Algorithmic Game Theory

Algorithmische Spieltheorie

Complexity of Problems for Weighted Voting Games

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Complexity of Problems for Weighted Voting Games

- Weighted voting games can be represented compactly, since only the weights of the n players and a quota need to be given.
- This implicitly tells us which of the altogether 2^n possible coalitions of players are winning and which are losing, and we don't have to explicitly list this information, which would require exponential space.
- Note that the weights and the quota of a weighted voting game (and also, e.g., the ε in ε-$Core(G)$) must be restricted to be rational numbers, for otherwise problem instances containing weighted voting games (or an ε) could not always be handled algorithmically.

Reminder: Many-One Reducibility and Completeness

Definition

- Let $\Sigma = \{0,1\}$ be a fixed alphabet, and let $A, B \subseteq \Sigma^*$.
- Let FP denote the set of polynomial-time computable total functions mapping from Σ^* to Σ^*.
- Let $\mathscr{C}$ be any complexity class.

1. Define the *polynomial-time many-one reducibility*, denoted by $\leq_{\mathrm{m}}^{\mathrm{p}}$, as follows: $A \leq_{\mathrm{m}}^{\mathrm{p}} B$ if there is a function $f \in \mathrm{FP}$ such that for each

$$(\forall x \in \Sigma^*)\,[x \in A \iff f(x) \in B].$$

Reminder: Many-One Reducibility and Completeness

Definition (continued)

2. A set B is *$\leq_{\mathrm{m}}^{\mathrm{p}}$-hard for $\mathscr{C}$* if $A \leq_{\mathrm{m}}^{\mathrm{p}} B$ for each $A \in \mathscr{C}$.

3. A set B is *$\leq_{\mathrm{m}}^{\mathrm{p}}$-complete for $\mathscr{C}$* if
    1. B is $\leq_{\mathrm{m}}^{\mathrm{p}}$-hard for $\mathscr{C}$ (lower bound) and
    2. $B \in \mathscr{C}$ (upper bound).

4. $\mathscr{C}$ is said to be *closed under the $\leq_{\mathrm{m}}^{\mathrm{p}}$-reducibility* ($\leq_{\mathrm{m}}^{\mathrm{p}}$-*closed*, for short) if for any two sets A and B,

$$\text{if } A \leq_{\mathrm{m}}^{\mathrm{p}} B \text{ and } B \in \mathscr{C}, \text{ then } A \in \mathscr{C}.$$

Reminder: Properties of $\leq_{\mathrm{m}}^{\mathrm{p}}$

Lemma

1. *$A \leq_{\mathrm{m}}^{\mathrm{p}} B$ implies $\overline{A} \leq_{\mathrm{m}}^{\mathrm{p}} \overline{B}$, yet in general it is not true that $A \leq_{\mathrm{m}}^{\mathrm{p}} \overline{A}$.*

2. *The relation $\leq_{\mathrm{m}}^{\mathrm{p}}$ is both reflexive and transitive, yet not antisymmetric.*

3. P *("deterministic polynomial time") and* NP *("nondeterministic polynomial time") are $\leq_{\mathrm{m}}^{\mathrm{p}}$-closed.*
   That is, upper bounds are inherited downward with respect to $\leq_{\mathrm{m}}^{\mathrm{p}}$.

4. *If $A \leq_{\mathrm{m}}^{\mathrm{p}} B$ and A is $\leq_{\mathrm{m}}^{\mathrm{p}}$-hard for some complexity class $\mathscr{C}$, then B is $\leq_{\mathrm{m}}^{\mathrm{p}}$-hard for $\mathscr{C}$.*
   That is, lower bounds are inherited upward with respect to $\leq_{\mathrm{m}}^{\mathrm{p}}$.

Reminder: Properties of $\leq_{\mathrm{m}}^{\mathrm{p}}$

Lemma (continued)

5. *Let $\mathscr{C}$ and $\mathscr{D}$ be any complexity classes. If $\mathscr{C}$ is $\leq_{\mathrm{m}}^{\mathrm{p}}$-closed and B is $\leq_{\mathrm{m}}^{\mathrm{p}}$-complete for $\mathscr{D}$, then*

$$\mathscr{D} \subseteq \mathscr{C} \iff B \in \mathscr{C}.$$

   In particular, if B is NP-complete, then

$$\mathrm{P} = \mathrm{NP} \iff B \in \mathrm{P}.$$

6. *For each nontrivial set $B \in \mathrm{P}$ (i.e., $\emptyset \neq B \neq \Sigma^*$) and for each set $A \in \mathrm{P}$, $A \leq_{\mathrm{m}}^{\mathrm{p}} B$. Thus, every nontrivial set in P is $\leq_{\mathrm{m}}^{\mathrm{p}}$-complete for P.*

Veto Player and Dummy Player

- Recall that it is common to assume that the grand coalition forms in simple games, just as in superadditive games.

VETO

Given: A weighted voting game $G = (w_1, w_2, \ldots, w_n; q)$ and a player i.

Question: Is i a veto player in G?

Theorem

VETO *is in* P.

Proof: Under the above assumption, it is enough to check whether the coalition $P \smallsetminus \{i\}$ is winning, i.e., whether $w(P \smallsetminus \{i\}) \geq q$. ❑

Veto Player and Dummy Player

DUMMY	
Given:	A weighted voting game $G = (w_1, w_2, \ldots, w_n; q)$ and a player i.
Question:	Is i a dummy player in G?

Theorem

DUMMY *is* coNP*-complete, where* $\mathrm{coNP} = \{\overline{L} \mid L \in \mathrm{NP}\}$*.*

Remark

1. *Our reduction will not give "strong* coNP*-completeness," i.e.,* coNP*-hardness is relevant only if the weights are fairly large.*
2. *While weights are rather small in parliamentary voting, they can be huge in other applications of weighted voting games, such as shareholder voting.*

Veto Player and Dummy Player

Proof:

- For proving that DUMMY is in coNP, it is enough to check that i is useless *for all coalitions $C \subseteq P$*: $v(C \cup \{i\}) = v(C)$.
- For the hardness proof, we reduce from the NP-complete problem

PARTITION	
Given:	A nonempty sequence $(k_1, k_2, \ldots, k_n)$ of positive integers satisfying that $\sum_{i=1}^{n} k_i$ is even.
Question:	Does there exist a subset $A \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in A} k_i = \sum_{i \in \{1,2,\ldots,n\} \smallsetminus A} k_i$?

to the complement of DUMMY. And now, see blackboard. ❑

WVG-Empty-Core

Recall that the core of a game $G = (P, v)$ is the set of imputations $\vec{a}$ such that $a(C) \geq v(C)$ for each $C \subseteq P$ (assuming the grand coalition forms).

WVG-EMPTY-CORE	
Given:	A weighted voting game $G = (w_1, w_2, \ldots, w_n; q)$.
Question:	Does it hold that $Core(G) = \emptyset$?

Theorem

WVG-EMPTY-CORE *is in* P.

Proof: Under our assumption that the grand coalition forms, we know that G has a nonempty core if and only if it has a veto player.
So it is enough to check for each player if she is a veto player. ❑

WVG-In-Core and WVG-Construct-Core

Remark

1. *Similarly, to check whether a given outcome $\vec{a} = (a_1, a_2, \ldots, a_n)$ (i.e., a payoff vector for the grand coalition) is in the core, it is enough to check that $a_i = 0$ for each player i that is not a veto player.*

2. *Also, a payoff vector $\vec{a} = (a_1, a_2, \ldots, a_n)$ in the core can be constructed if there exists one:*
   - *If there is no veto player, the core of G is empty, so we have a yes-instance of* WVG-EMPTY-CORE.
   - *On the other hand, if there is some veto player i, construct an imputation $\vec{a}$ with $a_i = 1$, $a_j = 0$ for $j \in P \smallsetminus \{i\}$.*

WVG-In-Core and WVG-Construct-Core

WVG-In-Core	
Given:	A weighted voting game $G = (w_1, \ldots, w_n; q)$ and an imputation $\vec{a}$.
Question:	Is $\vec{a}$ in the core of G?

WVG-Construct-Core	
Given:	A weighted voting game $G = (w_1, \ldots, w_n; q)$.
Task:	Construct an imputation $\vec{a}$ in the core of G.

Theorem

WVG-In-Core *and* WVG-Construct-Core *can be solved in polynomial time.*

WVG-CS-Core

Remark

1. **What if the grand coalition does not form?**

   If $q < {}^{w(P)}/_2$, there may be two or more disjoint winning coalitions.

   - *Such a quota doesn't make sense in a voting context.*
   - *However, it does make sense for multiagent task allocation, where disjoint teams of players tackle different tasks.*

2. *For a weighted voting game G, let $CS\text{-}Core(G)$ denote the set of outcomes $(\mathfrak{C}, \vec{a})$ with $\mathfrak{C} \in \mathscr{C}\mathscr{S}_P$ that are stable against deviation.*

WVG-CS-Core

WVG-CS-CORE	
Given:	A weighted voting game $G = (w_1, w_2, \ldots, w_n; q)$, where the players may form nontrivial coalition structures.
Question:	Does it hold that $\mathit{CS\text{-}Core}(G) \neq \emptyset$?

Theorem (Elkind, Chalkiadakis, and Jennings (2008))

Let $G = (w_1, \ldots, w_n; q)$ be a weighted voting game over $P = \{1, \ldots, n\}$. If there exists a coalition structure $\mathfrak{C} = \{C_1, \ldots, C_k\}$ in $\mathscr{CS}_P$ such that $w(C_j) = q$ for all j, $1 \leq j \leq k$, then $\mathit{CS\text{-}Core}(G) \neq \emptyset$.

Proof: See blackboard. ❑

WVG-CS-Core

Theorem (Elkind, Chalkiadakis, and Jennings (2008))

WVG-CS-CORE *is* NP*-hard.*

Proof: See blackboard. ❑

Remark

1. *It is not clear if* WVG-CS-CORE *is* NP*-complete (i.e., in* NP*):*
   - *After guessing an outcome, exponentially many checks are needed to verify stability.*
   - *When guessing an outcome $\vec{a} = (a_1, a_2, \ldots, a_n)$, can the a_i be written using $p(n, \log w_{\max})$ bits, where $w_{\max}$ is the largest weight?*
     Elkind et al. (2008): ***Yes!*** *If CS-Core$(G) \neq \emptyset$ then it contains such an outcome.* *So, we know that* WVG-CS-CORE *is in $\Sigma_2^p = \mathrm{NP}^{\mathrm{NP}}$.*
   - *Greco et al. (2011) improve this to:* WVG-CS-CORE *is in $\Delta_2^p = \mathrm{P}^{\mathrm{NP}}$.*

WVG-CS-Core

Remark

2. *Checking whether a given outcome $(\mathfrak{C}, \vec{a})$ with $\mathfrak{C} \in \mathscr{C}\mathscr{S}_P$ and $\vec{a} = (a_1, a_2, \ldots, a_n)$ is in CS-Core(G) is:*
   - coNP*-complete in general (reduction from* Partition*), but*
   - *in* P *if the weights are given in unary (reduction to* Knapsack*).*

Knapsack

Given: A list of k items with utilities $u_1, \ldots, u_k \in \mathbb{N}$ and sizes $s_1, \ldots, s_k \in \mathbb{N}$, the knapsack size S, and the target utility U.

Question: Is there a subset of indices $I \subseteq \{1, \ldots, k\}$ such that $\sum_{i \in I} s_i \leq S$ and $\sum_{i \in I} u_i \geq U$?

ε-Core and Least Core for Weighted Voting Games

WVG-EPSILON-CORE

Given: A weighted voting game $G = (w_1, w_2, \ldots, w_n; q)$ and a rational value $\varepsilon \geq 0$.

Question: Does it hold that ε-$Core(G) \neq \emptyset$?

WVG-IN-EPSILON-CORE

Given: A weighted voting game $G = (w_1, w_2, \ldots, w_n; q)$, a rational value $\varepsilon \geq 0$, and an efficient payoff vector $\vec{a}$.

Question: Is $\vec{a}$ in ε-$Core(G)$?

ε-Core and Least Core for Weighted Voting Games

Theorem (Elkind, Goldberg, Goldberg, and Wooldridge (2009))

1. WVG-EPSILON-CORE *is* coNP-*hard.*
2. WVG-IN-EPSILON-CORE *is* coNP-*complete.*

Remark

1. *It is not clear if* WVG-EPSILON-CORE *is* coNP-*complete (i.e., in* coNP*).*
2. *The best known upper bound for* WVG-EPSILON-CORE *is* $\Sigma_2^p = \mathrm{NP}^{\mathrm{NP}}$*:*
   - *Guess a solution and*
   - *verify that no coalition can gain more than* ε *by deviating.*

ε-Core and Least Core for Weighted Voting Games

Proof: We show

$$\overline{\text{PARTITION}} \leq_{\mathrm{m}}^{\mathrm{p}} \text{WVG-EPSILON-CORE}.$$

Given an instance $(k_1, k_2, \ldots, k_n)$ with $\sum_{i=1}^{n} k_i = 2K$ for some positive integer K, construct a WVG with $n+1$ players:

$$G = (w_1, \ldots, w_n, w_{n+1}; q) = (k_1, \ldots, k_n, K; K).$$

ε-Core and Least Core for Weighted Voting Games

Lemma (Elkind, Goldberg, Goldberg, and Wooldridge (2009))

1. *If* $(k_1, k_2, \ldots, k_n) \in \text{Partition}$ *then*
   (a) *the value of the least core of* G *is* $\frac{2}{3}$, *and*
   (b) *for each efficient payoff vector* $\vec{a} = (a_1, a_2, \ldots, a_{n+1})$ *in the least core of* G, *it holds that* $a_{n+1} = \frac{1}{3}$.
2. *If* $(k_1, k_2, \ldots, k_n) \notin \text{Partition}$ *then*
   (a) *the value of the least core of* G *is at most* $\frac{2}{3} - \frac{1}{6K}$, *and*
   (b) *for each efficient payoff vector* $\vec{a} = (a_1, a_2, \ldots, a_{n+1})$ *in the least core of* G, *it holds that* $a_{n+1} \geq \frac{1}{3} + \frac{1}{6K}$.

ε-Core and Least Core for Weighted Voting Games

Proof: of the lemma.

1. Define the payoff vector $\vec{a} = (a_1, a_2, \ldots, a_{n+1})$ by

$$a_i = \frac{w_i}{3K} \quad \text{for } 1 \leq i \leq n+1.$$

Note that $\vec{a}$ is efficient and $a_i > 0$ for each i.

Define the *excess of a coalition C w.r.t. $\vec{a}$* by

$$e(\vec{a}, C) = a(C) - v(c).$$

Note that $e(\vec{a}, C) \geq -\frac{2}{3}$ for all $C \subseteq P = \{1, \ldots, n+1\}$.

Hence, $\vec{a} \in \frac{2}{3}$-*Core*(G), so $\tilde{\varepsilon}(G) \leq \frac{2}{3}$.

ε-Core and Least Core for Weighted Voting Games

Since $(k_1, k_2, \ldots, k_n) \in \textsc{Partition}$, there are three disjoint winning coalitions:

$$\begin{aligned} C_1 &= J \subseteq \{1, \ldots, n\} \quad \text{with } \sum_{j \in J} k_j = K, \\ C_2 &= \{1, \ldots, n\} \setminus J, \\ C_3 &= \{n+1\}. \end{aligned}$$

Every efficient payoff vector $\vec{b} = (b_1, b_2, \ldots, b_{n+1})$ with $b_{n+1} \neq \frac{1}{3}$ satisfies $b(C_i) < \frac{1}{3}$ for some $i \in \{1, 2, 3\}$, and thus $e(\vec{b}, C_i) < -\frac{2}{3}$.

Hence, if some $\vec{a} = (a_1, a_2, \ldots, a_{n+1})$ maximizes its least excess, it must satisfy $a_{n+1} = \frac{1}{3}$.

Therefore, $\tilde{\varepsilon}(G) = \frac{2}{3}$ and every $\vec{a} = (a_1, a_2, \ldots, a_{n+1})$ in the least core of G satisfies $a_{n+1} = \frac{1}{3}$.

ε-Core and Least Core for Weighted Voting Games

2. Now suppose $(k_1, k_2, \ldots, k_n) \notin$ PARTITION.

   Modify the payoff vector $\vec{a} = (a_1, \ldots, a_{n+1}) = \left(\frac{k_1}{3K}, \ldots, \frac{k_n}{3K}, \frac{1}{3}\right)$ by setting

$$\vec{a}' = (a'_1, \ldots, a'_{n+1}) = \left(a_1 - \frac{1}{6nK}, \ldots, a_n - \frac{1}{6nK}, a_{n+1} + \frac{1}{6K}\right).$$

   Note that $\vec{a}'$ is efficient and $a'_i > 0$ for each i.

   One can show that

$$e(\vec{a}', C) \geq -\frac{2}{3} + \frac{1}{6K} \quad \text{for each } C \subseteq P = \{1, \ldots, n+1\}.$$

   See blackboard.

ε-Core and Least Core for Weighted Voting Games

Since $e(\vec{a}', C) \geq -\frac{2}{3} + \frac{1}{6K}$ for each $C \subseteq P = \{1, \ldots, n+1\}$, $\vec{a}'$ witnesses that

$$\tilde{\varepsilon}(G) \leq \frac{2}{3} - \frac{1}{6K}.$$

Hence, for each payoff vector $\vec{b}$ in the least core of G, we have

$$e(\vec{b}, C) \geq -\frac{2}{3} + \frac{1}{6K} \quad \text{for each } C \subseteq P = \{1, \ldots, n+1\}.$$

In particular, for $C_3 = \{n+1\}$:

$$b_{n+1} \geq \frac{1}{3} + \frac{1}{6K}. \quad \square \quad \text{Lemma}$$

And now see blackboard again for completing the proof of the theorem:

$$\overline{\text{Partition}} \leq_m^p \text{WVG-Epsilon-Core}. \quad \square$$

ε-Core and Least Core for Weighted Voting Games

WVG-IN-LEAST-CORE

Given: A weighted voting game $G = (w_1, w_2, \ldots, w_n; q)$ and an efficient payoff vector $\vec{a}$.

Question: Is $\vec{a}$ in the least core of G?

WVG-CONSTRUCT-LEAST-CORE

Given: A weighted voting game $G = (w_1, w_2, \ldots, w_n; q)$.

Task: Construct an efficient payoff vector $\vec{a}$ in the least core of G.

ε-Core and Least Core for Weighted Voting Games

Theorem (Elkind, Goldberg, Goldberg, and Wooldridge (2009))

1. WVG-In-Least-Core *is* NP*-hard.*
2. WVG-Construct-Least-Core *cannot be solved in deterministic polynomial time, unless* $\mathrm{P} = \mathrm{NP}$.

Proof: See blackboard. ❑

Remark

1. *It is not clear if* WVG-In-Least-Core *is* NP*-complete (i.e., in* NP*).*
2. *The best known upper bound for* WVG-In-Least-Core *is* $\Pi_2^p = \mathrm{coNP}^{\mathrm{NP}}$.

ε-Core and Least Core for Weighted Voting Games

Remark

- *On the other hand, despite their* NP*- or* coNP*-hardness, each of the problems*
  - WVG-Epsilon-Core,
  - WVG-In-Epsilon-Core,
  - WVG-In-Least-Core, *and*
  - WVG-Construct-Least-Core

  admits a pseudo-polynomial-time algorithm, which can then be converted to a fully polynomial-time approximation scheme (FPTAS).
- *Similarly, the value of the least core of a given weighted voting game with n players can be computed in time polynomial in n and $w_{\max}$.*
- *The proof makes use of the linear program for the least core.*

Cost of Stability for Weighted Voting Games

Let $G = (P, v)$ be a superadditive weighted voting game.

Recall the notion of the *(additive) cost of stability for G*, defined by

$$CoS(G) = \inf\{\Delta \,\big|\, \Delta \geq 0 \text{ and } Core(G_\Delta) \neq \emptyset\},$$

where the *adjusted game $G_\Delta = (P, v_\Delta)$* is given by

- $v_\Delta(C) = v(C)$ for $C \neq P$ and
- $v_\Delta(P) = v(P) + \Delta$.

Similarly, we can define the *multiplicative cost of stability* by

$$CoS^\times(G) = \frac{CoS(G) + v(P)}{v(P)}. \tag{1}$$

Cost of Stability for Weighted Voting Games

Remark

- *Results for the additive cost of stability can be restated for its multiplicative sibling, and vice versa.*
  For example, if $CoS(G) = v(P)$, we have $CoS^{\times}(G) = 2$.
- *Note that $CoS^{\times}(G) \geq 1$ for profit-sharing games.*
- *For cost-sharing games, the multiplicative cost of stability is also known as the cost recovery ratio, and we have $0 \leq CoS^{\times}(G) \leq 1$.*

Theorem (Bachrach et al. (2018))

For each superadditive weighted voting game $G = (P, v) = (w_1, \ldots, w_n; q)$,

$$CoS^{\times}(G) < 2.$$

Cost of Stability for Weighted Voting Games

Proof:

- Since G is a simple game, it is superadditive if and only if every pair of winning coalitions has a nonempty intersection.
- Recall that we assume that $w(P) \geq q$.
  Suppose that there is an agent i^* with weight $w_{i^*} \geq q$.
  Then, by superadditivity, i^* must be a veto player, so the core of G is nonempty and hence $CoS^{\times}(G) = 1$.
- Otherwise, let S be a minimum-weight winning coalition in G.
  Pick a player $j \in S$ such that $w_j \leq w_i$ for all $i \in S$, and set
  $$s = 1 - \frac{w(S \setminus \{j\})}{q}.$$
  Note that $s > 0$ by our choice of S.

Cost of Stability for Weighted Voting Games

- Define a payoff vector $\vec{a}$ by setting $a_j = s$, $a_i = \frac{w_i}{q}$ for $i \in P \setminus \{j\}$.
- We claim that $\vec{a}$ is stable.

  Indeed, consider a winning coalition R.

  If $j \notin R$, then $a(R) = \frac{w(R)}{q} \geq 1$, so R does not block $\vec{a}$.

  If $j \in R$, then (since $w(R) \geq w(S)$ by our choice of S) we have

$$a(R) = a(R \setminus \{j\}) + a_j = \frac{w(R \setminus \{j\})}{q} + a_j \geq \frac{w(S \setminus \{j\})}{q} + s = 1.$$

- It remains to bound the total payment:

$$a(P) = a(S \setminus \{j\}) + a_j + a(P \setminus S) = \frac{w(S \setminus \{j\})}{q} + s + \frac{w(P \setminus S)}{q}$$

$$= 1 + \frac{w(P \setminus S)}{q} < 1 + 1 = 2,$$

  where the inequality holds because $P \setminus S$ is a losing coalition. ❑

Cost of Stability for Weighted Voting Games

WVG-SUPER-IMPUTATION-STABILITY	
Given:	A weighted voting game G, a parameter $\Delta \geq 0$, and an imputation $\vec{a} = (a_1, a_2, \ldots, a_n)$ in the adjusted game G_Δ.
Question:	Is it true that $\vec{a} \in Core(G_\Delta)$?

WVG-COST-OF-STABILITY	
Given:	A weighted voting game G and a parameter $\Delta \geq 0$.
Question:	Is it true that $CoS(G) \leq \Delta$ (i.e., is it true that $Core(G_\Delta) \neq \emptyset$)?

Cost of Stability for Weighted Voting Games

Theorem (Bachrach et al. (2009))

1. WVG-SUPER-IMPUTATION-STABILITY *is* coNP-*complete.*
2. WVG-COST-OF-STABILITY *is* coNP-*hard.*

Proof: See blackboard. ❑

Remark

- *Again, if the weights and the quota of the given weighted voting game in these problems are represented in unary, then both problems can be solved in polynomial time.*
- *Bachrach et al. (2009) also showed that there is an FPTAS for computing $CoS(G)$.*

Complexity of Computing Power Indices

How hard is it to compute the Shapley–Shubik or Banzhaf index?

Definition

Define $\#\mathrm{P}$ as the class of functions that give the number of solutions of NP problems. $\#\mathrm{P}$ is also known as the *"counting version of* NP*."*

Example (of functions in $\#\mathrm{P}$)

- $\#\mathrm{SAT}$ maps each boolean formula to the number of its satisfying assignments.
- $\#\textsc{Partition}$ maps each instance $(k_1, k_2, \ldots, k_n)$ of $\textsc{Partition}$ to the number of subsets $A \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in A} k_i = \sum_{i \in \{1,2,\ldots,n\} \smallsetminus A} k_i .$$

Complexity of Computing Power Indices

Definition

Let f and g be two functions mapping from Σ^* to $\mathbb{N}$.

- We say *f (many-one) reduces to g* if there exist two polynomial-time computable functions, $\psi : \mathbb{N} \to \mathbb{N}$ and $\rho : \Sigma^* \to \Sigma^*$, such that for each $x \in \Sigma^*$,
$$f(x) = \psi(g(\rho(x))).$$
- We say *f parsimoniously reduces to g* if there exists a polynomial-time computable function ρ such that for each $x \in \Sigma^*$, $f(x) = g(\rho(x))$.
- g is *(parsimoniously) hard for $\#\mathrm{P}$* if every $f \in \#\mathrm{P}$ (parsimoniously) reduces to g, and g is *(parsimoniously) complete for $\#\mathrm{P}$* if $g \in \#\mathrm{P}$ and g is (parsimoniously) hard for $\#\mathrm{P}$.

Complexity of Computing Power Indices

Theorem

1. *Computing the (raw) Shapley–Shubik index of a player in a given weighted voting game is complete for* $\#\mathrm{P}$*.*
   (Deng and Papadimitriou (1994))

2. *Computing the (raw) Banzhaf index is parsimoniously complete for* $\#\mathrm{P}$*.*
   (Prasad and Kelly (1990))

3. *For both problems, there exist pseudo-polynomial-time algorithms.*
   (Matsui and Matsui (2000))

Complexity of Computing Power Indices

Proof: We show only the first statement: Computing the (raw) Shapley–Shubik index of a player in a given weighted voting game is #P-complete.

1. **Membership in #P.** Given a WVG G and a player i:
   - Nondeterministically guess all permutations π of the players in G.
   - For each permutation π guessed, accept if and only if $\Delta_{\pi}^{G}(i) = 1$.

Clearly, the number of accepting computation paths is

$$\sum_{\pi \in \Pi_P} \Delta_{\pi}^{G}(i) = \mathrm{SSI}^*(G, i).$$

Complexity of Computing Power Indices

2. #P-hardness. We reduce from the #P-complete problem #SubsetSum, the counting version of the NP-complete problem

SubsetSum	
Given:	A sequence $(a_1, \ldots, a_m)$ of positive integers and a positive integer K.
Question:	Does there exist a subset $A \subseteq \{1, \ldots, m\}$ such that $\sum_{i \in A} a_i = K$?

We work with a simplified but still #P-complete variant of this problem by assuming that:

- $K = \frac{M}{2}$, where $M = \sum_{i=1}^{m} a_i$ and
- all solutions A have the same size.

Complexity of Computing Power Indices

Given such an instance of #SubsetSum, with $(a_1, \ldots, a_m)$ and $K = \frac{M}{2}$, construct a weighted voting game $G = (P, v)$ with $n = m + 1$ players:

$$G = (w_1, \ldots, w_m, w_n; q) = (a_1, \ldots, a_m, 1; \frac{\sum_{i \in P} w_i}{2}).$$

Note that the quota is $\frac{M+1}{2}$ for an even number M.

For all $A \subseteq P$, we have $v(A) - v(A \setminus \{n\}) = 1$ if and only if the following conditions hold:

1. $n \in A$,
2. $\sum_{j \in A} w_j > \frac{M+1}{2}$, and
3. $\sum_{j \in A \setminus \{n\}} w_j < \frac{M+1}{2}$.

Complexity of Computing Power Indices

Since $w_n = 1$, this is equivalent to

$$\sum_{j \in A \setminus \{n\}} w_j = \frac{M}{2} = K.$$

In other words, $A \setminus \{n\}$ is a solution to the original SubsetSum instance.

Recall our assumption that all solutions have the same size: Letting $\|A\| = k$, we have $\|A \setminus \{n\}\| = k - 1$. Hence,

$$\begin{aligned} \mathrm{SSI}^*(G, n) &= \sum_{C \subseteq P \setminus \{n\}} \|C\|! \cdot (n - \|C\| - 1)! \cdot (v(C) - v(C \setminus \{n\})) \\ &= (k-1)!(n-k)! \cdot \left(\begin{array}{c} \text{“number of solutions to the} \\ \text{SubsetSum instance”} \end{array} \right). \quad \square \end{aligned}$$

Complexity of Power Comparison

For a power index $\mathbb{PI}$ (such as Shapley-Shubik or Banzhaf), define:

$\mathbb{PI}$-Power-Compare

Given: Two weighted voting games, G and G', and a player i occurring in both games.

Question: Is it true that $\mathbb{PI}(G, i) > \mathbb{PI}(G', i)$?

Theorem (Faliszewski & Hemaspaandra (2009))

Shapley-Shubik-Power-Compare *and* Banzhaf-Power-Compare *are* PP*-complete, where*

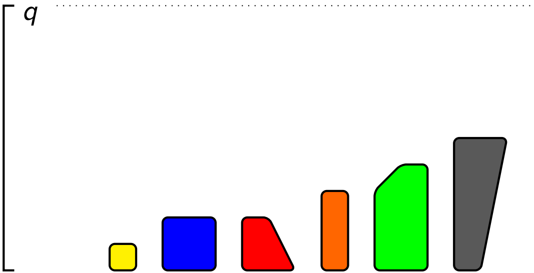
$$\mathrm{PP} = \left\{ A \;\middle|\; (\exists f \in \#\mathrm{P})(\forall x)\left[x \in A \Longleftrightarrow f(x) \geq 2^{p(|x|)-1}\right] \right\}$$

is "probabilistic polynomial time."

without proof

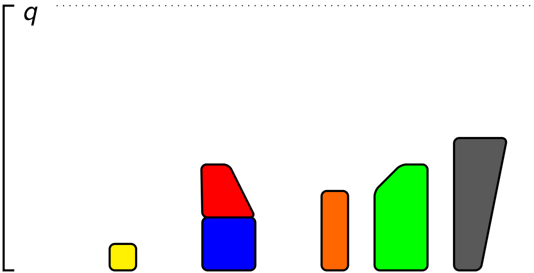
Beneficial Merging

Example



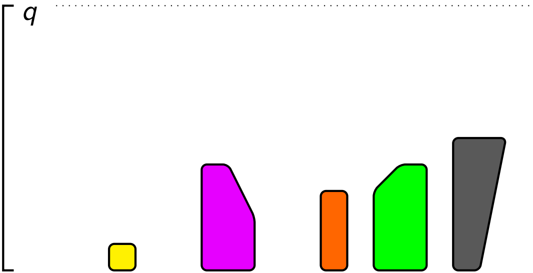
Beneficial Merging

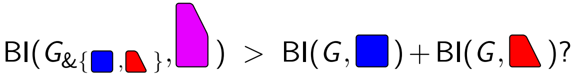
Example



Beneficial Merging

Example





Beneficial Merging

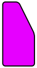
Example

$$\mathrm{BI}(G_{\&\{\ ,\ \}},\ ) \;=\; \mathrm{BI}(G,\ ) + \mathrm{BI}(G,\ )$$

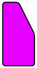
$$= \frac{6}{16} \qquad\qquad = \frac{6}{32} \qquad\qquad = \frac{6}{32}$$

Beneficial Merging

Example

BI($G_{\&\{$  ,  $\}}$, ) $=$ BI(G, ) $+$ BI(G, )

$= \frac{6}{16}$ $\qquad$ $= \frac{6}{32}$ $\qquad$ $= \frac{6}{32}$

SSI($G_{\&\{$  ,  $\}}$, ) $>$ SSI(G, ) $+$ SSI(G, ) ?

Beneficial Merging

Example

$$\underset{=\frac{6}{16}}{\mathrm{BI}(G_{\&\{\textcolor{blue}{\blacksquare},\textcolor{red}{\blacktriangle}\}},\textcolor{magenta}{\blacksquare})} = \underset{=\frac{6}{32}}{\mathrm{BI}(G,\textcolor{blue}{\blacksquare})} + \underset{=\frac{6}{32}}{\mathrm{BI}(G,\textcolor{red}{\blacktriangle})}$$

$$\underset{=\frac{14}{60}}{\mathrm{SSI}(G_{\&\{\textcolor{blue}{\blacksquare},\textcolor{red}{\blacktriangle}\}},\textcolor{magenta}{\blacksquare})} > \underset{=\frac{6}{60}}{\mathrm{SSI}(G,\textcolor{blue}{\blacksquare})} + \underset{=\frac{6}{60}}{\mathrm{SSI}(G,\textcolor{red}{\blacktriangle})}$$

Beneficial Merging

For a power index $\mathbb{PI}$ (such as Shapley-Shubik or Banzhaf), define:

$\mathbb{PI}$-Beneficial-Merge

Given: A weighted voting game $G = (w_1, \ldots, w_n; q)$ and a nonempty coalition $S \subseteq \{1, \ldots, n\}$.

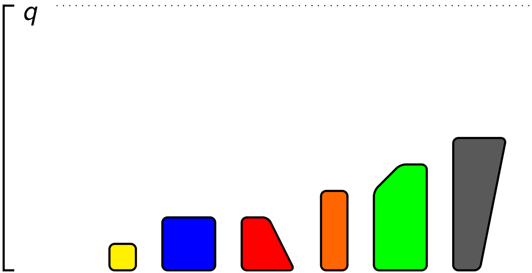
Question: Is it true that

$$\mathbb{PI}(G_{\&S}, 1) > \sum_{i \in S} \mathbb{PI}(G, i),$$

where $G_{\&S} = (\sum_{i \in S} w_i, w_{j_1}, \ldots, w_{j_{n-\|S\|}}; q)$ with $\{j_1, \ldots, j_{n-\|S\|}\} = \{1, \ldots, n\} \smallsetminus S$?

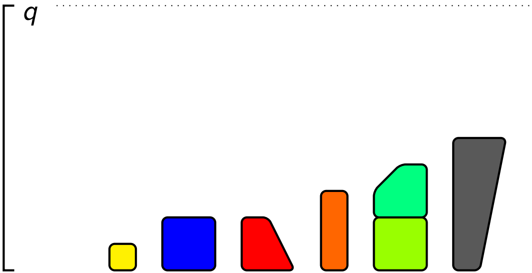
Beneficial Splitting

Example



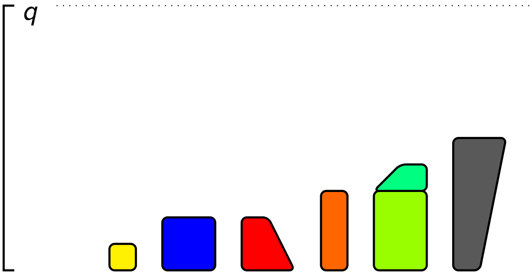
Beneficial Splitting

Example



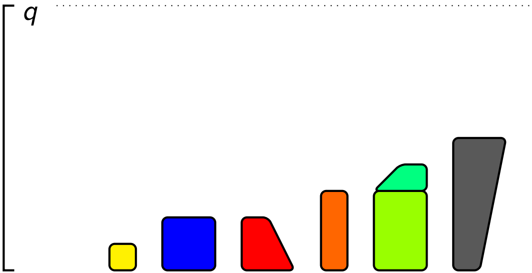
Beneficial Splitting

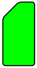
Example

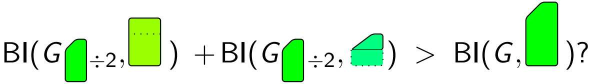


Beneficial Splitting

Example



Does there exist a split of  into two players such that



Beneficial Splitting

Example

$$\mathrm{BI}(G_{\div 2}, \;) + \mathrm{BI}(G_{\div 2}, \;) = \mathrm{BI}(G, \;)$$

$$= \frac{12}{64} \qquad\qquad = \frac{12}{64} \qquad\qquad = \frac{12}{32}$$

$$\mathrm{BI}(G_{\div 2}, \;) + \mathrm{BI}(G_{\div 2}, \;) = \mathrm{BI}(G, \;)$$

$$= \frac{19}{64} \qquad\qquad = \frac{5}{64} \qquad\qquad = \frac{12}{32}$$

Beneficial Splitting

Example

$$\mathrm{SSI}(G_{\div 2}, \;) + \mathrm{SSI}(G_{\div 2}, \;) < \mathrm{SSI}(G, \;)$$

$$= \frac{41}{420} \qquad = \frac{41}{420} \qquad = \frac{91}{420}$$

$$\mathrm{SSI}(G_{\div 2}, \;) + \mathrm{SSI}(G_{\div 2}, \;) < \mathrm{SSI}(G, \;)$$

$$= \frac{73}{420} \qquad = \frac{17}{420} \qquad = \frac{91}{420}$$

Beneficial Splitting

For a power index $\mathbb{PI}$ (such as Shapley-Shubik or Banzhaf), define:

$\mathbb{PI}$-BENEFICIAL-SPLIT

Given: A weighted voting game $G = (w_1, \ldots, w_n; q)$, a player $i \in \{1, \ldots, n\}$, and an integer $k \geq 2$.

Question: Is it possible to split i into k new players with positive integer weights $u_1, \ldots, u_k$ satisfying $\sum_{j=1}^{k} u_j = w_i$ so that

$$\sum_{j=0}^{k-1} \mathbb{PI}(G_{i \div k}, i+j) > \mathbb{PI}(G, i),$$

where $G_{i \div k} = (w_1, \ldots, w_{i-1}, u_1, \ldots, u_k, w_{i+1}, \ldots, w_n; q)$?

Complexity Classes

$$
\begin{array}{l}
\mathrm{PSPACE} \\
\quad | \\
\mathrm{NP}^{\mathrm{PP}} = \left\{ A \;\middle|\; (\exists \text{NPOTM } M)(\exists B \in \mathrm{PP}) \left[A = L(M^B) \right] \right\} \\
\quad | \\
\mathrm{PP} = \left\{ A \;\middle|\; (\exists f \in \#\mathrm{P})(\forall x) \left[x \in A \Longleftrightarrow f(x) \geq 2^{p(|x|)-1} \right] \right\} \\
\quad | \\
\mathrm{NP} \\
\quad | \\
\mathrm{P}
\end{array}
$$

Overview of Complexity Results

PI-BENEFICIAL-MERGE

PI-BENEFICIAL-SPLIT

Overview of Complexity Results

$\mathbb{PI}$-Beneficial-Merge

- open question [1]

$\mathbb{PI}$-Beneficial-Split

- SSI: NP-hard [1] $(k = 2)$

[1] Bachrach & Elkind, AAMAS 2008

Overview of Complexity Results

$\mathbb{PI}$-Beneficial-Merge

- open question [1]
- $\overline{\text{BI}}$, SSI: NP-hard [2] [3]

$\mathbb{PI}$-Beneficial-Split

- SSI: NP-hard [1] [3] ($k = 2$)
- $\overline{\text{BI}}$: NP-hard [2] [3]

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[1] Bachrach & Elkind, AAMAS 2008

[2] Aziz & Paterson, AAMAS 2009

[1] + [2] = [3] Aziz et al., JAIR 2011

Overview of Complexity Results

$\mathbb{PI}$-Beneficial-Merge

- open question [1]
- $\overline{\text{BI}}$, SSI: NP-hard [2] [3]
- SSI: $\|S\| = 2$: in PP [4]

$\mathbb{PI}$-Beneficial-Split

- SSI: NP-hard [1] [3] $(k = 2)$
- $\overline{\text{BI}}$: NP-hard [2] [3]

[1] Bachrach & Elkind, AAMAS 2008
[2] Aziz & Paterson, AAMAS 2009
[1] + [2] = [3] Aziz et al., JAIR 2011
[4] Faliszewski & Hemaspaandra, TCS 2009

Overview of Complexity Results

$\mathbb{PI}$-Beneficial-Merge

- open question [1]
- $\overline{\text{BI}}$, SSI: NP-hard [2] [3]
- SSI: $\|S\| = 2$: in PP [4]
- BI: $\|S\| = 2$: in P;
  $\|S\| \geq 3$: in PP, NP-hard [5]

$\mathbb{PI}$-Beneficial-Split

- SSI: NP-hard [1] [3] ($k = 2$)
- $\overline{\text{BI}}$: NP-hard [2] [3]
- BI: $k = 2$: in P;
  $k \geq 3$: in PP, NP-hard [5]

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[1] Bachrach & Elkind, AAMAS 2008
[2] Aziz & Paterson, AAMAS 2009
[1] + [2] = [3] Aziz et al., JAIR 2011
[4] Faliszewski & Hemaspaandra, TCS 2009
[5] Rey & Rothe, ECAI 2010

Overview of Complexity Results

PI-Beneficial-Merge

- open question [1]
- $\overline{\text{BI}}$, SSI: NP-hard [2] [3]
- SSI: $\|S\| = 2$: in PP [4]
- BI: $\|S\| = 2$: in P;
  $\|S\| \geq 3$: in PP, NP-hard [5]
- BI, SSI: PP-complete [6]

PI-Beneficial-Split

- SSI: NP-hard [1] [3] ($k = 2$)
- $\overline{\text{BI}}$: NP-hard [2] [3]
- BI: $k = 2$: in P;
  $k \geq 3$: in PP, NP-hard [5]
- BI, SSI: PP-hard, in NP^{PP} [6]

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[1] Bachrach & Elkind, AAMAS 2008
[2] Aziz & Paterson, AAMAS 2009
[1] + [2] = [3] Aziz et al., JAIR 2011
[4] Faliszewski & Hemaspaandra, TCS 2009
[5] Rey & Rothe, ECAI 2010
[6] Rey & Rothe, LATIN 2014 + JAIR 2014

Merging and Splitting Is Easy for Two Players

Fact

Let G be a weighted voting game and $S \subseteq \{1, \ldots, n\}$ be a coalition of its players.

1. BI-BENEFICIAL-MERGE *is in* P *for instances* (G, S) *with* $\|S\| = 2$.
2. BI-BENEFICIAL-SPLIT *is in* P *for instances* $(G, i, 2)$.

Proof:

1. Let $G = (w_1, \ldots, w_n; q)$ be a weighted voting game.

   Without loss of generality, let $S = \{1, n\}$.

   We obtain a new game $G_{\&S} = (w_1 + w_n, w_2, \ldots, w_{n-1}; q)$, where the first player is the new player merging S.

Merging and Splitting Is Easy for Two Players

Letting v_G and $v_{G_{\&S}}$ denote the corresponding coalitional functions, it holds that

$$
\begin{aligned}
&\mathrm{BI}(G_{\&S},1)-(\mathrm{BI}(G,1)+\mathrm{BI}(G,n)) \\
&= \frac{1}{2^{n-2}}\left(\sum_{C\subseteq\{2,\ldots,n-1\}}(v_{G_{\&S}}(C\cup\{1\})-v_{G_{\&S}}(C))\right) \\
&\quad -\frac{1}{2^{n-1}}\left(\sum_{C\subseteq\{2,\ldots,n\}}(v_G(C\cup\{1\})-v_G(C)) + \sum_{C\subseteq\{1,\ldots,n-1\}}(v_G(C\cup\{n\})-v_G(C))\right)
\end{aligned}
$$

Merging and Splitting Is Easy for Two Players

$$= \frac{1}{2^{n-1}} \left(\sum_{C \subseteq \{2,\dots,n-1\}} \big(2(v_{G_{\&S}}(C \cup \{1\}) - v_{G_{\&S}}(C)) \right.$$

$$- \; (v_G(C \cup \{1\}) - v_G(C)) \; - \; (v_G(C \cup \{1,n\}) - v_G(C \cup \{n\}))$$

$$\left. - \; (v_G(C \cup \{n\}) - v_G(C)) \; - \; (v_G(C \cup \{n,1\}) - v_G(C \cup \{1\}))\big) \right)$$

$$= \frac{1}{2^{n-1}} \left(\sum_{C \subseteq \{2,\dots,n-1\}} (2v_{G_{\&S}}(C \cup \{1\}) - 2v_G(C \cup \{1,n\}) + 2v_G(C) - 2v_{G_{\&S}}(C)) \right)$$

$$= \; 0.$$

Merging and Splitting Is Easy for Two Players

2. In the case of splitting, it similarly holds that

$$\mathrm{BI}(G_{n\div 2}, n+1) + \mathrm{BI}(G_{n\div 2}, n+2) - \mathrm{BI}(G, n) = 0$$

for a weighted voting game G, $k = 2$, and, without loss of generality, player n in G splitting into players $n+1$ and $n+2$ in a new game $G_{n\div 2}$. ❑

Merging Is Hard for More Than Two Players

Theorem (Rey & Rothe)

Banzhaf-BENEFICIAL-MERGE *is* PP-*hard.*

Proof Sketch.

COMPARE-#SUBSETSUM

COMPARE-#SUBSETSUM-R

COMPARE-#SUBSETSUM-RR

Banzhaf-BENEFICIAL-MERGE

Merging Is Hard for More Than Two Players

Theorem (Rey & Rothe)

Banzhaf-BENEFICIAL-MERGE *is* PP-*hard.*

Proof Sketch.

COMPARE-#SUBSETSUM

COMPARE-#SUBSETSUM-R

COMPARE-#SUBSETSUM-RR

Banzhaf-BENEFICIAL-MERGE

For F #P-parsimonious-complete,
$\text{COMPARE-}F = \{(x,y) \mid F(x) > F(y)\}$
is PP-complete.[4]

$\#\text{SUBSETSUM}((a_1,\ldots,a_n),q) = \|\{I \subseteq N \mid \sum_{i\in I} a_i = q\}\|.$

Merging Is Hard for More Than Two Players

Theorem (Rey & Rothe)

Banzhaf-Beneficial-Merge *is* PP-*hard.*

Proof Sketch.

Compare-#SubsetSum PP-complete ✓

Compare-#SubsetSum-R

Compare-#SubsetSum-RR

Banzhaf-Beneficial-Merge

Merging Is Hard for More Than Two Players

Theorem (Rey & Rothe)

Banzhaf-Beneficial-Merge *is* PP-*hard.*

Proof Sketch.

Compare-#SubsetSum

$\downarrow$

Compare-#SubsetSum-R

Compare-#SubsetSum-RR

Banzhaf-Beneficial-Merge

Given $A = (a_1, \dots, a_n)$, q_1, and q_2, is $\#\mathrm{S}(A, q_1) > \#\mathrm{S}(A, q_2)$?

$\leq_{\mathrm{m}}^{\mathrm{p}}$-reduction via:

$((x_1, \dots, x_m), q_x), \ ((y_1, \dots, y_n), q_y)$

$\mapsto A = (x_1, \dots, x_m, 2\alpha y_1, \dots, 2\alpha y_n),$

$\alpha = \sum_{i=1}^{m} x_i, \ q_1 = q_x, \text{ and } q_2 = 2\alpha q_y.$

Merging Is Hard for More Than Two Players

Theorem (Rey & Rothe)

Banzhaf-BENEFICIAL-MERGE *is* PP-*hard.*

Proof Sketch.

COMPARE-#SUBSETSUM PP-complete ✓

↓

COMPARE-#SUBSETSUM-R PP-hard ✓

COMPARE-#SUBSETSUM-RR

Banzhaf-BENEFICIAL-MERGE

Merging Is Hard for More Than Two Players

Theorem (Rey & Rothe)

Banzhaf-BENEFICIAL-MERGE *is* PP-*hard.*

Proof Sketch.

COMPARE-#SUBSETSUM PP-complete ✓

↓

COMPARE-#SUBSETSUM-R

↓

COMPARE-#SUBSETSUM-RR

Banzhaf-BENEFICIAL-MERGE

Given $A = (a_1, \ldots, a_n)$,
is $\#\mathrm{S}(A, \frac{\alpha}{2} - 2) > \#\mathrm{S}(A, \frac{\alpha}{2} - 1)$?

$\leq_{\mathrm{m}}^{\mathrm{p}}$-reduction via:

$((a_1, \ldots, a_n), q_1, q_2)$

$\mapsto B = (a_1, \ldots, a_n, 2\alpha{-}q_1, 2\alpha{+}1{-}q_2, 2\alpha{+}3{+}q_1{+}q_2, 3\alpha)$.

Merging Is Hard for More Than Two Players

Theorem (Rey & Rothe)

Banzhaf-Beneficial-Merge *is* PP-*hard.*

Proof Sketch.

$$\begin{array}{ll} \textsc{Compare-\#SubsetSum} & \text{PP-complete } \checkmark \\ \downarrow & \\ \textsc{Compare-\#SubsetSum}\text{-R} & \text{PP-hard } \checkmark \\ \downarrow & \\ \textsc{Compare-\#SubsetSum}\text{-RR} & \text{PP-hard } \checkmark \end{array}$$

Banzhaf-Beneficial-Merge

Merging Is Hard for More Than Two Players

Theorem (Rey & Rothe)

Banzhaf-BENEFICIAL-MERGE *is* PP-*hard.*

Proof Sketch.

COMPARE-#SUBSETSUM — PP-complete ✓

$\downarrow$

COMPARE-#SUBSETSUM-R — PP-hard ✓

$\downarrow$

COMPARE-#SUBSETSUM-RR

$\downarrow$

Banzhaf-BENEFICIAL-MERGE

$\leq_m^p$-reduction via:
$(a_1, \ldots, a_n)$
$\mapsto G = (2a_1, \ldots, 2a_n, 1, 1, 1, 1; \alpha)$,
$C = \{n+2, n+3, n+4\}$.

Merging Is Hard for More Than Two Players

Theorem (Rey & Rothe)

Banzhaf-Beneficial-Merge *is* PP*-hard.*

Proof Sketch.

Compare-#SubsetSum — PP-complete ✓

↓

Compare-#SubsetSum-R — PP-hard ✓

↓

Compare-#SubsetSum-RR — PP-hard ✓

↓

Banzhaf-Beneficial-Merge — PP-hard ✓

□

Merging Is Hard for More Than Two Players

Lemma (Faliszewski & Hemaspaandra, 2009)

Let F be a $\#\mathrm{P}$-parsimonious-complete function. The problem

$$\textsc{Compare}\text{-}F = \{(x,y) \mid F(x) > F(y)\}$$

is PP*-complete.*

$\#\textsc{SubsetSum}$ is known to be $\#\mathrm{P}$-parsimonious-complete.

Corollary

$\textsc{Compare}\text{-}\#\textsc{SubsetSum}$ *is* PP*-complete.*

Merging Is Hard for More Than Two Players

$\textsc{Compare-\#SubsetSum-R}$

Given: A sequence $A = (a_1, \ldots, a_n)$ of positive integers and two positive integers q_1 and q_2 with $1 \leq q_1, q_2 \leq \alpha - 1$, where $\alpha = \sum_{i=1}^{n} a_i$.

Question: Is the number of subsequences of A summing up to q_1 greater than the number of subsequences of A summing up to q_2, that is, does it hold that

$$\begin{aligned} & \#\textsc{SubsetSum}((a_1, \ldots, a_n), q_1) \\ > \quad & \#\textsc{SubsetSum}((a_1, \ldots, a_n), q_2) \ ? \end{aligned}$$

Merging Is Hard for More Than Two Players

Lemma (Rey & Rothe)

$\textsc{Compare-\#SubsetSum} \leq_{\mathrm{m}}^{\mathrm{p}} \textsc{Compare-\#SubsetSum-R}$.

Proof: Given an instance (X, Y) of $\textsc{Compare-\#SubsetSum}$, $X = ((x_1, \ldots, x_m), q_x)$ and $Y = ((y_1, \ldots, y_n), q_y)$, construct a $\textsc{Compare-\#SubsetSum-R}$ instance (A, q_1, q_2) as follows.

Let $\alpha = \sum_{i=1}^{m} x_i$ and define

$$A = (x_1, \ldots, x_m, 2\alpha y_1, \ldots, 2\alpha y_n), \quad q_1 = q_x, \quad \text{and} \quad q_2 = 2\alpha q_y.$$

This construction can obviously be achieved in polynomial time.

Merging Is Hard for More Than Two Players

It holds that integers from A can only sum up to $q_1 = q_x \leq \alpha - 1$ if they do not contain multiples of 2α, thus

$$\#\textsc{SubsetSum}(A, q_1) \;=\; \#\textsc{SubsetSum}((x_1, \ldots, x_m), q_x).$$

On the other hand, q_2 cannot be obtained by adding any of the x_i's, since this would yield a non-zero remainder modulo 2α, because $\sum_{i=1}^{m} x_i = \alpha$ is too small.

Thus, it holds that

$$\#\textsc{SubsetSum}(A, q_2) \;=\; \#\textsc{SubsetSum}((y_1, \ldots, y_n), q_y).$$

It follows that (X, Y) is in $\textsc{Compare-\#SubsetSum}$ if and only if (A, q_1, q_2) is in $\textsc{Compare-\#SubsetSum-R}$. ❑

Merging Is Hard for More Than Two Players

To perform the next step, we need to ensure that all integers in a COMPARE-#SUBSETSUM-R instance are divisible by 8.

This can easily be achieved, by multiplying each integer in an instance $((a_1, \ldots, a_n), q_1, q_2)$ by 8, obtaining

$$((8a_1, \ldots, 8a_n), 8q_1, 8q_2)$$

without changing the number of solutions for both related SUBSETSUM instances.

Thus, from now on, without loss of generality, we assume that for a given COMPARE-#SUBSETSUM-R instance $((a_1, \ldots, a_n), q_1, q_2)$, it holds that

$$a_i, q_j \equiv 0 \bmod 8 \quad \text{for } 1 \leq i \leq n \text{ and } j \in \{1, 2\}.$$

Merging Is Hard for More Than Two Players

COMPARE-#SUBSETSUM-RR

Given: A sequence $A = (a_1, \ldots, a_n)$ of positive integers.

Question: Is the number of subsequences of A summing up to $\frac{\alpha}{2} - 2$, where $\alpha = \sum_{i=1}^{n} a_i$, greater than the number of subsequences of A summing up to $\frac{\alpha}{2} - 1$, i.e., is it true that

$$\begin{aligned} & \#\textsc{SubsetSum}((a_1, \ldots, a_n), \frac{\alpha}{2} - 2) \\ > \quad & \#\textsc{SubsetSum}((a_1, \ldots, a_n), \frac{\alpha}{2} - 1) \text{ ?} \end{aligned}$$

Merging Is Hard for More Than Two Players

Lemma (Rey & Rothe)

$\textsc{Compare-\#SubsetSum-R} \leq_{\mathrm{m}}^{\mathrm{p}} \textsc{Compare-\#SubsetSum-RR}$.

Proof: Given an instance (A, q_1, q_2) of $\textsc{Compare-\#SubsetSum-R}$, where we assume that $A = (a_1, \ldots, a_n)$, q_1, and q_2 satisfy

$$a_i, q_j \equiv 0 \bmod 8 \quad \text{for } 1 \leq i \leq n \text{ and } j \in \{1, 2\},$$

we construct an instance B of $\textsc{Compare-\#SubsetSum-RR}$ as follows.

(This reduction is inspired by the standard reduction from $\textsc{SubsetSum}$ to $\textsc{Partition}$ due to Karp (1972).)

Merging Is Hard for More Than Two Players

Letting $\alpha = \sum_{i=1}^{n} a_i$, define

$$B = (a_1, \ldots, a_n,\ 2\alpha - q_1,\ 2\alpha + 1 - q_2,\ 2\alpha + 3 + q_1 + q_2,\ 3\alpha).$$

This instance can obviously be constructed in polynomial time.

Observe that

$$T = \left(\sum_{i=1}^{n} a_i\right) + (2\alpha - q_1) + (2\alpha + 1 - q_2) + (2\alpha + 3 + q_1 + q_2) + 3\alpha = 10\alpha + 4,$$

and therefore, $\frac{T}{2} - 2 = 5\alpha$ and $\frac{T}{2} - 1 = 5\alpha + 1$.

We show that (A, q_1, q_2) is in Compare-#SubsetSum-R if and only if B is in Compare-#SubsetSum-RR.

Merging Is Hard for More Than Two Players

First, we examine which subsequences of B sum up to 5α.

Case 1: If 3α is added, $2\alpha + 3 + q_1 + q_2$ cannot be added, as it would be too large.

Also, $2\alpha + 1 - q_2$ cannot be added, leading to an odd sum.

So, $2\alpha - q_1$ has to be added, as the remaining α are too small.

Since $3\alpha + 2\alpha - q_1 = 5\alpha - q_1$, 5α can be achieved by adding some a_i's if and only if there exists a subset $A' \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in A'} a_i = q_1$ (i.e., A' is a solution of the $\textsc{SubsetSum}$ instance (A, q_1)).

Merging Is Hard for More Than Two Players

Case 2: If 3α is not added, but $2\alpha+3+q_1+q_2$, an even number can only be achieved by adding $2\alpha+1-q_2$.

Thus, $\alpha-4-q_1$ remain.

$2\alpha-q_1$ is too large, while no subsequence of A sums up to $\alpha-4-q_1$, because of the assumption of divisibility by 8.

If neither 3α nor $2\alpha+3+q_1+q_2$ are added, the remaining $5\alpha+1-q_1-q_2$ are too small.

Thus, the only possibility to obtain 5α is to find a subsequence of A adding up to q_1. Thus,

$$\#\textsc{SubsetSum}(A, q_1) = \#\textsc{SubsetSum}(B, 5\alpha).$$

Merging Is Hard for More Than Two Players

Second, for similar reasons, a sum of $5\alpha+1$ can only be achieved by adding $3\alpha+(2\alpha+1-q_2)$ and a term $\sum_{i\in A'} a_i$, where A' is a subset of $\{1,\ldots,n\}$ such that $\sum_{i\in A'} a_i = q_2$.

Hence,

$$\#\textsc{SubsetSum}(A,q_2) = \#\textsc{SubsetSum}(B,5\alpha+1).$$

Thus,

$$\begin{array}{ccc} \#\textsc{SubsetSum}(A,q_1) & > & \#\textsc{SubsetSum}(A,q_2) \\ & \Longleftrightarrow & \\ \#\textsc{SubsetSum}(B,5\alpha) & > & \#\textsc{SubsetSum}(B,5\alpha+1), \end{array}$$

which completes the proof. ❑

Merging Is Hard for More Than Two Players

Theorem (Rey & Rothe)

BI-$\textsc{BeneficialMerge}$ *is* PP*-complete, even if only three players of equal weight merge.*

Proof: Membership of BI-$\textsc{BeneficialMerge}$ in PP follows from

- the fact that the raw Banzhaf index is in $\#\mathrm{P}$ and
- that $\#\mathrm{P}$ is closed under addition and
- since comparing the values of two $\#\mathrm{P}$ functions on two (possibly different) inputs reduces to a PP-complete problem and
- PP is closed under $\leq_{\mathrm{m}}^{\mathrm{p}}$-reducibility.

Merging Is Hard for More Than Two Players

We show PP-hardness of BI-BeneficialMerge by means of a $\leq_{\mathrm{m}}^{\mathrm{p}}$-reduction from Compare-#SubsetSum-RR, which is PP-hard by the previous lemmas.

Given an instance $A = (a_1, \ldots, a_n)$ of Compare-#SubsetSum-RR, construct the following instance for BI-BeneficialMerge.

Let $\alpha = \sum_{i=1}^{n} a_i$. Define the WVG

$$G = (2a_1, \ldots, 2a_n, 1, 1, 1, 1; \alpha)$$

with $n+4$ players, and let the merging coalition be

$$S = \{n+2, n+3, n+4\}.$$

Merging Is Hard for More Than Two Players

Letting $N = \{1, \ldots, n\}$, it holds that

$$
\begin{aligned}
&\mathrm{BI}(G, n+2) \\
&= \frac{1}{2^{n+3}} \left\| \left\{ C \subseteq \{1, \ldots, n+1, n+3, n+4\} \;\middle|\; \sum_{i \in C} w_i = \alpha - 1 \right\} \right\| \\
&= \frac{1}{2^{n+3}} \left(\left\| \left\{ A' \subseteq N \;\middle|\; \sum_{i \in A'} 2a_i = \alpha - 1 \right\} \right\| + 3 \cdot \left\| \left\{ A' \subseteq N \;\middle|\; 1 + \sum_{i \in A'} 2a_i = \alpha - 1 \right\} \right\| \right. \qquad (2) \\
&\qquad \left. + 3 \cdot \left\| \left\{ A' \subseteq N \;\middle|\; 2 + \sum_{i \in A'} 2a_i = \alpha - 1 \right\} \right\| + \left\| \left\{ A' \subseteq N \;\middle|\; 3 + \sum_{i \in A'} 2a_i = \alpha - 1 \right\} \right\| \right) \qquad (3) \\
&= \frac{1}{2^{n+3}} \left(3 \cdot \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{i \in A'} 2a_i = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{i \in A'} 2a_i = \alpha - 4 \right\} \right\| \right).
\end{aligned}
$$

Merging Is Hard for More Than Two Players

Explanation:

- The last equality holds since the $2a_i$'s can only add up to an even number.
- The first of the four sets in (2) and (3) refers to those coalitions that do not contain any of the players $n+1$, $n+3$, and $n+4$;
- the second, third, and fourth set in (2) and (3) refers to those coalitions containing either one, two, or three of them, respectively.

Since the players in S have the same weight, players $n+3$ and $n+4$ have the same probabilistic Banzhaf index as player $n+2$.

Merging Is Hard for More Than Two Players

The new game after merging is $G_{\&\{n+2,n+3,n+4\}} = (3, 2a_1, \ldots 2a_n, 1; \alpha)$ with $n+2$ players. Similarly as above, we calculate:

$$
\begin{aligned}
&\mathrm{BI}\left(G_{\&\{n+2,n+3,n+4\}}, 1\right)\\
&= \frac{1}{2^{n+1}} \left\| \left\{ C \subseteq \{2, \ldots, n+2\} \;\middle|\; \sum_{i \in C} w_i \in \{\alpha-3, \alpha-2, \alpha-1\} \right\} \right\|\\
&= \frac{1}{2^{n+1}} \left(\left\| \left\{ A' \subseteq N \;\middle|\; \sum_{i \in A'} 2a_i \in \{\alpha-3, \alpha-2, \alpha-1\} \right\} \right\| \right.\\
&\quad \left. + \left\| \left\{ A' \subseteq N \;\middle|\; 1 + \sum_{i \in A'} 2a_i \in \{\alpha-3, \alpha-2, \alpha-1\} \right\} \right\| \right)\\
&= \frac{1}{2^{n+1}} \left(2 \cdot \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{i \in A'} 2a_i = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{i \in A'} 2a_i = \alpha - 4 \right\} \right\| \right).
\end{aligned}
$$

Merging Is Hard for More Than Two Players

Altogether, it holds that

$$
\begin{aligned}
\mathrm{BI}\left(G_{\&\{n+2,n+3,n+4\}},1\right) \quad - \sum_{i\in\{n+2,n+3,n+4\}} \mathrm{BI}(G,i) \\
= \quad & \frac{1}{2^{n+1}}\left(2\cdot\left\|\left\{A'\subseteq N \;\middle|\; \sum_{i\in A'} 2a_i = \alpha-2\right\}\right\| + \left\|\left\{A'\subseteq N \;\middle|\; \sum_{i\in A'} 2a_i = \alpha-4\right\}\right\|\right) \\
& -\frac{3}{2^{n+3}}\left(3\cdot\left\|\left\{A'\subseteq N \;\middle|\; \sum_{i\in A'} 2a_i = \alpha-2\right\}\right\| + \left\|\left\{A'\subseteq N \;\middle|\; \sum_{i\in A'} 2a_i = \alpha-4\right\}\right\|\right) \\
= \quad & \left(\frac{1}{2^{n+1}}\cdot 2-\frac{3}{2^{n+3}}\cdot 3\right)\left\|\left\{A'\subseteq N \;\middle|\; \sum_{i\in A'} 2a_i = \alpha-2\right\}\right\| \\
& +\left(\frac{1}{2^{n+1}}-\frac{3}{2^{n+3}}\right)\left\|\left\{A'\subseteq N \;\middle|\; \sum_{i\in A'} 2a_i = \alpha-4\right\}\right\|
\end{aligned}
$$

Merging Is Hard for More Than Two Players

$$= \quad -\frac{1}{2^{n+3}} \cdot \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{i \in A'} a_i = \frac{\alpha}{2} - 1 \right\} \right\| + \frac{1}{2^{n+3}} \cdot \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{i \in A'} a_i = \frac{\alpha}{2} - 2 \right\} \right\|,$$

which is greater than zero if and only if

$$\left\| \left\{ A' \subseteq N \;\middle|\; \sum_{i \in A'} a_i = \frac{\alpha}{2} - 2 \right\} \right\| > \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{i \in A'} a_i = \frac{\alpha}{2} - 1 \right\} \right\|,$$

which in turn is the case if and only if the original instance A is in Compare-#SubsetSum-RR. ❑

Splitting Into More Than Two Players Is Hard

Theorem (Rey & Rothe)

Banzhaf-Beneficial-Split *is* PP-*hard, even if the given player can only split into three players of equal weight.*

Proof: We use the same techniques as in the previous proof, appropriately modified.

We show PP-hardness for $m = 3$ false identities.

(If $m > 3$, we split into $m - 3$ additional players of weight 0 each. Then the sum of all m new players' Banzhaf power is equal to the combined Banzhaf power of the three players.)

First, we slightly change the definition of Compare-#SubsetSum-RR by switching $\frac{\alpha}{2} - 2$ and $\frac{\alpha}{2} - 1$, yielding Compare-#SubsetSum-ЯЯ.

Splitting Into More Than Two Players Is Hard

Compare-#SubsetSum-RR

Given: A sequence $A = (a_1, \ldots, a_n)$ of positive integers.

Question: Is the number of subsequences of A summing up to $\frac{\alpha}{2} - 2$, where $\alpha = \sum_{i=1}^{n} a_i$, greater than the number of subsequences of A summing up to $\frac{\alpha}{2} - 1$, i.e., is it true that

$$\begin{aligned} & \#\textsc{SubsetSum}((a_1, \ldots, a_n), \frac{\alpha}{2} - 2) \\ > \quad & \#\textsc{SubsetSum}((a_1, \ldots, a_n), \frac{\alpha}{2} - 1) \text{ ?} \end{aligned}$$

Splitting Into More Than Two Players Is Hard

COMPARE-#SUBSETSUM-ЯЯ

Given: A sequence $A = (a_1, \ldots, a_n)$ of positive integers.

Question: Is the number of subsequences of A summing up to $\frac{\alpha}{2} - 1$, where $\alpha = \sum_{i=1}^{n} a_i$, greater than the number of subsequences of A summing up to $\frac{\alpha}{2} - 2$, i.e., is it true that

$$\begin{aligned} & \#\text{SUBSETSUM}((a_1, \ldots, a_n), \frac{\alpha}{2} - 1) \\ > \quad & \#\text{SUBSETSUM}((a_1, \ldots, a_n), \frac{\alpha}{2} - 2) \text{ ?} \end{aligned}$$

Splitting Into More Than Two Players Is Hard

We show

$$\text{Compare-\#SubsetSum-ЯЯ} \leq_{\mathrm{m}}^{\mathrm{p}} \text{Banzhaf-Beneficial-Split}.$$

Given an instance $A = (a_1, \ldots, a_n)$ of Compare-#SubsetSum-ЯЯ, construct the game $G = (2a_1, \ldots, 2a_n, 1, 3; \alpha)$, where $\alpha = \sum_{j=1}^{n} a_j$, and let $i = n+2$ be the player to be split.

G is (apart from the order of players) equivalent to the game obtained by merging in the previous proof.

Thus, letting $N = \{1, \ldots, n\}$, $\mathrm{BI}(G, n+2)$ equals

$$\frac{1}{2^{n+1}} \left(2 \cdot \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{j \in A'} 2a_j = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{j \in A'} 2a_j = \alpha - 4 \right\} \right\| \right).$$

Splitting Into More Than Two Players Is Hard

Allowing players with weight zero, there are different possibilities to split player $n+2$ into three players:

- Splitting $n+2$ into one player with weight 3 and two others with weight 0 is not beneficial, since adding a player with weight zero does not change the original players' power indices, and the new player's power index is zero.
- Likewise, splitting $n+2$ into two players with weights 1 and 2 and one player with weight 0 is not beneficial, since splitting into two players is not beneficial.
- Thus, the only possibility left is splitting $n+2$ into three players of weight 1 each.

Splitting Into More Than Two Players Is Hard

This corresponds to the original game in the previous proof:

$$G_{i \div 3} = (2a_1, \ldots, 2a_n, 1, 1, 1, 1; \alpha).$$

Therefore,

$$\mathrm{BI}(G_{i \div 3}, n+2) = \mathrm{BI}(G_{i \div 3}, n+3) = \mathrm{BI}(G_{i \div 3}, n+4) =$$
$$\frac{1}{2^{n+3}} \left(3 \cdot \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{j \in A'} 2a_j = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{j \in A'} 2a_j = \alpha - 4 \right\} \right\| \right).$$

Splitting Into More Than Two Players Is Hard

Altogether, as in the previous proof,

$$(\mathrm{BI}(G_{i\div 3}, n+2) + \mathrm{BI}(G_{i\div 3}, n+3) + \mathrm{BI}(G_{i\div 3}, n+4)) - \mathrm{BI}(G, n+2) > 0$$

if and only if

$$\left\| \left\{ A' \subseteq N \;\middle|\; \sum_{j \in A'} a_j = \frac{\alpha}{2} - 1 \right\} \right\| > \left\| \left\{ A' \subseteq N \;\middle|\; \sum_{j \in A'} a_j = \frac{\alpha}{2} - 2 \right\} \right\|,$$

which is true if and only if A is in Compare-#SubsetSum-ЯЯ. ❑

Structural Control by Adding or Deleting Players

Given a WVG G and a player i in G, can we

- increase,
- decrease, or
- maintain

i's power by adding players to G or deleting players from G?

Example

- Collective decision making: An organizer might invite further participants or might choose a certain meeting schedule to make sure that members originally expected to participate are now excluded.
- Machines may be needed to fulfill a certain task, independent of the number of currently available machines; some machines can be removed, new ones can be bought.

Deleting Players: Example

Example

Consider the WVG $G = (3,3,2,1;6)$. We have:

$$\mathrm{BI}(G,1) = \mathrm{BI}(G,2) = 1/2 \quad \text{and} \quad \mathrm{BI}(G,3) = \mathrm{BI}(G,4) = 1/4,$$
$$\mathrm{SSI}(G,1) = \mathrm{SSI}(G,2) = 1/3 \quad \text{and} \quad \mathrm{SSI}(G,3) = \mathrm{SSI}(G,4) = 1/6.$$

If we remove player 4, we obtain the new game $G_{\setminus\{4\}} = (3,3,2;6)$ with

$$\mathrm{BI}(G,1) = \mathrm{BI}(G,2) = 1/2 \quad \text{and} \quad \mathrm{BI}(G,3) = 0,$$
$$\mathrm{SSI}(G,1) = \mathrm{SSI}(G,2) = 1/2 \quad \text{and} \quad \mathrm{SSI}(G,3) = 0.$$

Players 1 and 2 have increased their SSI while maintaining their BI.

At the same time, both power indices of player 3 have decreased to 0.

Deleting Players: Change of PIs

Theorem (Rey & Rothe, 2018; Kaczmarek & Rothe, 2022)

After deleting the players of a subset $M \subseteq N \setminus \{i\}$ of size $m \geq 1$ from a WVG G with $n = |N|$ players, the difference between player i's old and new

1. *Penrose-Banzhaf index is at most $1 - 2^{-m}$ and is at least $-1 + 2^{-m}$;*
2. *Shapley-Shubik index is at most $1 - \frac{(n-m+1)!}{2n!}$ and is at least $-1 + \frac{(n-m+1)!}{2n!}$.*

Deleting Players: Change of PIs

Theorem (Kaczmarek & Rothe, 2022)

Let $G = (w_1, \ldots, w_n; q)$ be a WVG with players N. Let $M \subseteq N \setminus \{i\}$ be a set of players which are going to be deleted and $m = |M|$.

1. $\mathrm{BI}(G,i) - \mathrm{BI}(G_{\setminus M},i) \geq \max((1-2^m)\mathrm{BI}(G,i), \mathrm{BI}(G,i)-1)$,
2. $\mathrm{SSI}(G,i) - \mathrm{SSI}(G_{\setminus M},i) \geq \max((1-\binom{n}{m})\mathrm{SSI}(G,i), \mathrm{SSI}(G,i)-1)$

and

3. $\mathrm{BI}(G,i) - \mathrm{BI}(G_{\setminus M},i) \leq \min\left(\mathrm{BI}(G,i), \sum_{j \in M} \mathrm{BI}(G,j) + \frac{(2^m-1)^2}{2^{n-1}}\right)$,
4. $\mathrm{SSI}(G,i) - \mathrm{SSI}(G_{\setminus M},i) \leq \min\left(\mathrm{SSI}(G,i), \sum_{j \in M} \mathrm{SSI}(G,j) + \frac{1}{(n-m)!}\right)$.

Deleting Players: Change of PIs

Example

$G = (4, \mathbf{2}, 1, 1, 1; 4)$: Let $M = \{5\}$. Then

$$\mathrm{BI}(G,2) = \frac{1}{4} \quad \text{and} \quad \mathrm{BI}(G_{\setminus M},2) = \frac{1}{8}.$$

The upper bound from the first theorem is

$$\mathrm{BI}(G,2) - \mathrm{BI}(G_{\setminus M},2) \leq 1 - \frac{1}{2} = \frac{1}{2}$$

and that from the second theorem is

$$\mathrm{BI}(G,2) - \mathrm{BI}(G_{\setminus M},2) \leq \min(\frac{1}{4}, \frac{1}{8} + \frac{1}{16}) = \frac{3}{16}.$$

Deleting Players: Change of PIs

Example

$G = (4, \mathbf{2}, 1, 1, 1; 4)$: Let $M = \{5\}$. Then

$$\mathrm{SSI}(G,2) = \frac{11}{60} \quad \text{and} \quad \mathrm{SSI}(G_{\setminus M},2) = \frac{5}{60}.$$

The upper bound from the first theorem is

$$\mathrm{SSI}(G,2) - \mathrm{SSI}(G_{\setminus M},2) \leq 1 - \frac{(5-1+1)!}{2 \cdot 5!} = \frac{1}{2}$$

and that from the second theorem is

$$\mathrm{SSI}(G,2) - \mathrm{SSI}(G_{\setminus M},2) \leq \min(\frac{11}{60}, \frac{1}{10} + \frac{1}{4!}) = \frac{17}{120}.$$

Deleting Players: Control Problem

CONTROL BY DELETING PLAYERS TO INCREASE $\mathbb{PI}$

Given:
- A WVG G with players $N = \{1, \ldots, n\}$,
- a distinguished player $p \in N$, and
- a positive integer k.

Question: Can at most k players $M \subseteq N \setminus \{p\}$ be deleted from G such that for the new game $G_{\setminus M}$, it holds that

$$\mathbb{PI}(G_{\setminus M}, p) > \mathbb{PI}(G, p)?$$

Deleting Players: Overview of Complexity Results

Goal		**Control by deleting players**
Decrease	BI	$\mathrm{P}^{\mathrm{NP}[\log]}$-hard (Kaczmarek and Rothe, 2022)
	SSI	NP-hard (Kaczmarek and Rothe, 2022)
Increase	BI	DP-hard (Kaczmarek and Rothe, 2022)
	SSI	NP-hard (Rey and Rothe, 2018)
Maintain	BI	coNP-hard (Rey and Rothe, 2018)
	SSI	coNP-hard (Rey and Rothe, 2018)

Weighted Voting Games with Changing Quota

Definition (weighted voting game with quota change)

A **weighted voting game** **with changing quota** $G = (w_1, \ldots, w_n; r)$ is a simple coalitional game that consists of

- the players $N = \{1, \ldots, n\}$,
- weights $w_i \in \mathbb{R}_{\geq 0}$, $i \in N$, where w_i is the i-th player's weight, and
- a quota $q = r \sum_{i=1}^{n} w_i$ (i.e., a given threshold) for $r \in (0, 1]$.

Weighted Voting Games with Changing Quota

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- weights $w_i \in \mathbb{R}_{\geq 0}$, $i \in N$, where w_i is the i-th player's weight, and
- a quota $q = r \sum_{i=1}^{n} w_i$ (i.e., a given threshold) for $r \in (0, 1]$.

Again, for each coalition $S \subseteq N$, S *wins* if $w_S \geq q$, and *loses* otherwise:

$$v(S) = \begin{cases} 1 & \text{if } w_S \geq q, \\ 0 & \text{otherwise.} \end{cases}$$

Weighted Voting Games with Changing Quota

Example

Let $G = (10, 3, 10; 12)$ be a WVG without changing quota. Let us consider the following weighted voting games with changing quota:

- $H_1 = (10, 3, 10; \frac{12}{23})$:

$$q(H_1) = \frac{12}{23} \sum_{i=1}^{3} w_i = \frac{12}{23} \cdot 23 = 12,$$

- $H_2 = (10, 3, 10; \frac{1}{2})$:

$$q(H_2) = \frac{1}{2} \cdot 23 = 11.5.$$

Without any manipulation, G, H_1, and H_2 define the same game.

Adding Players in WVGs with Changing Quota

Example

$G = (\mathbf{1}, 2, 1, 1; \frac{1}{2})$:

$$q(G) = r \sum_{i=1}^{4} w_i = \frac{1}{2} \cdot 5 = 2.5,$$

$$\mathrm{BI}(G, 1) = \frac{1}{4}, \quad \mathrm{SSI}(G, 1) = \frac{1}{6}.$$

Then

$$q(G_{\cup\{5\}}) = r \sum_{i=1}^{5} w_i = \frac{1}{2} \cdot 8 = 4,$$

$$\mathrm{BI}(G_{\cup\{5\}}, 1) = \frac{3}{16}, \quad \mathrm{SSI}(G_{\cup\{5\}}, 1) = \frac{7}{60}.$$

Adding Players in WVGs with Changing Quota

Theorem (Kaczmarek & Rothe, 2022)

Let $G = (w_1, \ldots, w_n; r)$ be a WVG with changing quota with $q_1 = r\sum_{i=1}^{n} w_i$. Let N be a set of the players and M be a set of players which are added to the game G. Next, let $G_{\cup M}$ be a new game with a set of players $N \cup M$, $q_2 = r\sum_{j \in N \cup M} w_j$ and $m = |M|$. Then

1. $-1 + 2^{-m} \leq \mathrm{BI}(G, i) - \mathrm{BI}(G_{\cup M}, i) \leq 1$,
2. $-1 + \frac{(n+1)!}{2(n+m)!} \leq \mathrm{SSI}(G, i) - \mathrm{SSI}(G_{\cup M}, i) \leq 1$.

Adding Players in WVGs with Changing Quota

Example

$G = (\mathbf{2}, 1; \frac{2}{3})$:

$$\mathrm{BI}(G,1) = \mathrm{SSI}(G,1) = 1.$$

Let us add two players with weights $w_3 = w_4 = 4$. Then

$$q(G_{\cup\{3,4\}}) = \frac{22}{3}$$

and in $G_{\cup\{3,4\}} = (\mathbf{2}, 1, 4, 4; \frac{2}{3})$,

$$\mathrm{BI}(G_{\cup\{3,4\}},1) = \mathrm{SSI}(G_{\cup\{3,4\}},1) = 0.$$

Adding Players in WVGs with Changing Quota

CONTROL BY ADDING PLAYERS WITH CHANGING QUOTA TO INCREASE $\mathbb{PI}$

Given:
- A WVG G with players $N = \{1, \ldots, n\}$, a quota $r\sum_{i=1}^{n} w_i$ ($r \in (0,1]$),
- a set M of unregistered players with weights $w_{n+1}, \ldots, w_{n+m}$,
- a distinguished player $p \in N$, and
- a positive integer k.

Question: Can at most k players $M' \subseteq M$ be added to G such that for the new game $G_{\cup M'}$ with the new quota $r\sum_{i \in N \cup M'} w_i$, it holds that

$$\mathbb{PI}(G_{\cup M'}, p) > \mathbb{PI}(G, p)?$$

Adding Players in WVGs with Changing Quota: Complexity

Goal		Control by adding players
Decrease	BI	PP-hard
	SSI	PP-hard
Increase	BI	PP-hard
	SSI	PP-hard
Maintain	BI	coNP-hard
	SSI	coNP-hard

All results are due to Kaczmarek and Rothe (2022).

Deleting Players in WVGs with Changing Quota

Example

$G = (\mathbf{1}, 2, 1, 1; \frac{1}{2})$:

$$q(G) = r \sum_{i=1}^{4} w_i = \frac{1}{2} \cdot 5 = 2.5,$$

$$\mathrm{BI}(G, 1) = \frac{1}{4}, \quad \mathrm{SSI}(G, 1) = \frac{1}{6}.$$

Let $M = \{2\}$. Then

$$q(G_{\setminus M}) = \frac{1}{2} \cdot 3 = 1.5,$$

$$\mathrm{BI}(G_{\setminus M}, 1) = \frac{1}{2}, \quad \mathrm{SSI}(G_{\setminus M}, 1) = \frac{1}{3}.$$

Deleting Players in WVGs with Changing Quota

Theorem (Kaczmarek & Rothe, 2022)

Let $G = (w_1, \dots, w_n; r)$ be a WVG with changing quota with $q_1 = r\sum_{i=1}^{n} w_i$. Let N be a set of the players and $M \subseteq N \setminus \{i\}$ a set of players which are going to be deleted. Next, let $G_{\setminus M}$ be a new game with a set of players $N \setminus M$, $q_2 = r\sum_{j \in N \setminus M} w_j$ and $m = |M|$. Then

1. $-1 \leq \mathrm{BI}(G,i) - \mathrm{BI}(G_{\setminus M},i) \leq 1 - 2^{-m}$,
2. $-1 \leq \mathrm{SSI}(G,i) - \mathrm{SSI}(G_{\setminus M},i) \leq 1 - \frac{(n-m+1)!}{2n!}$.

Deleting Players in WVGs with Changing Quota

Example

$G = (\mathbf{3}, 5, 5, 3, 1, 1; \frac{5}{9})$: Let $M = \{3, 4\}$. Then

$$q(G) = 10, \quad q(G_{\setminus M}) = \frac{5}{9} \cdot 10 = \frac{50}{9},$$

$$\mathrm{BI}(G, 1) = \frac{1}{4} \quad \text{and} \quad \mathrm{BI}(G_{\setminus M}, 1) = \frac{1}{8}.$$

The upper bound from the theorem is

$$\mathrm{BI}(G, 1) - \mathrm{BI}(G_{\setminus M}, 1) \leq 1 - 2^{-2} = \frac{3}{4}.$$

Deleting Players in WVGs with Changing Quota

Example

$G = (\mathbf{3}, 5, 5, 3, 1, 1; \frac{5}{9})$: Let $M = \{3, 4\}$. Then

$$q(G) = 10, \quad q(G_{\setminus M}) = \frac{5}{9} \cdot 10 = \frac{50}{9},$$

$$\mathrm{SSI}(G, 1) = \frac{2}{15} \quad \text{and} \quad \mathrm{SSI}(G_{\setminus M}, 1) = \frac{1}{12}.$$

The upper bound from the theorem is

$$\mathrm{SSI}(G, 1) - \mathrm{SSI}(G_{\setminus M}, 1) \leq 1 - \frac{5!}{2 \cdot 6!} = \frac{11}{12}.$$

Deleting Players in WVGs with Changing Quota

Control by Deleting Players with Changing Quota to Increase $\mathbb{PI}$

Given:
- A WVG G with players $N = \{1, \ldots, n\}$, a quota $r \sum_{i=1}^{n} w_i$ ($r \in (0,1]$),
- a distinguished player $p \in N$, and
- a positive integer $k < |N|$.

Question: Can at most k players $M \subseteq N \setminus \{p\}$ be deleted from G such that for the new game $G_{\setminus M}$ with the new quota $r \sum_{i \in N \setminus M'} w_i$, it holds that

$$\mathbb{PI}(G_{\setminus M}, p) > \mathbb{PI}(G, p)?$$

Deleting Players in WVGs with Changing Quota: Complexity

Goal		Control by deleting players
Decrease	BI	DP-hard
	SSI	NP-hard
Increase	BI	DP-hard
	SSI	NP-hard
Maintain	BI	coNP-hard
	SSI	coNP-hard

All results are due to Kaczmarek and Rothe (2022).