Dependence in Games and Dependence Games

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Abstract

The paper provides a formal analysis of a notion of dependence between players in a game. We will show: first, how this notion of dependence allows for an elegant characterization of a property of reciprocity for the outcomes of a game; and second, how it can be used to ground new cooperative solution concepts for strategic games, where coalitions can force outcomes only in the presence of reciprocal dependences.

1 Introduction

The paper outlines a theory of dependence for strategic games. It moves from the following definition of dependence, inspired by foundational literature on multi-agent systems (see for instance [2, p.4]): player i depends on player j for reaching outcome s, within a given game, if and only if j plays a strategy, in the profile determining s, which is a best response (or a dominant strategy) not for j itself, but instead for i (Definition 8).

The aim of the paper is to provide a thorough analysis of the above definition. Concretely, it presents two results. First, it shows that this notion of dependence allows for the characterization of an original notion of *reciprocity* for strategic games (Theorem 1). Second, it shows that this notion of dependence can be fruitfully applied to ground cooperative solution concepts. These solution concepts are characterizable as the core of a specific class of coalitional games—here called *dependence games*—where coalitions can force outcomes only in the presence of reciprocity (Theorems 2 and 3). The paper generalizes and extends results presented in [3].

2 Dependence in games

The section introduces some preliminary notions and notation from game theory and proceeds to the definition and analysis of the notion of dependence.

2.1 Preliminary definitions and notation

The present section introduces the basic game-theoretic notions used in the paper. All definitions will be based on an ordinal notion of preference. Our main sources are [5] and [4].

Definition 1 (Game) *A (strategic form) game is a tuple $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ where: N is a set of players; S is a set of outcomes; Σ_i is a set of strategies for player $i \in N$; $\succeq_i$ is a total preorder on S; $o : \bigtimes_{i \in N} \Sigma_i \rightarrow S$ is a bijective function from the set of strategy profiles to S. Strategy profiles will be denoted $\sigma, \sigma', \ldots$.*

We will also make use of the notion of sub-game.

Definition 2 (Sub-game of a strategic game) *Let $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ be a game, σ be a strategy profile, and $C \subseteq N$. The subgame of $\mathcal{G}$ defined by σ_C is a game $\mathcal{G} \downarrow \sigma_c = (N', S', \Sigma'_i, \succeq'_i, o')$ such* *that:* $N' = N - C$; $S' = S - \{s \mid \exists\sigma' \text{ s.t. } s = o(\sigma') \text{ and } \sigma'_C \neq \sigma_C\}$; *for all* $i \in N - C$, $\Sigma'_i = \sigma_i$; *for all* $i \in N - C$, $\Sigma'_i = \succeq_i$; $o' : \bigtimes_{i \in N-C} \Sigma_i \to S'$ *is a bijection such that for all* $\sigma' \in \bigtimes_{i \in N-C} \Sigma_i$, $o'(\sigma') = o(\sigma', \sigma_C)$.

To put it in words, a subgame of $\mathcal{G}$ is nothing but what it is obtained from $\mathcal{G}$ once the strategies of a set of players in C are fixed, or what is still 'left to play' once the players in C have made their choice.

As to the solution concepts, we will work with Nash equilibrium, which we will refer to also as best response equilibrium (BR-equilibrium), and the dominant strategy equilibrium (DS-equilibrium).

Definition 3 (Equilibria) *Let* $\mathcal{G}$ *be a game. A strategy profile* σ *is: a BR-equilibrium (Nash equilibrium) iff* $\forall i \in N, \forall \sigma'_i \in \Sigma_i : o(\sigma) \succeq_i o(\sigma'_i, \sigma_{-i})$; *it is a DS-equilibrium iff* $\forall i \in N, \sigma' \in \bigtimes_{i \in N} \Sigma_i : o(\sigma_i, \sigma'_{-i}) \succeq_i o(\sigma')$.

In addition to the games in strategic form (Definition 1) we will also work with coalitional games, i.e., cooperative games with non-transferable pay-offs abstractly represented by effectivity functions [4].

Definition 4 (Coalitional game) *A coalitional game is a tuple* $\mathcal{C} = (N, S, E, \succeq_i)$ *where:* N *is a set of players;* S *is a set of outcomes;* E *is function* $E : 2^N \to 2^{2^S}$; $\succeq_i$ *is a total preorder on* S.

An effectivity function associates to a coalition a set of sets of outcomes and the fact that $X \in E(C)$ is usually understood as the coalition C being able to force the interaction to end up in an outcome in X. This intuition can be given a concrete semantics in terms of strategic games, from which a coalitional games can be obtained in a canonical way (cf. [4]). These games in particular will be object of study in Section 3.

Definition 5 (Coalitional games from strategic ones) *Let* $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ *be a game. The coalitional game* $\mathcal{C}^{\mathcal{G}} = (N, S, E^{\mathcal{G}}, \succeq_i)$ *of* $\mathcal{G}$ *is a coalitional game where the effectivity function* $E^{\mathcal{G}}$ *is defined as follows:*

$$X \in E^{\mathcal{G}}(C) \Leftrightarrow \exists\sigma_C \forall\sigma_{\overline{C}}\ o(\sigma_C, \sigma_{\overline{C}}) \in X.$$

As it can be observed from the translation, the effectivity function of $\mathcal{C}^{\mathcal{G}}$ contains those sets in which a coalition C can force the game to end up, no matter what strategies $\overline{C}$ decides to play.

Finally we consider the standard solution concept for coalitional games.

Definition 6 (The Core) *Let* $\mathcal{C} = (N, S, E, \succeq_i)$ *be a coalitional game. We say that a state* s *is dominated in* $\mathcal{C}$ *if for some* C *and* $X \in E(C)$ *it holds that* $x \succ_i s$ *for all* $x \in X, i \in C$. *The core of* $\mathcal{C}$, *in symbols* $CORE(\mathcal{C})$ *is the set of undominated states.*

Intuitively, the core is the set of those states in the game that are stable, i.e., for which there is no coalition that is at the same time able and interested to deviate from them.

2.2 Dependence

We will work with the following notion of dependence: a player i depends on a player j for the realization of an outcome s, i.e., of the strategy profile σ such that $o(\sigma) = s$, when, in order for σ to occur, j has to favour i, that is, it has to play in i's interest. To put it otherwise, i depends on j for σ when, in order to achieve σ, j has to do a favour to i by playing σ_j (which is obviously not under i's control). This intuition is made clear in the following definition:

	g	$\neg g$
g	3,3,3	2,4,2
$\neg g$	4,2,2	1,1,0

g

	g	$\neg g$
g	2,2,4	0,1,1
$\neg g$	1,0,1	1,1,1

$\neg g$

Figure 1: A three person Prisoner's dilemma.

Definition 7 (Best for someone else) *Assume a game $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ and let $i, j \in N$. 1) Player j's strategy in σ is a* best response *for i iff $\forall \sigma'$, $o(\sigma) \succeq_i o(\sigma'_j, \sigma_{-j})$. 2) Player j's strategy in σ is a* dominant strategy *for i iff $\forall \sigma'$, $o(\sigma_j, \sigma'_{-j}) \succeq_i o(\sigma')$.*

Definition 7 generalizes the standard definitions of best response and dominant strategy by allowing the player holding the preference to be different from the player whose strategies are considered.

Definition 8 (Dependence) *Let $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ be a game and $i, j \in N$. 1) Player i* BR-depends *on j for strategy σ—in symbols, $iR^{BR}_\sigma j$—if and only if σ_j is a best response for i in σ. 2) Player i* DS-depends *on j for strategy σ—in symbols, $iR^{DS}_\sigma j$—if and only if σ_j is a dominant strategy for i.*

Intuitively, i depends on j for profile σ in a best response sense if, in σ, j plays a strategy which is a best response for i given the strategies in σ_{-j} (and hence given the choice of i itself), and similarly for dominant strategy dependence.

In general, relations R^{BR}_σ and R^{DS}_σ do not enjoy any particular structural property. However, the following simple fact is of interest as it shows a direct connection between dependence graphs and underlying games.

Fact 1 (Reflexive dependencies) *Let $\mathcal{G}$ be a game and (N, R^x_σ) be its dependence structure for outcome σ with $x \in \{BR, DS\}$. It holds that R^x_σ is reflexive iff σ is an x-equilibrium.*

The proof is omitted for space reasons. The relation of dependence acquires interest for cooperative interaction when a structural property, namely the presence of cycles, suggests the possibility of players acting for each other. The following three sections study this property.

2.3 Cycles

As also emphasized by related contributions (see for instance [1]), cycles in dependence graphs represent the possibility of social interaction between players of a *do-ut-des* (give-to-get) type. In a cycle, the first player of the cycle could be prone to do what the last player asks since it can obtain something from the second player who, in turn, can obtain something from the third and so on.

Definition 9 (Dependence cycles) *Let $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ be a game, (N, R^x_σ) be its dependence structure for profile σ with $x \in \{BR, DS\}$, and let $i, j \in N$. An R^x_σ-dependence cycle $\mathfrak{c}$ of length $k-1$ in $\mathcal{G}$ is a tuple $(a_1, \ldots, a_k)$ such that: $a_1, \ldots, a_k \in N$; $a_1 = a_k$; $\forall a_i, a_j$ with $1 \leq i \neq j < k$, $a_i \neq a_j$; $a_1 R^x_\sigma a_2 R^x_\sigma \ldots R^x_\sigma a_{k-1} R^x_\sigma a_k$. Given a cycle $\mathfrak{c} = (a_1, \ldots, a_k)$, its orbit $O(\mathfrak{c}) = \{a_1, \ldots, a_{k-1}\}$ denotes the set of its elements.*

In other words, cycles are sequences of pairwise different players, except for the first and the last which are equal, such that all players are linked by a dependence relation. Note that the definition allows for cycles of length 1, whose orbit is a singleton, i.e., loops. Those are the cycles occurring at reflexive points in the graph.

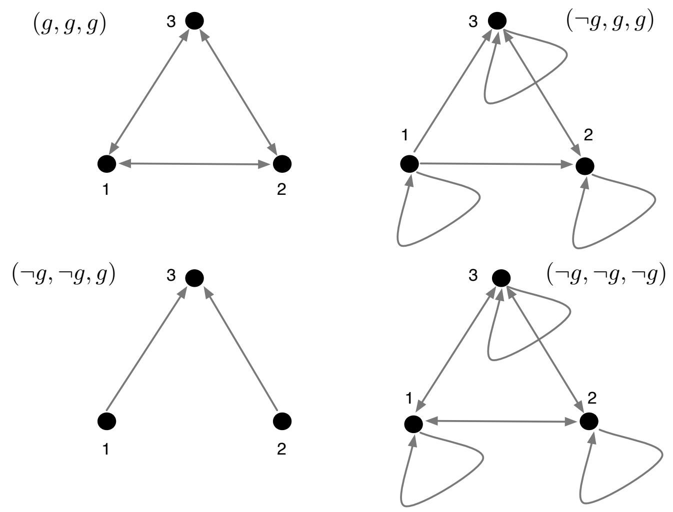

Figure 2: Some BR-dependences of Example 1.

Cycles become of particular interest in games with more than two players, so let us illustrate the definition by the following example.

Example 1 (Cycles in three person games.) *Consider the following three-person variant of the Prisoner's dilemma. A committee of three juries has to decide whether to declare a defendant in a trial guilty or not. All the three juries want the defendant to be found guilty, however, all three prefer that the others declare her guilty while she declares her innocent. Also, they do not want to be the only ones declaring her guilty if the other two declare her innocent. They all know each other's preferences. Figure 1 gives a payoff matrix for such game. Figure 2 depicts some cyclic BR-dependencies inherent in the game presented. Player 1 is Row, player 2 Column, and player 3 picks the right or left table. Among the ones depicted, the reciprocal profiles are clearly* (g, g, g), $(\neg g, \neg g, \neg g)$ *(which is also universal) and* $(\neg g, g, g)$, *only the last two of which are Nash equilibria (reflexive). Looking at the cycles present in these BR-reciprocal profiles, we notice that* (g, g, g) *contains the* 2×3 *cycles of length 3, all yielding the partition* $\{\{1, 2, 3\}\}$ *of the set of players* $\{1, 2, 3\}$. *Profile* $(\neg g, g, g)$, *instead, yields two partitions:* $\{\{1\}, \{2\}, \{3\}\}$ *and* $\{\{1\}, \{2, 3\}\}$. *The latter is determined by the cycles* $(1, 1)$ *and* $(2, 3, 2)$ *or* $(1, 1)$ *and* $(3, 2, 3)$. *Finally, profile* $(\neg g, \neg g, g)$ *is such that both 1 and 2 depend on 3. Yet, neither of them plays a best response strategy.*

The notion of reciprocity obtains a formal definition in the following section.

2.4 Reciprocity

Depending on the properties of the dependence cycles of a given profile, we can distinguish between several notions of reciprocity capturing different ways in which players are interconnected via a dependence structure.

Definition 10 (Types of reciprocity in profiles) *Let* $\mathcal{G}$ *be a game and* (N, R^x_σ) *be its dependence structure with* $x \in \{BR, DS\}$ *and* σ *be a profile, and* $C \subseteq N$.

i) A profile σ *is* x-reciprocal *if and only if there exists a partition* $P(N)$ *of* N *such that each element* p *of the partition is the orbit of some* R^x_σ*-cycle, i.e., a cycle in the directed graph* (N, R^x_σ);

ii) A profile σ is partially x-reciprocal in C *(or C-x-reciprocal) if and only if C is the orbit of some* R^x_σ*-cycle, i.e., a cycle in the directed graph* (N, R^x_σ)*;*

iii) A profile σ is trivially x-reciprocal *if and only if it yields only x-cycles whose orbits are singletons;*

iv) A profile σ is fully x-reciprocal *if and only if it yields an x-cycle with orbit N (i.e., a Hamiltonian cycle) or, equivalently, if and only if it is N-x-reciprocal.*

Let us explain the above definitions by referring to BR-dependence. A profile σ is BR-reciprocal if all players belong to some cycle of BR-dependence. Along the same lines, a profile σ is partially BR-reciprocal in coalition C (or C-BR-reciprocal) if the all the members of C are linked by a cycle of BR-dependence. This means, intuitively, that independently on whether the players outside of coalition C are linked by dependencies or not, the members of C are in a situation of reciprocity in which everybody plays a dominant strategy for somebody else in the coalition. To put it yet otherwise, a profile is reciprocal when the corresponding dependence relation, be it a *BR*- or *DS*-dependence, clusters the players into non-overlapping groups whose members are all part of some cycle of dependencies (including degenerate ones such as reflexive links). It is partially reciprocal if its dependence graph contains at least one cycle. Finally, trivial and full *BR*-reciprocity refer to two extreme cases of *BR*-reciprocity. In the first case the cycles are loops, that is, all players play their own dominant strategy, in the second case there exists one Hamiltonian cycle, that is, all players are connected to one another by a path of *BR*-dependence. For example, inspecting the BR-dependencies in the Prisoner Dilemma (Figure 3) it can be observed that: (U, L) is fully BR-reciprocal, (D, R) is trivially *BR*-reciprocal, (U, R) is $\{2\}$-BR-reciprocal and (D, L) is $\{1\}$-BR-reciprocal.

It is worth noting that x-reciprocity is a more demanding requirement than C-x-reciprocity as it is easy to see that if σ is x-reciprocal, then for each $C \in P(N)$ σ is C-x-reciprocal. Also, here below we report a few simple but relevant facts concerning the logical relationship between DS- and BR-reciprocity.

Fact 2 (DS- vs. BR-reciprocity) *Let* $\mathcal{G}$ *be a game and* (N, R^x_σ) *be its dependence structure with* $x \in \{BR, DS\}$*, σ be a profile, and* $C \subseteq N$*. The following holds:*

i) σ is C-BR-reciprocal iff σ_C *is BR-reciprocal in* $\mathcal{G} \downarrow \sigma_{\overline{C}}$*;*

ii) σ is C-BR-reciprocal iff σ_C *is DS-reciprocal in* $\mathcal{G} \downarrow \sigma'_{\overline{C}}$ *for any* σ'*;*

iii) if σ is C-DS-reciprocal, then σ is C-BR-reciprocal, but not vice versa;

iv) if σ is DS-reciprocal, then σ is BR-reciprocal, but not vice versa.

The proof is omitted for space reasons. The first claim suggests that a *C-DS*-reciprocal profile σ can be referred to simply by the partial profile σ_C without loss of information. The second and third claims point out, as expected, that DS-reciprocity is a stronger notion than BR-reciprocity.

2.5 Reciprocity and equilibrium

We provide a characterization of reciprocity as defined in Definition 10 in terms of standard solution concepts. However, we first have to complement the set of notions provided in Section 2.1 with the notion of permuted game.

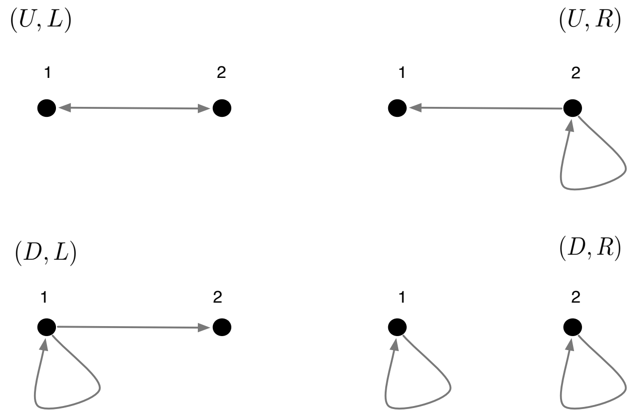

Figure 3: BR-dependences in the Prisoner Dilemma: 1 and 2 denote row and column.

Definition 11 (Permuted games) *Let $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ be a game and $\mu : N \mapsto N$ a bijection on N. The μ-permutation of game $\mathcal{G}$ is the game $\mathcal{G}^\mu = (N^\mu, S^\mu, \Sigma^\mu, \succeq_i^\mu, o^\mu)$ such that: $N^\mu = N$; $S^\mu = S$; for all $i \in N$, $\Sigma_i^\mu = \Sigma_{\mu(i)}$; for all $i \in N$, $\succeq_i^\mu = \succeq_i$; $o_\mu : \bigtimes_{i \in N} \Sigma_{\mu(i)} \rightarrow S$ is such that $o_\mu(\mu(\sigma)) = o(\sigma)$ with $\mu(\sigma)$ denoting the permutation of σ according to μ.*

Intuitively, a permuted game $\mathcal{G}^\mu$ is therefore a game where the strategies of each player are redistributed according to μ in the sense that i's strategies become $\mu(i)$'s strategies, where players keep the same preferences over outcomes, and where the outcome function assigns same outcomes to same profiles.

Example 2 (Two horsemen [6]) *"Two horsemen are on a forest path chatting about something. A passerby M, the mischief maker, comes along and having plenty of time and a desire for amusement, suggests that they race against each other to a tree a short distance away and he will give a prize of \$100. However, there is an interesting twist. He will give the \$100 to the owner of the slower horse. Let us call the two horsemen Bill and Joe. Joe's horse can go at 35 miles per hour, whereas Bill's horse can only go 30 miles per hour. Since Bill has the slower horse, he should get the \$100. The two horsemen start, but soon realize that there is a problem. Each one is trying to go slower than the other and it is obvious that the race is not going to finish. [...] Thus they end up [...] with both horses going at 0 miles per hour. [...] However, along comes another passerby, let us call her S , the problem solver, and the situation is explained to her. She turns out to have a clever solution. She advises the two men to switch horses. Now each man has an incentive to go fast, because by making his competitor's horse go faster, he is helping his own horse to win!" [6, p. 195-196].*

Once the game of the example is depicted as the left-hand side game in Figure 4, it is possible to view the second passerby's solution as a bijection μ which changes the game to the right-hand side version. Now Row can play Column's moves and Column can play Row's moves. The result is a swap of (D, L) with (U, R), since (D, L) in $\mathcal{G}^\mu$ corresponds

	L	R
U	0, 0	1, 0
D	0, 1	1, 0

$\mathcal{G}$

	L	R
U	0, 0	0, 1
D	1, 0	1, 0

$\mathcal{G}^\mu$

Figure 4: The two horsemen game and its transposition.

to (U, R) in $\mathcal{G}$ and vice versa. On the other hand, (U, L) and (D, R) stay the same, as the exchange of strategies do not affect them. As a consequence, profile (D, R), in which both horsemen engage in the race, becomes a dominant strategy equilibrium.

On the ground of these intuitions, it is possible to obtain a simple characterization of the different notions of reciprocity given in Definition 10 as the existence of equilibria in appropriately permuted games.

Theorem 1 (Reciprocity in equilibrium) *Let $\mathcal{G}$ be a game and (N, R^x_σ) be its dependence structure with $x \in \{BR, DS\}$ and σ be a profile. It holds that:*

i) σ is x-reciprocal iff there exists a bijection $\mu : N \mapsto N$ s.t. σ is a x-equilibrium in the permuted game $\mathcal{G}^\mu$;

ii)
- *σ is partially BR-reciprocal in C (or C-BR-reciprocal) iff there exists a bijection $\mu : C \mapsto C$ s.t. σ_C is a BR-equilibrium in the permuted subgame $(\mathcal{G} \downarrow \sigma_{\overline{C}})^\mu$;*
- *σ is partially DS-reciprocal in C (or C-DS-reciprocal) iff there exists a bijection $\mu : C \mapsto C$ s.t. σ_C is a DS-equilibrium in all permuted subgames $(\mathcal{G} \downarrow \rho_{\overline{C}})^\mu$, for $\rho_{\overline{C}} \in \bigtimes_{j \in \overline{C}} \rho_j$ and $\rho_j \in \Sigma_j$;*

iii) σ is trivially x-reciprocal iff σ is an x-equilibrium in $\mathcal{G}^\mu$ where μ is the identity over N;

iv) σ is fully x-reciprocal iff there exists a bijection $\mu : N \mapsto N$ s.t. σ is a x-equilibrium in the permuted game $\mathcal{G}^\mu$ and μ is such that $\{(i, j) \mid i \in N \;\&\; j = \mu(i)\}$ is a Hamiltonian cycle in N.

The proof is omitted for space reasons. From the foregoing result, it follows that permutations can be fruitfully viewed as ways of *implementing*—in a social software sense [6]—a reciprocal profile. This is terminology is worth casting in the following definition.

Definition 12 (Implementation as game permutation) *Let $\mathcal{G}$ be a game, (N, R^x_σ) be its dependence structure in σ with $x \in \{BR, DS\}$, and $\mu : N \mapsto N$ and $\mu' : C \mapsto C$ with $C \subseteq N$ be two bijections. We say that:*

i) μ x-implements σ iff σ is an x-equilibrium in $\mathcal{G}^\mu$;

ii)
- *μ' C-BR-implements σ iff σ_C is an BR-equilibrium in $(\mathcal{G} \downarrow \sigma_{\overline{C}})^{\mu'}$;*
- *μ' C-DS-implements σ iff σ_C is a DS-equilibrium in all $(\mathcal{G} \downarrow \rho_{\overline{C}})^{\mu'}$.*

Intuitively, implementation is here understood as a way of transforming a game in such a way that the desirable outcomes, in the transformed game, are brought about at an equilibrium point. In this sense we talk about *BR*- or *DS*-implementation. The difference between the two arises in the implementation of partial agreements where the locality of partial BR-reciprocity becomes apparent vis-à-vis the global character of partial DS-reciprocity.

3 Solving dependencies: dependence games

The previous section has shown how reciprocity can be given two corresponding formal characterization: existence of cycles in a dependence structure, and existence of equilibria in a suitably permuted game (Theorem 1). In the present section we take the notion of reciprocity as the basis upon which to define two new solution concepts, of a cooperative kind, for games in strategic form.

	L	R
U	2,2	0,3
D	3,0	1,1

$\mathcal{G}^{\mu}$

	L	R
U	2,2	3,0
D	0,3	1,1

$\mathcal{G}^{\nu}$

Figure 5: Agreements between the prisoners

3.1 Agreements

The intuition is that, given a reciprocal profile (of some sort according to Definition 10), the players can fruitfully *agree* to transform the game by some suitable permutation of strategy sets.

Definition 13 (Agreements and partial agreements) *Let $\mathcal{G}$ be a game, (N, R^x_σ) be its dependence structure in σ with $x \in \{BR, DS\}$, and let $i, j \in N$. A pair (σ, μ) is:*

i) an x-agreement for $\mathcal{G}$ if σ is an x-reciprocal profile, and $\mu : N \mapsto N$ a bijection which x-implements σ;

ii) a partial x-agreement in C (or a C-x-agreement) for $\mathcal{G}$, if σ is a C-x-reciprocal profile and $\mu : C \mapsto C$ a bijection which C-x-implements σ.

The set of x-agreements of a game $\mathcal{G}$ is denoted x-AGR($\mathcal{G}$) and the set of partial x-agreements, that is the set of pairs (σ, μ) for which there exists a C such that μ C-x-implements σ, is denoted x-pAGR($\mathcal{G}$).

Intuitively, a (partial) agreement, of BR or DS type, can be seen as the result of coordination (endogenous, via the players themselves, or exogenous, via a third party like in Example 2) selecting a desirable outcome and realizing it by an appropriate exchange of strategies.

Example 3 (Agreements in PD) *In the game Prisoner's Dilemma two DS-agreements can be observed, whose permutations give rise to the games depicted in Figure 5. Agreement $((D, R), \mu)$ with $\mu(i) = i$ for all players, is the standard DS-equilibrium of the strategic game. But there is another possible agreement, where the players swap their strategies: it is $((U, L), \nu)$, for which $\nu(i) = N \backslash \{i\}$. Here Row plays cooperatively for Column and Column plays cooperatively for Row. Of the same kind is the agreement arising in Example 2. Notice that in such example, the agreement is the result of coordination mediated by a third party (the second passerby). Analogous considerations can also be done about Example 1 where, for instance, $((g, g, g), \mu)$ with $\mu(1) = 2, \mu(2) = 3, \mu(3) = 1$ is a BR- agreement.*

As we might expect, BR- and DS-agreements are related in the same way as BR- and DS-reciprocity (Fact 2). In what follows we will focus only on DS-agreements and partial DS-agreements so, whenever we talk about agreements and partial agreements, we mean DS-agreements and partial DS-agreements, unless stated otherwise.

3.2 Dominance

As there can be several possible agreements in a game, the natural issue arises of how to order them. We will do that by defining a natural notion of dominance between agreements, but first we need some auxiliary notions.

Definition 14 (C-candidates and C-variants) *Let $\mathcal{G} = (N, S, \Sigma_i, \geq_i, o)$ be a game and C a non-empty subset of N. An agreement (σ, μ) for $\mathcal{G}$ is a* C-candidate *if C is the union of some members of the partition induced by μ, that is: $C = \bigcup X$ where $X \subseteq P_\mu(N)$. An agreement (σ, μ) for $\mathcal{G}$ is a*

C-variant *of an agreement* (σ', μ') *if* $\sigma_C = \sigma'_C$ *and* $\mu_C = \mu'_C$*, where* μ_C *and* μ'_C *are the restrictions of* μ *to* C*. As a convention we take the set of* $\emptyset$*-candidate agreements to be empty and an agreement* (σ, ν) *to be the only* $\emptyset$*-variant of itself.*

In other words, an agreement (σ, μ) is a C-candidate if $\{C, \overline{C}\}$ is a bipartition of $P_\mu(N)$, and it is a C-variant of (σ', μ') if it differs from this latter at most in its C-part. We can now define the following notions of dominance between agreements and between partial agreements.

Definition 15 (Dominance) *Let* $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ *be a game and* $C \subseteq N$ *be a coalition. We say that:*

i) An agreement (σ, μ) *is dominated iff for some coalition* C *there exists a* C*-candidate agreement* (σ', μ') *for* $\mathcal{G}$ *such that for all agreements* (ρ, ν) *which are* $\overline{C}$*-variants of* (σ', μ')*,* $o(\rho) \succ_i o(\sigma)$ *for all* $i \in C$*.*

ii) A partial agreement (σ_C, μ) *in* C *is dominated iff for some coalition* $D \subseteq N$ *there exists* (τ_D, ν) *which is a* D*-agreement such that for all* σ', τ'*,* $o(\tau_D, \tau'_{\overline{D}}) \succ_i o(\sigma_C, \sigma'_{\overline{C}})$ *for all* $i \in D$*.*

The set of undominated agreements of $\mathcal{G}$ *is denoted* $DEP(\mathcal{G})$ *and the set of undominated partial agreements is denoted* $pDEP(\mathcal{G})$*.*

Intuitively, an agreement is undominated when a coalition C can force all possible agreements to yield outcomes which are better for all the members of the coalition, regardless of what the rest of the players can agree to do, that is, regardless of the $\overline{C}$-variants of their agreements. A partial agreement in coalition C is undominated when C can, by means of a partial permutation, force the game to end up in a set of states which are better for the member of the coalition no matter what the players in $\overline{C}$ do.

It is worth stressing the critical difference between the two notions of dominance. This difference resides in the fact that while dominance between agreements only considers deviations which are the results of agreements, dominance between partial agreements considers any form of possible deviation.

Example 4 (Dominance between partial agreements) *In the three persons Prisoner Dilemma (see Figure 1),* $((g_1, g_2), (\mu(1) := 2, \mu(2) := 1))$ *is a partial DS-agreement in* $\{1, 2\}$*. This agreement, which represents a form of dependence-based cooperation between* 1 *and* 2 *dominates the partial DS-agreement in* N*—on a trivially DS-reciprocal profile—*$((\neg g_1, \neg g_2, \neg g_3), (\mu(1) := 1, \mu(2) := 2, \mu(3) := 3))$*. In fact, it is undominated, since even the partial DS-agreement in* N $((g_1, g_2, g_3), (\mu(1) := 2, \mu(2) := 3, \mu(3) := 1))$ *(which is also a DS-agreement) does not dominate it.*

3.3 Dependence-based coalitional games

Now the question is, can we characterize the notion of dominance for agreements and partial agreements (Definition 15) in terms of a suitable notion of stability in appropriately defined games?

In order to answer this question we proceed as follows. First, starting from a game $\mathcal{G}$, we consider its representation $C^{\mathcal{G}}$ as a coalitional game as illustrated in Section 2.1 (Definition 5). As Definition 5 abstracts from dependence-theoretic considerations we refine it in two ways, corresponding to the two different sorts of dependence upon which we want to build the coalitional game:

1. The first refinement is obtained by defining a coalitional game $C^{\mathcal{G}}_{DEP}$ capturing the intuition that coalitions form only by means of *agreements* (Definition 13). Such games are called *dependence games*.

2. The second one is obtained by defining a coalitional game $C^{\mathcal{G}}_{pDEP}$ capturing the intuition that coalitions form only by means of *partial agreements* (Definition 13). Such games are called *partial dependence games*.

Having done this, we show that the core of $C^{\mathcal{G}}_{DEP}$ coincides with the set of undominated agreements of $\mathcal{G}$ (Theorem 2) and, respectively, that the core of $C^{\mathcal{G}}_{pDEP}$ coincides with the set of undominated partial agreements of $\mathcal{G}$ (Theorem 3). We thereby obtain a game-theoretical characterization of Definition 15.

3.3.1 Dependence games

Definition 16 (Dependence games from strategic ones) *Let $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ be a game. The dependence game $C^{\mathcal{G}}_{DEP} = (N, S, E^{\mathcal{G}}_{DEP}, \succeq_i)$ of $\mathcal{G}$ is a coalitional game where the effectivity function $E^{\mathcal{G}}_{DEP}$ is defined as follows:*

$$
\begin{aligned}
X \in E^{\mathcal{G}}_{DEP}(C) \quad \Leftrightarrow \quad & \exists \sigma_C, \mu_C \text{ S.T.} \\
& \exists \sigma_{\overline{C}}, \mu_{\overline{C}} : [((\sigma_C, \sigma_{\overline{C}}), (\mu_C, \mu_{\overline{C}})) \in AGR(\mathcal{G})] \\
& \text{AND } [\forall \sigma_{\overline{C}}, \mu_{\overline{C}} : [((\sigma_C, \sigma_{\overline{C}}), (\mu_C, \mu_{\overline{C}})) \in AGR(\mathcal{G}) \\
& \text{IMPLIES } o(\sigma_C, \sigma_{\overline{C}}) \in X]].
\end{aligned}
$$

where $\mu : N \to N$ is a bijection.

This somewhat intricate formulation states nothing but that the effectivity function $E^{\mathcal{G}}_{DEP}(C)$ associates with each coalition C the states which are outcomes of agreements (and hence of reciprocal profiles), and which C can force via partial agreements (σ_C, μ_C) regardless of the partial agreements $(\sigma_{\overline{C}}, \mu_{\overline{C}})$ of $\overline{C}$.

We have the following theorem.

Theorem 2 (*DEP* vs. *CORE*) *Let $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ be a game. It holds that, for all agreements (σ, μ):*

$$(\sigma, \mu) \in DEP(\mathcal{G}) \;\Leftrightarrow\; o(\sigma) \in CORE(C^{\mathcal{G}}_{DEP}).$$

where $\mu : N \to N$.

The proof is omitted for space reasons. Put it otherwise, here is what Theorem 2 states. Given a game $\mathcal{G}$, a profile σ which is partially DS-implemented by μ (Definition 12) forms an undominated partial agreement (σ, μ) if and only if σ is in the core of the dependence game of $\mathcal{G}$. By taking Definition 10 and Theorem 1 into the picture, we thus see that Theorem 2 connects three apparently rather different properties of a strategic game $\mathcal{G}$: the existence of reciprocal profiles, the existence of DS-equilibria in permutations of $\mathcal{G}$, and the core of the dependence game built on $\mathcal{G}$.

3.3.2 Partial dependence games

Definition 17 (Partial dependence games from strategic ones) *Let $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ be a game. The partial dependence game $C^{\mathcal{G}}_{pDEP} = (N, S, E^{\mathcal{G}}_{pDEP}, \succeq_i)$ of $\mathcal{G}$ is a coalitional game where the effectivity function $E^{\mathcal{G}}_{pDEP}$ is defined as follows:*

$$
\begin{aligned}
X \in E^{\mathcal{G}}_{pDEP}(C) \quad \Leftrightarrow \quad & \exists \sigma_C, \mu_C \text{ S.T.} \\
& (\sigma_C, \mu_C) \in pAGR(\mathcal{G}) \\
& \text{AND } [\forall \sigma_{\overline{C}} : o(\sigma_C, \sigma_{\overline{C}}) \in X]].
\end{aligned}
$$

where $\mu_C : C \to C$ is a bijection.

Partial dependence games are defined by just looking at the set of outcomes that each coalition can force by means of a partial agreement. Unlike Definition 16, Definition 17 is much closer to the standard definition of coalitional games based on strategic ones (Definition 5).

Like for dependence games, we have a characterization of the set of undominated partial agreements.

Theorem 3 (*pDEP* vs. *CORE*) *Let $\mathcal{G} = (N, S, \Sigma_i, \succeq_i, o)$ be a game. It holds that, for all agreements (σ, μ):*

$$(\sigma, \mu) \in pDEP(\mathcal{G}) \;\Leftrightarrow\; o(\sigma) \in CORE(C^{\mathcal{G}}_{pDEP}).$$

where $\mu : C \to C$ is a bijection with $C \subseteq N$.

The proof is omitted for space reasons. Like Theorem 2, Theorem 3 establishes a precise connection between the notions of partial reciprocity in a strategic game $\mathcal{G}$, the existence of DS-equilibria in all permuted subgames of $\mathcal{G}$, and the core of the partial dependence game built on $\mathcal{G}$.

3.4 Coalitional, dependence, partial dependence effectivity

The coalitional game $C^{\mathcal{G}}$ built on a strategic game $\mathcal{G}$ and its dependence-based counterparts $C^{\mathcal{G}}_{DEP}$ and $C^{\mathcal{G}}_{pDEP}$ are clearly related. The following fact shows how.

Fact 3 (Effectivity functions related) *The following relations hold:*

i) For all $\mathcal{G}$: $E^{\mathcal{G}}_{pDEP} \subseteq E^{\mathcal{G}}$;

ii) It does not hold that for all $\mathcal{G}$: $E^{\mathcal{G}}_{DEP} \subseteq E^{\mathcal{G}}_{pDEP}$; nor it holds that for all $\mathcal{G}$: $E^{\mathcal{G}}_{pDEP} \subseteq E^{\mathcal{G}}_{DEP}$;

iii) It does not hold that for all $\mathcal{G}$: $E^{\mathcal{G}}_{DEP} \subseteq E^{\mathcal{G}}$; nor it holds that for all $\mathcal{G}$: $E^{\mathcal{G}} \subseteq E^{\mathcal{G}}_{DEP}$.

The proof is omitted for space reasons. The fact shows that dependence games are not just a refinement of coalitional ones, which instead holds for partial dependence games. In other words dependence-based effectivity function considerably modify the powers assigned to coalitions by the standard definition of coalitional games on strategic ones (Definition 5).

4 Conclusions

The contribution of the paper is two-fold. On the one hand it has been shown that central dependence-theoretic notions such as the notion of cycle are amenable to a game-theoretic characterization (Theorem 1). On the other hand dependence theory has been demonstrated to give rise to types of cooperative games where solution concepts such as the core can be applied. The relation between the various forms of cooperative games where coalitions undertake agreements (dependence and partial dependence) have been analyzed, together with the dominance they induce on agreements (Theorem 2 and 3).

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References

[1] G. Boella, L. Sauro, and L. van der Torre. Strengthening admissible coalitions. In *Proceeding of the 2006 conference on ECAI 2006: 17th European Conference on Artificial Intelligence*, pages 195–199. ACM, 2006.

[2] C. Castelfranchi, A. Cesta, and M. Miceli. Dependence relations among autonomous agents. In E. Werner and Y. Demazeau, editors, *Decentralized A.I.3*. Elsevier, 1992.

[3] D. Grossi and P. Turrini. Dependence theory via game theory. In W. van der Hoek and G. Kaminka, editors, *Proceedings of AAMAS 2010*, 2010.

[4] H. Moulin and B. Peleg. Cores of effectivity functions and implementation theory. *Journal of Mathematical Economics*, 10:115–145, 1982.

[5] M. J. Osborne and A. Rubinstein. *A Course in Game Theory*. MIT Press, 1994.

[6] R.Parikh. Social software. *Synthese*, 132(3):187–211, 2002.

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