The Efficiency of Fair Division with Connected Pieces

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Cakes

- A metaphor for any divisible, heterogeneous good that people share
- People may have different preferences regarding different parts of the cake

"I want lots of chocolate flakes!"

"I want as much cream as possible!"

"I want a piece that didn't even *touch* a cherry!"

A Fair Division?

- We want to share the cake fairly
  - But what should be considered "fair"?

Proportionality

Every player gets a piece he considers as worth at least 1/n.

Envy-Freeness

No player values the piece of any other player more than his own.

Equitability

All players have the same valuation of their own piece.

The Formal Setting

- Cake:
  - One-dimensional
  - Simply the interval [0,1]
- Preferences:
  - Non-atomic probability measures on [0,1]
- Division:
  - Arbitrary pieces, *or*
  - Connected intervals

Previous Work

- Problem first presented in the 1940s by H. Steinhaus
- Algorithms for different variants of the problem:
  - Finite algorithms (e.g. [Ste49,EP84])
  - "Moving knife" algorithms (e.g. [Str80])
- (Non-constructive) existence therems (e.g. [DS61,Str80])
- Lower bounds on the number of steps required for division (e.g. [SW03,EP06,Pro09])
- Books: [BT96,RW98,Mou04]

Economic Efficiency

- Besides fairness, we also want to maximize social welfare
- What is the trade-off between these desiderata?
- [CKKK09]: Let's define the "Price of Fairness"
  - Measures how much efficiency we need to give up for fairness
  - "Formally":

"Price of Propotionality"

"Price of Envy-Freeness"

Different welfare functions

$$\frac{\text{Highest possible welfare}}{\text{Welfare in best "fair" division}}$$

"Price of Equitability"

  - [CKKK09] considered utilitarian welfare, and allowed divisions with arbitrary pieces

Our Work

- Division:
  - Connected = Every player gets a single interval
  - This is required *both* in the fair divisions, *and* in the socially optimal ones
- Social welfare:
  - Utilitarian (sum of players' utilities)
  - Egalitarian (utility of the worst-off player)

Results

Price of:	Propotionality	Envy-Freeness	Equitability
Utilitarian	u.b: $\frac{\sqrt{n}}{2}+1-o(1)$ l.b: $\frac{\sqrt{n}}{2}$		u.b: n l.b: $n-1+o(1)$
Egalitarian	1	$\frac{n}{2}$	1
Utilitarian **Non-connected [CKKK09]**	u.b: $2\sqrt{n}-1$ l.b: $\frac{\sqrt{n}}{2}$	u.b: $n-1/2$ l.b: $\frac{\sqrt{n}}{2}$	u.b: n l.b: $\frac{(n+1)^2}{4n^2}$

Highlights of this Work

- A non-trivial $\frac{\sqrt{n}}{2}+1-o(1)$ upper bound on the Price of Envy-Freeness (for utilitarian welfare)
  - These are usually hard to obtain – we don't have good methods for finding EF divisions
- The egalitarian Price of Equitability is 1
  - In particular, every cake instance has an egalitarian-optimal (connected) equitable division
  - First proof for existence of equitable divisions with connected pieces

An Upper Bound on utilitarian PoEF

<u>Theorem 1:</u> For every cake-cutting instance with n players, there is an envy-free division with utilitarian welfare within a factor of at most $\frac{\sqrt{n}}{2}+1-o(1)$ *of the highest welfare possible for this instance.*

- Moreover, *any* envy-free division is never far from utilitarian optimality by more than this!

An Upper Bound on utilitarian PoEF

- Some notation:
  - x: an envy-free division
  - y: a utilitarian-optimal division

    (Since we consider connected pieces, a division is simply the positions of all n-1 cuts + a permutation that indicates who gets what)
  - $u_i(z)$: the utility of player i from her piece in division z
- The key observation:

  “Since x is envy-free, if $u_i(y) \geq \beta \cdot u_i(x)$ then the portion of the cake that was given to player i in the division y had to be divided between at least $\lceil \beta \rceil$ different players (possibly including i) in x"

An Upper Bound on utilitarian PoEF

- We can reduce the problem to finding $u_i(x)$ values and α_i values (no. of cuts given to player i) that maximize the ratio u(y)/u(x):

$$\text{maximize} \quad \frac{\sum (\alpha_i+1)\cdot u_i(x)}{\sum u_i(x)}$$

$$\begin{aligned} s.t. \quad & \sum \alpha_i = n-1 & \\ & u_i(x) \geq 1/n & \forall i \\ & (\alpha_i+1)\cdot u_i(x) \leq 1 & \forall i \\ & \alpha_i \in \{1,\dots,n-1\} & \forall i \end{aligned}$$

Total: n-1 cuts

x is proportional

y gives at most 100% to every player

no. of cuts a player may get is integer

- With some more work, it can be shown that the solution to this problem is bounded by $\frac{\sqrt{n}}{2}+1-\frac{n}{4n^2-4n+2\sqrt{n}}$

Open Question #1

- Egalitarian Price of Fairness for arbitrary pieces:
  - [CKKK09] analyzed only the utilitarian Price of Fairness
  - **What is the egalitarian Price of Envy-Freeness?**
    - u.b.: n/2 (trivial)
    - l.b.: > 1,
      can be shown by a rather simple example
  - That's quite a gap!
    - What is the right bound?

Open Question #2

- One extreme:

  Allow arbitrary pieces (like [CKKK09] did)
- The other extreme:

  Require that pieces are single intervals (like we did)
- A natural middle ground:
  - Pieces that are a bounded union of intervals
  - **Can we analyze the Price of Fairness as a function of the number of pieces players may get?**

Open Question #3

- Connected pieces may also apply to *chores*
  - E.g. a group of workers have to keep a beach strip clean
    - Some parts have more rocks, some have more plants, some are more popular by visitors, etc.
  - Every worker should be responsible for some (connected) part of the beach strip
  - We want to divide the work fairly
- **What can be said about the Price of Fairness here?**

Thank You!

Any Questions?