Group-Strategyproof Irresolute Social Choice Functions

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Abstract

We axiomatically characterize the class of pairwise irresolute social choice functions that are group-strategyproof according to Kelly's preference extension. The class is narrow but contains a number of appealing Condorcet extensions such as the *minimal covering set* and the *bipartisan set*, thereby answering a question raised independently by Barberà (1977) and Kelly (1977). These functions furthermore encourage participation and thus do not suffer from the no-show paradox (under Kelly's extension).

1 Introduction

One of the central results in social choice theory is that every social choice function (SCF)—a function mapping individual preferences to a collective choice—is susceptible to strategic manipulation (Gibbard, 1973; Satterthwaite, 1975). However, the classic result by Gibbard and Satterthwaite only applies to *resolute*, i.e., single-valued, SCFs. The notion of a resolute SCF is rather restricted and artificial.[1] For example, consider a situation with two voters and two alternatives such that each voter prefers a different alternative. The problem is not that a resolute SCF has to pick a single alternative (which is a well-motivated practical requirement), but that it has to pick a single alternative *based on the individual preferences alone* (see, e.g., Kelly, 1977). As a consequence, resoluteness is at variance with such elementary notions as neutrality and anonymity.

In order to remedy this shortcoming, Gibbard (1977) strengthened his impossibility to social choice functions that yield probability distributions over the set of alternatives rather than single alternatives. While this impossibility result is sweeping, it makes relatively strong assumptions on the voters' preferences. In contrast to the traditional setup in social choice theory, which usually only involves ordinal preferences, Gibbard's result relies on the axioms of von Neumann and Morgenstern (1947) (or an equivalent set of axioms) in order to compare lotteries over alternatives.[2]

The gap between Gibbard and Satterthwaite's theorem for resolute social choice functions and Gibbard's theorem for probabilistic social choice functions has been filled by a number of impossibility results with varying underlying notions of how to compare sets of alternatives with each other (e.g., Barberà, 1977; Kelly, 1977; Gärdenfors, 1976; Duggan and Schwartz, 2000). In this paper, we will be concerned with the weakest (and therefore least controversial) preference extension from alternatives to sets due to Kelly (1977). According to this definition, a set of alternatives is preferred to another set of alternatives if all elements of the former are preferred to all elements of the latter. Barberà (1977) and Kelly (1977) have shown independently that, for more than two alternatives, all social choice functions that are rationalizable via a binary preference relation are manipulable. Kelly (1977) concludes his paper by contemplating that "one plausible interpretation of such a theorem is that, rather than demonstrating the impossibility of reasonable strategy-proof social choice

[1] For example, Gärdenfors (1976) refers to resolute SCFs as "unnatural" and Kelly (1977) calls them "unreasonable."

[2] Gibbard (1978) later strengthened his impossibility theorem by generalizing it to choice mechanisms that do not necessarily take preference relations as inputs.

functions, it is part of a critique of the regularity [rationalizability] conditions" and Barberà (1977) states that "whether a nonrationalizable collective choice rule exists which is not manipulable and always leads to nonempty choices for nonempty finite issues is an open question." Also referring to nonrationalizable choice functions, Kelly (1977) writes: "it is an open question how far nondictatorship can be strengthened in this sort of direction and still avoid impossibility results."

In this paper, we characterize a class of social choice functions that cannot be manipulated by groups of voters who misrepresent their strict preferences. As a corollary of this characterization, all monotonic social choice functions that satisfy the *strong superset property* are group-strategyproof. The strong superset property goes back to early work by Chernoff (1954) (see also Bordes, 1979; Aizerman and Aleskerov, 1995) and requires that choice sets are invariant under the removal of unchosen alternatives. It has recently been used to characterize so-called set-rationalizable choice functions (Brandt and Harrenstein, 2009). The class of social choice functions satisfying the strong superset property is narrow but contains appealing Condorcet extensions such as *weak closure maximality* (also known as the *top cycle*, *GETCHA*, or the *Smith set*), the *minimal covering set*, the *bipartisan set*, and their generalizations (see Bordes, 1976; Laslier, 1997; Dutta and Laslier, 1999; Laslier, 2000).[3] Strategyproofness (according to Kelly's preference extension) thus draws a sharp line within the space of social choice functions as many established social choice functions (such as plurality, Borda's rule, and all weak Condorcet extensions) are known to be manipulable (Taylor, 2005) (and also fail to satisfy the strong superset property (Brandt and Harrenstein, 2009)). We furthermore show that our characterization is complete for *pairwise* social choice functions, i.e., social choice functions whose outcome only depends on the comparisons between pairs of alternatives.

Kelly's conservative preference extension has previously been primarily invoked in impossibility theorems because it is independent of the voters' attitude towards risk and the mechanism that eventually picks a single alternative from the choice set. Its interpretation in positive results, such as in this paper, is more debatable. Gärdenfors (1979) has shown that Kelly's extension is appropriate in a probabilistic context when voters are unaware of the lottery that will be used to pick the winning alternative. (Whether they are able to attach utilities to alternatives or not does not matter.) Alternatively, one can think of an independent chairman or a black-box that picks alternatives from choice sets in a way that prohibits a meaningful prior distribution. Whether these assumptions can be reasonably justified or such a device can actually be built is open to discussion. In particular, the study of distributed protocols or computational selection devices that justify Kelly's extension appears to be promising.

Remarkably, the robustness of the minimal covering set and the bipartisan set with respect to strategic manipulation also extends to *agenda manipulation.* The strong superset property precisely states that a social choice function is resistant to adding and deleting losing alternatives (see also the discussion by Bordes, 1983). Moreover, both choice rules are composition-consistent, i.e., they are strongly resistant to the introduction of clones (Laffond et al., 1993b, 1996).[4] Scoring rules like plurality and Borda's rule are prone to both types of agenda manipulation (Laslier, 1996; Brandt and Harrenstein, 2009) as well as to strategic manipulation.

We conclude the paper by pointing out that voters can never benefit from abstaining strategyproof pairwise SCFs. This does not hold for *resolute* Condorcet extensions, which

[3] If we assume an odd number of voters with strict preferences, the *tournament equilibrium set* (Schwartz, 1990) and the *minimal extending set* (Brandt, 2009) are conjectured to satisfy the strong superset property. Whether this is indeed the case depends on a certain graph-theoretic conjecture (Laffond et al., 1993a; Brandt, 2009).

[4] In addition to these attractive properties, the minimal covering set and the bipartisan set can be computed efficiently using non-trivial algorithms (Brandt and Fischer, 2008).

is commonly known as the *no-show paradox* (Moulin, 1988).

2 Related Work

Apart from the mentioned theorems by Barberà (1977) and Kelly (1977), there are numerous impossibility results concerning strategyproofness based on other—stronger—types of preferences over sets (see, e.g., Gärdenfors, 1976; Duggan and Schwartz, 2000; Barberà et al., 2001; Ching and Zhou, 2002; Sato, 2008; Umezawa, 2009), many of which are surveyed by Taylor (2005) and Barberà (2010). To the best of our knowledge, Jimeno et al. (2009) provide the only extension of Moulin's theorem on abstention for resolute Condorcet extensions (Moulin, 1988) to irresolute SCFs. Interestingly, they use stronger assumptions on preferences over sets and therefore obtain a negative result whereas our result is positive.

Inspired by early work by Bartholdi, III et al. (1989), recent research in computer science investigated how to use computational hardness—namely NP-hardness—as a barrier against manipulation (see, e.g., Conitzer and Sandholm, 2003; Conitzer et al., 2007; Faliszewski et al., 2009). However, NP-hardness is a worst-case measure and it would be much preferred if manipulation is hard on average. Recent negative results on the hardness of typical cases have cast doubt on this strand of research (see, e.g., Conitzer and Sandholm, 2006; Friedgut et al., 2008; Walsh, 2009), but more work remains to be done to settle the question completely. The current state of affairs is surveyed by Faliszewski and Procaccia (2010). If computational protocols or devices can be used to justify Kelly's extension by making "unpredictable" random selections, this might be an interesting alternative application of computational techniques to obtain strategyproofness.

3 Preliminaries

In this section, we provide the terminology and notation required for our results. We will use the standard model of social choice functions with a variable agenda (see, e.g., Taylor, 2005).

3.1 Social Choice Functions

Let U be a universe of alternatives over which voters entertain preferences. The preferences of voter i are represented by a complete preference relation $R_i \subseteq U \times U$.[5] We have $a \; R_i \; b$ denote that voter i values alternative a at least as much as alternative b. In compliance with conventional notation, we write P_i for the strict part of R_i, i.e., $a \; P_i \; b$ if $a \; R_i \; b$ but not $b \; R_i \; a$. Similarly, I_i denotes i's indifference relation, i.e., $a \; I_i \; b$ if both $a \; R_i \; b$ and $b \; R_i \; a$. The set of all preference relations over the universal set of alternatives U will be denoted by $\mathcal{R}(U)$. The set of *preference profiles*, i.e., finite vectors of preference relations, will be denoted by $\mathcal{R}^*(U)$. The typical element of $\mathcal{R}^*(U)$ is $R = (R_1, \ldots, R_n)$ and the typical set of voters is $N = \{1, \ldots, n\}$.

Any subset of U from which alternatives are to be chosen is a *feasible set* (sometimes also called an *issue* or *agenda*). Throughout this paper we assume the set of feasible subsets of U to be given by $\mathcal{F}(U)$, the set of finite and non-empty subsets of U, and generally refer to finite non-empty subsets of U as feasible sets. Our central object of study are *social choice*

[5]Transitivity of individual preferences is not necessary for our results to hold. In fact, Theorem 2 is easier to prove for general—possibly intransitive—preferences. Theorem 3, on the other hand, would require a more cumbersome case analysis for transitive preferences.

functions, i.e., functions that map the individual preferences of the voters and a feasible set to a set of socially preferred alternatives.[6]

Definition 1. A *social choice function (SCF)* is a function $f : \mathcal{R}^*(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ such that $f(R, A) \subseteq A$ and $f(R, A) = f(R', A)$ for all feasible sets A and preference profiles R, R' such that $R|_A = R'|_A$.

A *Condorcet winner* is an alternative a that, when compared with every other alternative, is preferred by more voters, i.e., $|\{i \in N \mid a\ R_i\ b\}| > |\{i \in N \mid b\ R_i\ a\}|$ for all alternatives $b \neq a$. An SCF is called a *Condorcet extension* if it uniquely selects the Condorcet winner whenever one exists.

The following notational convention will turn out to be useful throughout the paper. For a given preference profile R, $R_{i:(a,b)}$ denotes the preference profile $(R_1, \ldots, R_{i-1}, R'_i, R_{i+1}, \ldots, R_n)$ where $R'_i = R_i \cup \{(a, b)\}$ if $b\ P_i\ a$ and $R'_i = R_i \setminus \{(b, a)\}$ otherwise. That is, $R_{i:(a,b)}$ is identical to R except that alternative a is (weakly) strengthened with respect to b within voter i's preference relation.

A standard property of SCFs that is often considered is monotonicity. An SCF is monotonic if a chosen alternative remains in the choice set when it is strengthened in individual preference relations while leaving everything else unchanged.

Definition 2. An SCF f is *monotonic* if for all feasible sets A, preference profiles R, voters i, and alternatives $a, b \in A$, $a \in f(R, A)$ implies $a \in f(R'_{i:(a,b)}, A)$.

The strong superset property requires that a choice set is invariant under the removal of unchosen alternatives (Chernoff, 1954; Bordes, 1979; Aizerman and Aleskerov, 1995).

Definition 3. An SCF f satisfies the *strong superset property* (SSP) if for all feasible sets A, B and preference profiles R such that $f(R, A) \subseteq B \subseteq A$, $f(R, A) = f(R, B)$.

An SCF satisfies set-independence if the choice set is invariant under modifications of the preference profile with respect to unchosen alternatives (Laslier (1997) used the natural analog of this definition in the context of tournament solutions).

Definition 4. An SCF f satisfies *set-independence* if for all feasible sets A, preference profiles R, voters i, and alternatives $a, b \in A \setminus f(R, A)$, $f(R, A) = f(R_{i:(a,b)}, A)$.

The following proof is adapted from Laslier (1997), who showed the equivalent statement for tournament solutions.

Proposition 1. *Monotonicity and SSP imply set-independence.*

Proof. We show that every monotonic SCF f that satisfies SSP also satisfies set-independence. Let A be a feasible set, R a preference profile, i a voter, and $a, b \in A \setminus f(R, A)$. Furthermore, let $R' = R_{i:(b,a)}$. In case $a \in f(R', A)$, monotonicity yields a contradiction because a is strengthened in R but $a \notin f(R, A)$. Therefore, $a \notin f(R', A)$. SSP implies that $f(R, A) = f(R, A \setminus \{a\})$ and $f(R', A) = f(R', A \setminus \{a\})$. Moreover, $f(R, A \setminus \{a\}) = f(R', A \setminus \{a\})$ since R and R' are completely identical on $A \setminus \{a\}$. Hence, $f(R, A) = f(R', A)$ and f satisfies set-independence. $\square$

[6] This definition incorporates an independence condition that Bordes (1976) refers to as *independence of irrelevant alternatives (IIA)* and that resembles Arrow's IIA condition for social welfare functions.

3.2 Strategyproofness

An SCF is manipulable if one or more voters can misrepresent their preferences in order to obtain a more preferred outcome. Whether one choice set is preferred to another depends on how the preferences over individual alternatives are to be extended to sets of alternatives. In the absence of information about the mechanism that eventually picks a single alternative from any choice set, preferences over choice sets are typically obtained by the conservative extension $\widehat{R}_i$ (Barberà, 1977; Kelly, 1977), where for any pair of feasible sets A and B and preference relation R_i,

$$A \ \widehat{R}_i \ B \text{ if and only if } a \ R_i \ b \text{ for all } a \in A \text{ and } b \in B.$$

Clearly, in all but the simplest cases, $\widehat{R}_i$ is incomplete, i.e., many pairs of feasible sets are incomparable. $\widehat{P}_i$ denotes the strict part of relation $\widehat{R}_i$, i.e., $A \ \widehat{P}_i \ B$ if and only if $A \ \widehat{R}_i \ B$ and $a \ P_i \ b$ for at least one pair of $a \in A$ and $b \in B$.

Definition 5. An SCF f is *manipulable by a group of voters* $G \subseteq N$ if there exists a feasible set A and preference profiles R, R' with $R_i = R'_i$ for all $i \notin G$ such that

$$f(R', A) \ \widehat{P}_i \ f(R, A) \text{ for all } i \in G.$$

An SCF is *strategyproof* if it is not manipulable by single voters. An SCF is *group-strategyproof* if it is not manipulable by any group of voters.

It will turn out that many SCFs that fail to be strategyproof can only be manipulated by breaking ties strategically, i.e., voters can obtain a more preferred outcome by only misrepresenting their *indifference* relation. In many settings, for instance when the choice infrastructure requires a strict ranking of the alternatives, this may be deemed acceptable. Please observe that letting voters misrepresent their indifference relation is a weaker requirement than simply assuming that voters have linear preferences, which is often made in other results on strategyproofness (see, e.g., Taylor, 2005). Accordingly, we obtain the following definition.

Definition 6. An SCF is *strongly manipulable by a group of voters* $G \subseteq N$ if there exists a feasible set A and preference profiles R, R' with $R_i = R'_i$ for all $i \notin G$ and $I_i \subseteq I'_i$ for all $i \in G$ such that

$$f(R', A) \ \widehat{P}_i \ f(R, A) \text{ for all } i \in G.$$

An SCF is *weakly group-strategyproof* if it is not strongly manipulable by any group of voters.

In other words, every strongly manipulable SCF admits a manipulation in which voters only misrepresent their strict preferences.[7]

4 Results

We will present three main results. First, we show that no Condorcet extension is group-strategyproof. The proof of this claim, however, relies on breaking ties strategically. We therefore study *weak* group-strategyproofness and obtain a much more positive characterization result. Finally, we show that the two conditions used in our characterization are necessary and sufficient in the case of pairwise SCFs.

[7]Besides characterizing a class of SCFs that does not admit a strong manipulation, Theorem 2 shows something stronger about this class: In every manipulation where voters misrepresent strict preferences as well as indifferences, modifying the strict preferences is not necessary. The same outcome can be obtained by only misrepresenting the indifference relation.

2	...	2	1	...	1	1
$a_2, \ldots, a_m$	...	$a_1, \ldots, a_{m-1}$	$a_3, \ldots, a_m$	...	$a_1, \ldots, a_{m-2}$	$a_2, \ldots, a_{m-1}$
			a_1	...	a_{m-1}	a_m
a_1	...	a_m	a_2	...	a_m	a_1

Table 1: Preference profile R for $3m$ voters where $A = \{a_1, \ldots, a_m\}$

4.1 Manipulation of Condorcet Extensions

We begin by showing that all Condorcet extensions are weakly manipulable, which strengthens previous results by Gärdenfors (1976) and Taylor (2005) who showed the same statement for a weaker notion of manipulability and weak Condorcet extensions, respectively.[8]

Theorem 1. *Every Condorcet extension is manipulable when there are more than two alternatives.*

Proof. Let $A = \{a_1, \ldots, a_m\}$ with $m \geq 3$ and consider the preference profile R given in Table 1. For every alternative a_i, there are two voters who prefer every alternative to a_i and who are indifferent between the other alternatives. Moreover, there is one voter for every alternative a_i who ranks a_{i+1} below a_i and prefers every other alternative to both of them. Again, the voter is completely indifferent between these other alternatives.

Since $f(R, A)$ yields a non-empty choice set, there has to be some $1 \leq i \leq m$ such that $a_i \in f(R, A)$. Let $j = ((i - 2) \mod m) + 1$. Now, let R' be identical to R, except that the preferences of voter $2i - 1$ (i.e., the first voter who ranks a_i last) changed such that $a_j \; P'_{2i-1} \; a_k$ for all $k \neq j$. Furthermore, let R'' be identical to R, except that the preferences of voters $2i-1$ and $2i$ (i.e., the first two voters who rank a_i last) changed such that $a_j \; P''_{2i} \; a_k$ for all $k \neq j$.

In the case that $a_i \notin f(R', A)$, voter $2i - 1$ can manipulate as follows. Suppose R is the true preference profile. Then, the least favorable alternative of voter $2i - 1$ is chosen (possibly among other alternatives). He can misstate his preferences as in R' such that a_i is not chosen. Since he is indifferent between all other alternatives, $f(R', A) \; \widehat{P}_{2i-1} \; f(R, A)$.

If $a_i \in f(R', A)$, voter $2i$ can manipulate similarly. Suppose R' is the true preference profile. Again, the least favorable alternative of voter $2i$ is chosen. By misstating his preferences as in R'', he can assure that one of his preferred alternatives, namely a_j, is selected exclusively. This is the case because a_j is the Condorcet winner in R''. Hence, $f(R'', A) \; \widehat{P}'_{2i} \; f(R', A)$.

□

4.2 Weakly Group-Strategyproof SCFs

The previous statement showed that no Condorcet extension is group-strategyproof. For our characterization of *weakly* group-strategyproof SCFs, we require set-independence and a new property that we call *set-monotonicity*. Set-monotonicity requires that a choice set should be invariant under the strengthening of chosen alternatives with respect to unchosen ones.

[8] A weak Condorcet winner is an alternative that is preferred by *at least* as many voters than any other alternative in pairwise comparisons. In contrast to Condorcet winners, weak Condorcet winners need not be unique. An SCF is called a *weak Condorcet extension* if it chooses the set of weak Condorcet winners whenever this set is non-empty. A large number of reasonable Condorcet extensions (including the minimal covering set and the bipartisan set) are not weak Condorcet extensions. Taylor (2005) calls the definition of weak Condorcet extensions "really quite strong" and refers to Condorcet extensions as "much more reasonable."

Definition 7. An SCF f is *set-monotonic* if for all feasible sets A, preference profiles R, voters i, and alternatives $a \in f(R,A)$, $b \in A \setminus f(R,A)$, $f(R,A) = f(R_{i:(a,b)},A)$.

The conjunction of set-independence and set-monotonicity is stronger than monotonicity.

Proposition 2. *Set-independence and set-monotonicity imply monotonicity.*

Proof. Let f be a set-monotonic SCF, A a feasible set, R a preference profile, i a voter, and $a, b \in A$ such that $a \in f(R,A)$. Furthermore, let $R' = R_{i:(a,b)}$. Clearly, in case $b \notin f(R,A)$, set-monotonicity implies that $f(R',A) = f(R,A)$ and thus $a \in f(R',A)$. If, on the other hand, $b \in f(R,A)$, assume for contradiction that $a \notin f(R',A)$. If $b \in f(R',A)$, b is strengthened with respect to outside alternative a when moving from R' to R, and set-monotonicity again implies that $f(R,A) = f(R',A)$. Otherwise, if $b \notin f(R',A)$, it follows from set-independence that $f(R,A) = f(R',A)$, a contradiction. $\square$

Set-monotonicity can be connected to existing well-established properties via the following proposition, whose proof runs along the same lines as that of Proposition 1.

Proposition 3. *Monotonicity and SSP imply set-monotonicity.*

Proof. We show that every monotonic SCF f that satisfies SSP also satisfies set-monotonicity. Let A be a feasible set, R a preference profile, i a voter, $a \in f(R,A)$, and $b \in A \setminus f(R,A)$. Furthermore, let $R' = R_{i:(a,b)}$. In case $b \in f(R',A)$, monotonicity yields a contradiction because b is strengthened in R but $b \notin f(R,A)$. Therefore, $b \notin f(R',A)$. SSP implies that $f(R,A) = f(R,A \setminus \{b\})$ and $f(R',A) = f(R',A \setminus \{b\})$. Moreover, $f(R,A \setminus \{b\}) = f(R',A \setminus \{b\})$ because R and R' are completely identical on $A \setminus \{b\}$. As a consequence, $f(R,A) = f(R',A)$ and f satisfies set-monotonicity. $\square$

We are now ready to state the main result of this section.

Theorem 2. *Every SCF that satisfies set-monotonicity and set-independence is weakly group-strategyproof.*

Proof. Let f be an SCF that satisfies set-monotonicity and set-independence and assume for contradiction that f is not weakly group-strategyproof. Then, there has to be a feasible set A, a group of voters $G \subseteq N$, and two preference profiles R and R' with $R_i = R'_i$ for all $i \notin G$ and $I_i \subseteq I'_i$ for all $i \in G$ such that $f(R',A)\ \widehat{P}_i\ f(R,A)$ for all $i \in G$. We choose R and R' such that the union of the symmetric differences of individual preferences $\bigcup_{i \in N} (R_i \setminus R'_i) \cup (R'_i \setminus R_i)$ is inclusion-minimal, i.e., we look at a "smallest" counterexample in the sense that R and R' coincide as much as possible. Let $f(R,A) = X$ and $f(R',A) = Y$. Now, consider a pair of alternatives $a, b \in A$ such that, for some $i \in G$, $a\ P_i\ b$ and $b\ R'_i\ a$, i.e., voter i misrepresents his strict preference relation by strengthening b. The following case analysis will show that no such a and b exist, which implies that R and R' and consequently X and Y are identical, a contradiction.

Case 1 ($a, b \notin X$): It follows from set-independence that $R_{i:(b,a)}$ and R' yield a smaller counterexample since $f(R_{i:(b,a)},A) = f(R,A) = X$.

Case 2 ($a, b \notin Y$): It follows from set-independence that R and $R'_{i:(a,b)}$ yield a smaller counterexample since $f(R'_{i:(a,b)},A) = f(R',A) = Y$.

Case 3 ($a \in X$ and $b \in Y$): $Y\ \widehat{P}_i\ X$ implies that $b\ R_i\ a$, a contradiction.

Case 4 ($a \notin X$ and $b \in X$): It follows from set-monotonicity that $f(R_{i:(b,a)},A) = f(R,A) = X$. Consequently, $R_{i:(b,a)}$ and R' constitute a smaller counterexample.

Case 5 ($a \in Y$ and $b \notin Y$): It follows from set-monotonicity that $f(R'_{i:(a,b)}, A) = f(R', A) = Y$. Consequently, R and $R'_{i:(a,b)}$ constitute a smaller counterexample.

It is easily verified that this analysis covers all possible cases. Hence, R and R' have to be identical, which concludes the proof. □

As mentioned above, when assuming that voters have strict preferences, weak strategyproofness can be replaced with strategyproofness in Theorem 2.

Theorem 2 and Propositions 1 and 3 entail the following useful corollary.

Corollary 1. *Every monotonic SCF that satisfies SSP is weakly group-strategyproof.*

As mentioned in the introduction, there are few—but nevertheless quite attractive—SCFs that satisfy monotonicity and SSP, namely the top cycle, the minimal covering set, and the bipartisan set.[9]

4.3 Weakly Group-Strategyproof Pairwise SCFs

In this section, we identify a natural and well-known class of SCFs for which the characterization given in the previous section is complete. A SCF f is said to be based on pairwise comparisons (or simply *pairwise*) if, for all preference profiles R, R' and feasible sets A, $f(R, A) = f(R', A)$ if and only if

$$|\{i \in N \mid a\ P_i\ b\}| - |\{i \in N \mid b\ P_i\ a\}| = |\{i \in N \mid a\ P'_i\ b\}| - |\{i \in N \mid b\ P'_i\ a\}| \text{ for all } a, b \in A.$$

In other words, the outcome of a pairwise SCF only depends on the comparisons between pairs of alternatives (see, e.g., Young, 1974; Zwicker, 1991). The class of pairwise SCFs is quite natural and contains a large number of well-known voting rules such as Kemeny's rule, Borda's rule, Maximin, ranked pairs, and all rules based on simple majority rule (e.g., the Slater set, the uncovered set, the Banks set, the minimal covering set, and the bipartisan set). We now show that set-monotonicity and set-independence are *necessary* for the strategyproofness of pairwise SCFs.

Theorem 3. *Every weakly strategyproof pairwise SCF satisfies set-monotonicity and set-independence.*

Proof. We need to show that every pairwise SCF that fails to satisfy set-monotonicity or set-independence is strongly manipulable. Suppose SCF f does not satisfy set-monotonicity or set-independence. In either case, there exists a feasible set A, a preference profile R, a voter i, and two alternatives $a, b \in A$ with $a\ R_i\ b$ and $a \notin f(R, A) = X$ such that $f(R', A) = Y \neq X$ where $R' = R_{i:(b,a)}$. Let R_{n+1}, R_{n+2}, and R'_{n+2} be preference relations with indifferences between all pairs of alternatives except

$$
\begin{aligned}
&x\ P_{n+1}\ y \text{ for all } (x, y) \in (((X \setminus Y) \times Y) \cup (X \times (Y \setminus X))),\\
&y\ P_{n+2}\ x \text{ for all } (x, y) \in (((X \setminus Y) \times Y) \cup (X \times (Y \setminus X))) \setminus \{(b, a)\},\\
&a\ R_{n+2}\ b \text{ if and only if } a\ R_i\ b,\\
&b\ R_{n+2}\ a \text{ if and only if } b\ R_i\ a,\\
&y\ P'_{n+2}\ x \text{ for all } (x, y) \in (((X \setminus Y) \times Y) \cup (X \times (Y \setminus X))) \setminus \{(b, a)\},\\
&a\ R'_{n+2}\ b \text{ if and only if } a\ R'_i\ b, \text{ and}\\
&b\ R'_{n+2}\ a \text{ if and only if } b\ R'_i\ a.
\end{aligned}
$$

[9] SSP and monotonicity do not completely characterize weak strategyproofness. SCFs that satisfy set-monotonicity and set-independence but fail to satisfy SSP can easily be constructed.

We now define two preference profiles with $n+2$ voters where voter i is indifferent between a and b and the crucial change in preference between a and b has been moved to voter $n+2$. Let

$$S = (R_1, \dots, R_{i-1}, R_i \cup \{(b,a)\}, R_{i+1}, \dots, R_n, R_{n+1}, R_{n+2}) \text{ and}$$
$$S' = (R_1, \dots, R_{i-1}, R_i \cup \{(b,a)\}, R_{i+1}, \dots, R_n, R_{n+1}, R'_{n+2}).$$

Observe that all preferences between alternatives other than a and b cancel out each other in the preference relations of voter $n+1$ and $n+2$. It thus follows from the definition of pairwise SCFs that $f(S,A) = f(R,A) = X$ and $f(S',A) = f(R',A) = Y$. If $X \cup Y \neq \{a,b\}$ or $a\ P_i\ b$, we have $Y\ \widehat{P}_{n+2}\ X$ and f can be manipulated by voter $n+2$ at preference profile S by misstating his strict preference $a\ P_{n+2}\ b$ as $a\ I'_{n+2}\ b$. If, on the other hand, $X \cup Y = \{a,b\}$ and $a\ I_i\ b$, we have $X\ \widehat{P}'_{n+2}\ Y$ and f can be manipulated by voter $n+2$ at preference profile S' (by misstating his strict preference $b\ P'_{n+2}\ a$ as $a\ I_{n+2}\ b$). Hence, f is strongly manipulable. □

We can now completely characterize weak group-strategyproofness of pairwise SCFs using these two properties.

Corollary 2. *A pairwise SCF is weakly group-strategyproof if and only if it satisfies set-monotonicity and set-independence.*

This shows that many pairwise SCFs are not weakly group-strategyproof because they are known to fail set-independence (Laslier, 1997). Notable exceptions are the top cycle, the minimal covering set, and the bipartisan set mentioned above.

Brams and Fishburn (1983) introduced a particularly natural variant of strategic manipulation where voters obtain a more preferred outcome by abstaining the election. A SCF is said to satisfy *participation* if voters are never better off by abstaining. A common criticism of Condorcet extensions is that they do not satisfy participation and thus suffer from the so-called *no-show paradox* (Moulin, 1988). However, Moulin's proof strongly relies on resoluteness. Irresolute Condorcet extensions that satisfy participation do exist and, in the case of pairwise SCFs, there is a close connection between strategyproofness and participation as shown by the following simple observation.[10]

Proposition 4. *Every strategyproof pairwise SCF satisfies participation.*

Proof. Let f be a pairwise SCF that fails participation, i.e., there exists a feasible set A, a preference profile R, and a preference relation R_{n+1} such that $f(R,A)\ \widehat{P}_{n+1}\ f((R_1, \dots, R_n, R_{n+1}), A)$. Let furthermore R'_{n+1} be a preference relation that expresses complete indifference over all alternatives. Since f is pairwise, $f((R_1, \dots, R_n, R'_{n+1}), A) = f(R,A)$ and f can be manipulated at profile $(R_1, \dots, R_n, R_{n+1})$ by voter $n+1$ because by changing his preferences to R'_{n+1} he obtains the more preferred outcome $f(R,A)$. □

It follows that all SCFs satisfying set-monotonicity and set-independence, which includes the Condorcet extensions mentioned earlier, satisfy participation according to Kelly's preference extension.

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[10]Proposition 4 holds for any preference extension, not just Kelly's.

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