rhd c \rhd d \rhd e \rhd f \qquad \mathcal{N}_2\colon a \rhd d \rhd b \rhd c \rhd e \rhd f$$
$$\mathcal{N}_3\colon b \rhd a \rhd c \rhd d \rhd f \rhd e \qquad \mathcal{N}_4\colon b \rhd a \rhd d \rhd e \rhd f \rhd c$$

$$(k = 2;\ m = 6 \geq 2n - k)$$

Consider the agents in order $1 > 2 > 3 > 4$:

- *first step*: give a to 1; give b to 3; 1 and 3 leave the room;
- *second step*: give d to 2; give c to 4;
- *third step*: give e to 4; give f to 2.

PPE-PEF allocations

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PEF	X	X	X
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PPE-PEF allocations

	complete	PPE	NPE
PEF	X	X	X
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Result

∃ PPE-PEF allocation ⇔ ∃ complete, PEF allocation.

Based on the characterization of efficient allocations in [Brams and King, 2005].

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NPE-PEF allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

NPE-PEF allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Complexity of the existence of NPE-PEF allocations: *open*.

Complete NEF allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

- Two easy necessary conditions:
  - distinct top ranked objects;
  - m is a multiple of n.

Complete allocation

- deciding whether there exists a complete NEF allocation is NP-complete (even if $m = 2n$).
- the problem falls down in P for two agents

(hardness by reduction from [X3C])

Pareto-efficient-NEF allocations

	complete	PPE	NPE
PEF	X	X	X
NEF	X	X	X

Possible and necessary Pareto-efficiency

- existence of a PPE-NEF allocation: NP-complete
- existence of a NPE-NEF allocation: NP-hard but probably not in NP (Σ_2^p-completeness conjectured).

Results and beyond

Fair division with incomplete ordinal preferences:

- separable and monotone ordinal preferences;
- modal Pareto-efficiency and Envy-freeness.

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PEF	P (algorithm)	P (algorithm)	?
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Beyond separable preferences ? CI-nets [Bouveret et al., 2009].
$\rightsquigarrow$ Even dominance is PSPACE-complete.

Bouveret, S., Endriss, U., and Lang, J. (2009).
Conditional importance networks: A graphical language for representing ordinal, monotonic preferences over sets of goods.
In *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI'09)*, pages 67–72, Pasadena, California.