Algorithmic Game Theory

Algorithmische Spieltheorie

Foundations of Cooperative Game Theory

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Cooperative versus Noncooperative Game Theory

Noncooperative Games

- Players compete against each other, selfishly seeking to realize their own goals and to maximize their own profit,
- everybody fights for herself, and
- nobody coalesces.
- However, players may also "cooperate" (e.g., preferring the **dove** over the **hawk** strategy in the chicken game).

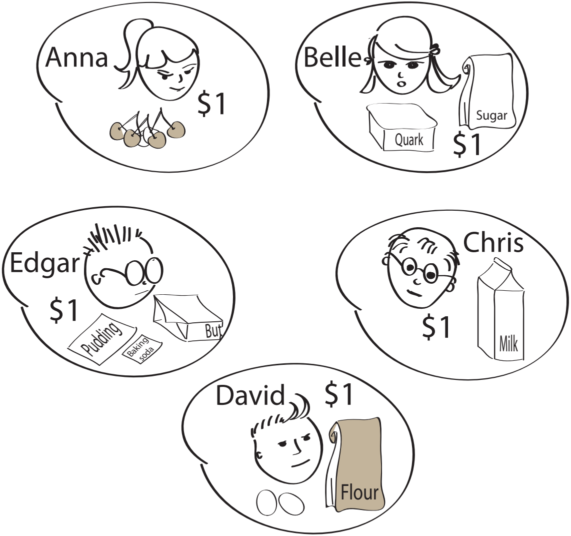
Cooperative Games

- Players work together by joining up in groups, so-called **coalitions**,
- they take **joint** actions so as to realize their goals, and
- they benefit from cooperating in coalitions if this helps them to raise their individual profit.
- However, players may join or leave a coalition to maximize their own, individual profit.

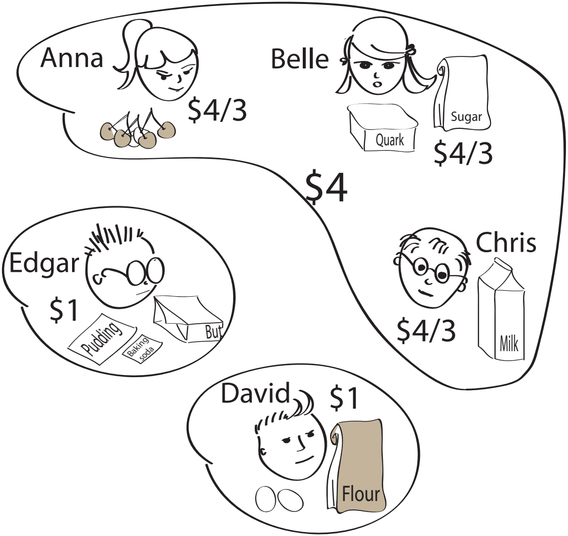
Cooperative versus Noncooperative Game Theory

- Both theories are concerned with certain aspects of cooperation as well as competition amongst players.
- Cooperative games may be viewed as the more general concept, since a noncooperative game may be seen as a cooperative game whose coalitions are singletons.
- Note, however, that there are various ways to generalize noncooperative games to cooperative games.
- The following example shows how coalitions of players can raise the profit of each member of a coalition by working together.

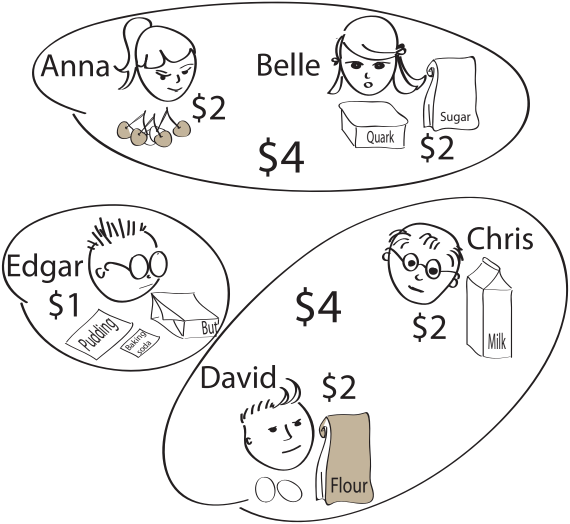
Story: Coalition Structure 1



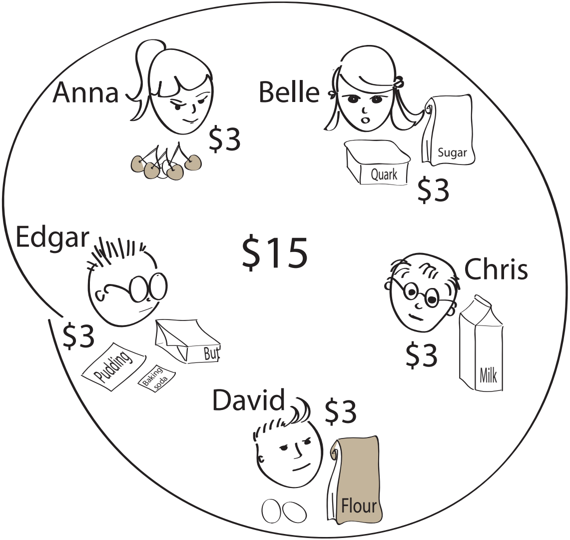
Story: Coalition Structure 2



Story: Coalition Structure 3



Story: Coalition Structure 4



Games with Transferable Utility

Definition (TU game)

A *cooperative game with transferable utility (TU game)* is given by a pair $G = (P, v)$,

- with the set $P = \{1, 2, \ldots, n\}$ of players and
- the *characteristic function*

$$v : 2^P \rightarrow \mathbb{R}^+$$

  (sometimes also referred to as the *coalitional function*), which for each subset (or *coalition*) $C \subseteq P$ of players indicates the utility (or gain) $v(C)$ that they attain by working together. Here, 2^P is the power set of P and $\mathbb{R}^+$ the set of nonnegative real numbers.

Games with Transferable Utility: Coalition Structure

It is common to assume that the characteristic function v of a TU game $G = (P, v)$ satisfies the following properties:

1. **Normalization**: $v(\emptyset) = 0$.
2. **Monotonicity**: $v(C) \leq v(D)$ for all coalitions C and D with $C \subseteq D$.

Definition (coalition structure)

- A *coalition structure* of a cooperative game $G = (P, v)$ with transferable utility is a partition $\mathfrak{C} = \{C_1, C_2, \ldots, C_k\}$ of P into pairwise disjoint coalitions, i.e., $\bigcup_{i=1}^{k} C_i = P$ and $C_i \cap C_j = \emptyset$ for $i \neq j$.
- The simplest coalition structure consists of only one coalition, the so-called *grand coalition*, embracing all players.
- For $C \subseteq P$, let $\mathscr{C}\mathscr{S}_C$ be the *set of coalition structures over C*.

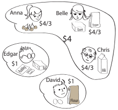
Games with Transferable Utility: Coalition Structure

Example

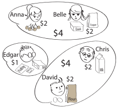
The four coalition structures from our example are represented as follows:

1. 

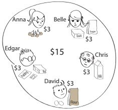
is $\mathfrak{C}_1 = \{\{\text{Anna}\}, \{\text{Belle}\}, \{\text{Chris}\}, \{\text{David}\}, \{\text{Edgar}\}\}$,

2. 

is $\mathfrak{C}_2 = \{\{\text{Anna, Belle, Chris}\}, \{\text{David}\}, \{\text{Edgar}\}\}$,

3. 

is $\mathfrak{C}_3 = \{\{\text{Anna, Belle}\}, \{\text{Chris, David}\}, \{\text{Edgar}\}\}$, and

4. 

is $\mathfrak{C}_4 = \{\{\text{Anna, Belle, Chris, David, Edgar}\}\}$.

Games with Transferable Utility: Coalition Structure

For n players, there are

- 2^n possible coalitions and
- $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$ possible coalition structures, where $B_0 = B_1 = 1$ and B_n is referred to as the *n-th Bell number*.

n	0	1	2	3	4	5	6	7	8	9	10
2^n	1	2	4	8	16	32	64	128	256	512	1 024
B_n	1	1	2	5	15	52	203	877	4 140	21 147	115 975

Games with Transferable Utility: Outcome

- For each coalition C, the value $v(C)$ merely indicates the joint gains of the players in C. However, it is also necessary to determine how these gains are then to be divided amongst them.

Definition (outcome of a TU game)

An *outcome* of a cooperative game $G=(P,v)$ with transferable utility is given by a pair $(\mathfrak{C},\vec{a})$, where $\mathfrak{C}$ is a coalition structure and $\vec{a}=(a_1,a_2,\ldots,a_n)\in\mathbb{R}^n$ is a payoff vector such that

$$a_i \geq 0 \text{ for each } i \in P \quad \text{and} \quad \sum_{i\in C} a_i \leq v(C) \text{ for each coalition } C \in \mathfrak{C}.$$

An outcome is said to be *efficient* if $\sum_{i\in C} a_i = v(C)$ for each $C\in\mathfrak{C}$. Abusing notation, we write $v(\mathfrak{C})=\sum_{C\in\mathfrak{C}} v(C)$ to denote the *social welfare of coalition structure* $\mathfrak{C}\in\mathscr{C}\mathscr{S}_P$.

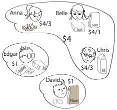
Games with Transferable Utility: Outcome

Example

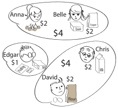
In our example, the four coalition structures have the following outcomes:

1. 

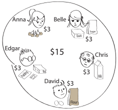
has the outcome $(\mathfrak{C}_1, \vec{a}_1)$ with $\vec{a}_1 = (1, 1, 1, 1, 1)$,

2. 

has the outcome $(\mathfrak{C}_2, \vec{a}_2)$ with $\vec{a}_2 = (4/3, 4/3, 4/3, 1, 1)$,

3. 

has the outcome $(\mathfrak{C}_3, \vec{a}_3)$ with $\vec{a}_3 = (2, 2, 2, 2, 1)$, and

4. 

has the outcome $(\mathfrak{C}_4, \vec{a}_4)$ with $\vec{a}_4 = (3, 3, 3, 3, 3)$.

Games with Nontransferable Utility: Story

There are also *cooperative games with nontransferable utility*.

Example:

- Think of n huskies that are supposed to drag several sledges from a research ship to a research station in Antarctica.
- Every husky has a different owner, and every sledge is dragged by a pack of huskies that tackle their task jointly.
- Depending on how such a pack (or "coalition") of huskies is assembled, they can solve their task more or less successfully.
- However, every husky will be rewarded only by its own owner, for example by getting more or less food, depending on how fast this husky's sledge has reached its destination.
- That means that gains are not transferable within a coalition.

Superadditive Games

Definition (superadditive game)

A cooperative game $G = (P, v)$ is said to be *superadditive* if for any two disjoint coalitions C and D, we have

$$v(C \cup D) \geq v(C) + v(D). \tag{1}$$

Example

If the characteristic function of G is defined by, say, $v(C) = \|C\|^2$, then G is superadditive because for any two disjoint coalitions C and D, we have:

$$v(C \cup D) = \|C \cup D\|^2 = (\|C\| + \|D\|)^2 \geq \|C\|^2 + \|D\|^2 = v(C) + v(D).$$

Superadditive Games: Properties

Fact

1. *Every superadditive game is monotonic.*
2. *There are monotonic games that are not superadditive.*

Proof:

1. Let $G = (P, v)$ be superadditive.

   Then, for all coalitions C, D with $C \subseteq D$:

   $$v(C) \leq v(D) - v(D \setminus C) \leq v(D),$$

   so $G = (P, v)$ is monotonic.

   First inequality: $D = C \cup (D \setminus C)$ and C and $(D \setminus C)$ are disjoint.

   Second inequality: $v(D \setminus C) \geq 0$.

Superadditive Games: Properties

2. Consider $G = (P, v)$ with $v(C) = \log \|C\|$.

   Then for $C \subseteq D$:

   $$v(C) = \log \|C\| \leq \log \|D\| = v(D),$$

   so $G = (P, v)$ is monotonic.

   However, if C' and D' are disjoint with, say, $\|C'\| = 4 = \|D'\|$, we have:

   $$\begin{aligned} v(C' \cup D') &= \log \|C' \cup D'\| = \log 8 = 3 < 4 = 2 + 2 \\ &= \log 4 + \log 4 = \log \|C'\| + \log \|D'\| = v(C') \cup v(D'), \end{aligned}$$

   so $G = (P, v)$ is not superadditive. ❑

Superadditive Games: Properties

- In superadditive games one may predict that the grand coalition will be formed, since any two coalitions can merge without loss.
- That is why one may identify the outcomes in superadditive cooperative games with the payoff vectors of the grand coalition and does not need to consider more complicated coalition structures.
- In practice, non-superadditive games may result from anti-trust or anti-monopolity laws.
- Every non-superadditive game $G = (P, v)$ can be transformed into a related superadditive game, its *superadditive cover* $G^* = (P, v^*)$ with

$$v^*(C) = \max_{\mathfrak{C} \in \mathscr{C}\mathscr{S}_C} v(\mathfrak{C})$$

for each $C \subseteq P$.

Stability Concepts for Cooperative Games

- In a cooperative game, payoff vectors specify how to distribute the jointly made profits within the coalitions, influencing their stability.
- In noncooperative game theory, we have already seen that stability concepts, such as the Nash equilibrium, play an important role.
- Players in a cooperative game are primarily interested in maximizing their own profit as well, and join a suitable coalition to this end.
- Suppose the grand coalition has formed. However, if some player can benefit from leaving the grand coalition, thus increasing her own profit, then she will do so, irrespective of the other players.
- If that happens, the game is *instable* and the grand coalition breaks up into several smaller coalitions.

Payoff Vectors and Imputations

- Players have an incentive to join the grand coalition in a cooperative game $G = (P, v)$ if the profit $v(P)$ of the grand coalition can be distributed among the single players by a *payoff vector* $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ such that the following properties hold:
  1. *efficiency*: $\sum_{i=1}^{n} a_i = v(P)$ and
  2. *individual rationality*: $a_i \geq v(\{i\})$ for all $i \in P$.
- No player can then make more profit alone than as a member of the grand coalition. This can be achieved exactly if $v(P) \geq \sum_{i \in P} v(\{i\})$.
- We collect all such payoff vectors, which are called the *imputations for G*, in the following set:

$$\mathscr{I}(G) = \left\{ (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} a_i = v(P) \text{ and } a_i \geq v(\{i\}),\ i \in P \right\}.$$

The Core of a Cooperative Game

- **Which imputations in $\mathscr{I}(G)$ make G stable?**
- There are various stability concepts for cooperative games, and a quite central one is the core of G.

Definition (core of a game)

Let $G = (P, v)$ be a cooperative game. For a coalition $C \subseteq P$ and a payoff vector $\vec{a} = (a_1, a_2, \ldots, a_n)$, let $a(C) = \sum_{i \in C} a_i$ denote the total payoff of C.

The *core of G* is defined as:

$$Core(G) = \left\{ \vec{a} \in \mathscr{I}(G) \;\middle|\; a(C) \geq v(C) \text{ for all coalitions } C \subseteq P \right\}.$$

Remark: We focus here on stabilizing the grand coalition.

More generally, stabilizing coalition structures, one can define the core as the set of outcomes $(\mathfrak{C}, \vec{a})$ with $a(C) \geq v(C)$ for each $C \subseteq P$.

The Core of a Cooperative Game

Example (Osborne and Rubinstein (1994))

- Consider a game with $n \geq 3$ players who want to play chess.
- Every pair of players appointed to play against each other receives one dollar.
- That is, the characteristic function of this game $G = (P, v)$ is given by

$$v(C) = \begin{cases} \|C\|/2 & \text{if } \|C\| \text{ is even} \\ (\|C\|-1)/2 & \text{if } \|C\| \text{ is odd} \end{cases}$$

for each coalition $C \subseteq P$.

The Core of a Cooperative Game

Example (Osborne and Rubinstein (1994), continued)

- If $n \geq 4$ is even, we have $(1/2, \ldots, 1/2) \in Core(G)$:
- It suffices to consider deviations by pairs of players (do you see why?),
- and any two players (whether or not they are currently matched to play with each other) jointly receive one dollar under this imputation, so they cannot do better by deviating.
- In fact, it can be shown that $(1/2, \ldots, 1/2)$ is the only imputation in the core of G, i.e., $Core(G) = \{(1/2, \ldots, 1/2)\}$ (exercise!).

The Core of a Cooperative Game

Example (Osborne and Rubinstein (1994), continued)

- However, if $n \geq 3$ is odd, one player remains without a partner.
- This implies that the core of G is empty in this case.
- Indeed, if the core of G were not empty for, say, $n = 3$ players, but would contain a vector $\vec{a} = (a_1, a_2, a_3)$, then since
$$a_1 + a_2 + a_3 = v(\{1,2,3\}) = 1,$$
at least one of the values a_i would be positive, say $a_1 > 0$.
- Thus $a_2 + a_3 < 1$.
- However, since $v(\{2,3\}) = 1$, we have a contradiction to the assumption that $\vec{a}$ is in $Core(G)$. Hence, $Core(G)$ is empty for $n = 3$ (and, by a similar argument, for any odd n).

The Core of a Cooperative Game

Theorem

Every outcome in the core of a cooperative game maximizes social welfare.
More precisely, if $(\mathfrak{C}, \vec{a}) \in Core(G)$ for a cooperative game $G = (P, v)$, then

$$v(\mathfrak{C}) \geq v(\mathfrak{C}')$$

for each coalition structure $\mathfrak{C}' \in \mathscr{C}\mathscr{S}_P$.

Proof: For a contradiction, suppose $v(\mathfrak{C}) < v(\mathfrak{C}')$ for some coalition structure $\mathfrak{C}' \in \mathscr{C}\mathscr{S}_P$. Then we have

$$\sum_{C' \in \mathfrak{C}'} a(C') = \sum_{i \in P} a_i = \sum_{C \in \mathfrak{C}} v(C) = v(\mathfrak{C}) < v(\mathfrak{C}') = \sum_{C' \in \mathfrak{C}'} v(C'). \qquad (2)$$

The Core of a Cooperative Game

But since $(\mathfrak{C}, \vec{a}) \in Core(G)$, we have $a(C') \geq v(C')$ for all $C' \in \mathfrak{C}'$.

Hence,

$$\sum_{C' \in \mathfrak{C}'} a(C') \geq \sum_{C' \in \mathfrak{C}'} v(C'),$$

contradicting (2). ❑

Theorem

There are cooperative games whose core is empty.

Proof: Consider the *3-player majority game* $G = (P, v)$ with $P = \{1, 2, 3\}$ and v defined for each coalition $C \subseteq P$ by

$$v(C) = \begin{cases} 1 & \text{if } \|C\| \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

The Core of a Cooperative Game

We show that $Core(G) = \emptyset$.

For a contradiction, suppose that $Core(G) \neq \emptyset$.

Since $v(P) = 1$, every $(\mathfrak{C}, (a_1, a_2, a_3)) \in Core(G)$ must satisfy:

$$a_1 \geq 0, \quad a_2 \geq 0, \quad a_3 \geq 0, \quad \text{and} \quad a_1 + a_2 + a_3 \geq 1.$$

Hence, $a_i \geq 1/3$ for some $i \in \{1, 2, 3\}$.

But since $v(\mathfrak{C}) \leq 1$ for each coalition structure $\mathfrak{C} \in \mathscr{CS}_P$ (because at most one coalition C can satisfy $\|C\| \geq 2$), we also have

$$a_1 + a_2 + a_3 \leq 1.$$

For $C = P \setminus \{i\}$, we have $v(C) = 1$ and $a(C) \leq 2/3$.

Hence, $a(C) < v(C)$, so $(\mathfrak{C}, (a_1, a_2, a_3)) \notin Core(G)$, a contradiction. ❑

The Core of Superadditive Games

- Recall that we identify the outcomes in superadditive games with the payoff vectors of the grand coalition and do not need to consider more complicated coalition structures.
- **Does this restriction eliminate some of the core outcomes? That is, can the core of a superadditive game contain an outcome where the grand coalition does not form?**
- **No: For any such outcome, there is an essentially equivalent outcome (with the same payoff vector) where the grand coalition forms.**
- However, if a game is not superadditive, its core can be nonempty, even though no outcome in which the grand coalition forms is stable.

The Core of Superadditive Games

Theorem

A cooperative game $G = (P, v)$ has a nonempty core if and only if its superadditive cover $G^* = (P, v^*)$ has a nonempty core.

Proof: $(\Rightarrow)$ Suppose $Core(G) \neq \emptyset$ and let $(\mathfrak{C}, \vec{a}) \in Core(G)$. We know:

Every outcome in the core of a cooperative game maximizes social welfare.

Hence, $v^*(P) = v(\mathfrak{C})$. Thus $\vec{a}$ is a payoff vector for P in G^*.

We show that $\vec{a}$ satisfies the core constraints in (G^*, v^*).

For a contradiction, suppose $a(C) < v^*(C)$ for some coalition $C \subseteq P$.

The Core of Superadditive Games

It holds that

$$
\begin{array}{lll}
 & v^*(C) = v(\mathfrak{C}') & \text{for some coalition structure } \mathfrak{C}' \in \mathscr{C}\mathscr{S}_P \\
\Rightarrow & a(C) < v(\mathfrak{C}') & \text{for some coalition structure } \mathfrak{C}' \in \mathscr{C}\mathscr{S}_P \\
\Rightarrow & a(C') < v(C') & \text{for some coalition } C' \in \mathfrak{C}',
\end{array}
$$

which contradicts $(\mathfrak{C}, \vec{a}) \in Core(G)$.

Hence, $\vec{a} \in Core(G^*)$.

The Core of Superadditive Games

($\Rightarrow$) Conversely, suppose $Core(G^*) \neq \emptyset$ and let $\vec{a} \in Core(G^*)$.

Let $\mathfrak{C}$ be any coalition structure with $v(\mathfrak{C}) = v^*(P)$.

Then $v(\mathfrak{C}) \geq a(P)$.

But since $\vec{a} \in Core(G^*)$, we have $a(C) \geq v^*(C) \geq v(C)$ for each $C \in \mathfrak{C}$.

If we add these $\|\mathfrak{C}\|$ inequalities, we get $a(P) \geq v(\mathfrak{C})$, so they are equations: $a(C) = v(C)$ for each $C \in \mathfrak{C}$.

Hence, $\vec{a}$ is a payoff vector for $\mathfrak{C}$.

Again, since $\vec{a} \in Core(G^*)$, we have $a(C) \geq v^*(C) \geq v(C)$ for all $C \subseteq P$.

Thus $(\mathfrak{C}, \vec{a}) \in Core(G)$. ❑

The Core of Superadditive Games

Remark

- *Superadditivity is often assumed and justified by the above result.*
- *However, this approach can be problematic*
  - *for other solution concepts (such as the Shapley value) if cross-coalitional transfers are not allowed;*
  - *because the characteristic function v^* of G^* may be hard to compute.*

The Core of Superadditive Games

- Cooperative games whose core is empty are unstable.
- **When does a game have a nonempty core?**
- For superadditive games $G = (P, v)$, this question boils down to checking whether the following linear program with variables $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $2^n + n + 1$ constraints has a feasible solution:

$$
\begin{array}{rcll}
a_i & \geq & 0 & \text{for all } i \in P \\
\sum_{i \in P} a_i & = & v(P) & \\
\sum_{i \in C} a_i & \geq & v(C) & \text{for all } C \subseteq P.
\end{array}
$$

- For some classes of cooperative games, it is possible to solve this linear program efficiently (i.e., in time polynomial in the number n of players), even though the number of constraints in it is exponential in n.

The ε-Core of a Superadditive Game

Shapley and Shubik (1966) introduced the following relaxation of the core:

Definition (ε-core of a superadditive game)

The *(strong) ε-core of a superadditive game $G = (P, v)$* is defined as:

$$\varepsilon\text{-}Core(G) = \left\{ \vec{a} \in \mathscr{P}\mathscr{V}(G) \;\middle|\; a(P) = v(P),\ a(C) \geq v(C) - \varepsilon \text{ for all } C \subset P \right\},$$

where $\mathscr{P}\mathscr{V}(G)$ denotes the set of all payoff vectors for P.

- If $\varepsilon > 0$, ε-$Core(G)$ contains those payoff vectors for which a coalition $C \subset P$ willing to leave the grand coalition has to pay a penalty of ε.
- For $\varepsilon = 0$, we have 0-$Core(G) = Core(G)$.
- General case not considered here: If $\varepsilon < 0$, then deviating from the grand coalition is made easier for C by paying a bonus of ε.

The Least Core of a Superadditive Game

- Certainly, even if the core of a game is empty, one can ensure this game to have a **nonempty ε-core** by choosing ε large enough.
- On the other hand, choosing ε small enough (indeed, negative), one can ensure this game to have an **empty ε-core**, even though its core may be nonempty.
- Maschler, Peleg, and Shapley (1979) generalized this idea by introducing the least core of a game.

Definition (least core of a superadditive game)

The *least core of a superadditive game G* is defined to be the intersection of all nonempty ε-cores of G.

The Least Core of a Superadditive Game

- Alternatively, the *least core of G* can be defined as its $\tilde{\varepsilon}$-core, where $\tilde{\varepsilon}$ is chosen so that
  - the $\tilde{\varepsilon}$-core of G is nonempty, but
  - the ε-core of G is empty for all values $\varepsilon < \tilde{\varepsilon}$.
- By definition, the least core of a game is never empty. Intuitively, the least core consists of the most stable outcomes of the game.
- It can be shown that the *value $\tilde{\varepsilon}$ of the least core* is well-defined. It can be computed by a linear program similar to the one for the core:

$$
\begin{aligned}
\min \quad & \varepsilon & \\
a_i &\geq 0 & \text{for all } i \in P \\
\textstyle\sum_{i \in P} a_i &= v(P) & \\
\textstyle\sum_{i \in C} a_i &\geq v(C) - \varepsilon & \text{for all } C \subset P.
\end{aligned}
$$

Cost of Stability for Superadditive Games: Ice Story

Two types of ice cream tubs:

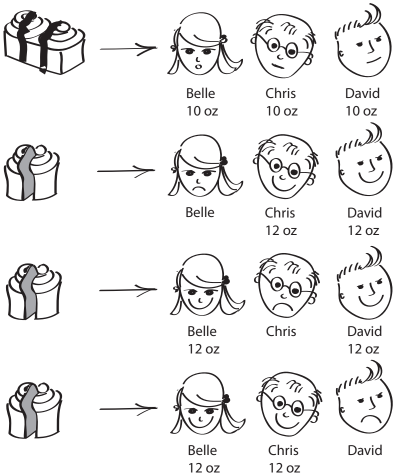
a small one (24 oz) for $ 8

a large one (30 oz) for $ 10

Belle has $ 5

Chris has $ 4

David has $ 4



Cost of Stability for Superadditive Games: Ice Story

Two types of ice cream tubs:

a small one (24 oz) for \$ 8

a large one (30 oz) for \$ 10

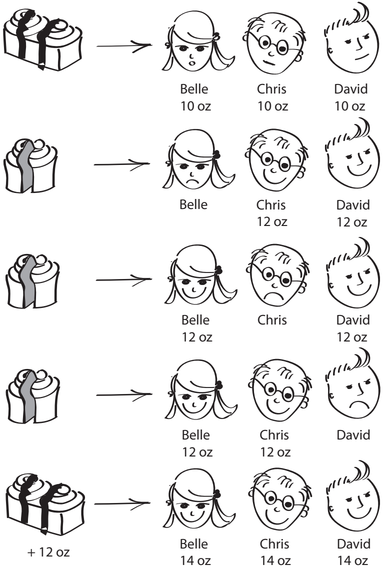
Belle has \$ 5

Chris has \$ 4

David has \$ 4

Chris's mother supplements it with 12 oz of ice cream from the freezer

David: "10 oz extra would do the trick, or maybe even less?"



Cost of Stability for Superadditive Games: X-Ray Story

- Suppose three private hospitals in the same community want to purchase an X-ray machine:
  - The **standard X-ray machine costs \$5 million**, but can comply with the requirements of only two hospitals.
  - A **more advanced machine**, which is capable of fulfilling the needs of all three hospitals, **costs \$9 million**.
- If all three hospitals join their forces and buy the more advanced, though more expensive, X-ray machine, this will cost them less than buying two standard X-ray machines.
- However, the three hospital managers cannot settle on how to distribute the cost for this more expensive machine among each other:

  If each hospital pays one third of the \$9 million, every pair of hospitals would be better off by leaving the grand coalition and to be content with the cheaper machine for themselves.

Cost of Stability for Superadditive Games: X-Ray Story

- Luckily for the three, the municipal council decides to solve this issue by subsidizing the more advanced X-ray machine with a supplement payment of \$3 million, so each hospital needs to add only \$2 million.
- This means that every pair of hospitals now pays only \$4 million together, and so has no longer an incentive to leave the grand coalition and to buy the less efficient, cheaper X-ray machine.
- **But wait!** Isn't that an incredible waste of tax money?
  Indeed, the positive effect of subsidizing the more advanced X-ray machine so as to stabilizing the grand coalition could have been achieved as well at a lower cost:

  > A subsidy of **\$1.5 million would have been enough** to ensure that no hospital manager has an incentive to leave the grand coalition.

Cost of Stability for Superadditive Games

- **Questions:**
  - What is the *minimum* external supplement payment needed to stabilize the grand coalition?
  - How hard is it to determine these cost of stability?
- Bachrach, Elkind, Meir, Pasechnik, Zuckerman, Rothe, and Rosenschein (2009) study such questions as follows:
  - Suppose an external party is interested in stabilizing the grand coalition in a game with an empty core, and is willing to pay for that.
  - The payment is done only if the players do not deviate from the grand coalition.
  - This amount plus the actual gains of the grand coalition will then be distributed among the players to ensure stability.
  - The *cost of stability for G* are defined to be the amount of the smallest supplement payment stabilizing G.

Cost of Stability for Superadditive Games

Definition (cost of stability)

For a superadditive game $G = (P, v)$ and a supplement payment of $\Delta \geq 0$, the *adjusted game* $G_\Delta = (P, v_\Delta)$ is given by

- $v_\Delta(C) = v(C)$ for $C \neq P$ and
- $v_\Delta(P) = v(P) + \Delta$.

The *cost of stability for G* is defined by

$$CoS(G) = \inf\{\Delta \mid \Delta \geq 0 \text{ and } Core(G_\Delta) \neq \emptyset\}.$$

An imputation $\vec{a}$ for G_Δ is not an imputation for G since $a(P) > v(P)$ is possible. Therefore, we call it a *super-imputation for G*.

Cost of Stability for Superadditive Games

Remark

- **What is the relationship between the value of the least core and the cost of stability?**
  - *Both of these quantities are strictly positive if and only if the core is empty. That is, $\tilde{\varepsilon} > 0$ if and only if $CoS(G) > 0$.*
  - *However, they capture two very different approaches to dealing with coalitional instability:*
    - *The least core corresponds to punishing undesirable behavior (i.e., making deviations more costly).*
    - *The cost of stability corresponds to encouraging desirable behavior (i.e., making staying in the grand coalition more attractive).*

Cost of Stability for Superadditive Games

Fact

In general, $CoS(G) \leq n \cdot \tilde{\varepsilon}(G)$, and there are examples where this bound is tight.

Proof: Clearly, if $\tilde{\varepsilon}(G) = 0$, we have $CoS(G) = 0$.

Now, assume $\tilde{\varepsilon}(G) > 0$.

Let $\vec{a}$ be an imputation in the least core of G.

For any $C \subseteq P$, we have $a(C) \geq v(C) - \tilde{\varepsilon}(G)$.

Consider a super-imputation $\vec{a}^*$ given by $a_i^* = a_i + \tilde{\varepsilon}(G)$.

Clearly, we have $a^*(C) \geq v(C)$ for any $C \subseteq P$ such that $C \neq \emptyset$.

Further, it is easy to see that $a^*(P) = v(P) + n\tilde{\varepsilon}(G)$, so $CoS(G) \leq n\tilde{\varepsilon}(G)$.

Cost of Stability for Superadditive Games

To see that this bound is tight, consider the game $G = (P, v)$ with $\|P\| = n$ and

$$v(C) = \begin{cases} 0 & \text{if } C = \emptyset \\ 1 & \text{if } C \neq \emptyset. \end{cases}$$

It is easy to see that

(a) $\tilde{\varepsilon}(G) = \frac{n-1}{n}$, since the imputation $(\frac{1}{n}, \ldots, \frac{1}{n})$ is in the least core of G.

(b) On the other hand, $CoS(G) = n - 1 = n\tilde{\varepsilon}(G)$. ❏

Cost of Stability for Superadditive Games

Remark (continued)

- *This notion can be extended to games that are not superadditive:*
  - *either aiming at stabilizing the grand coalition as well,*
  - *or looking for a coalition structure that is a cheapest to stabilize.*
- *The cost of stability can be bounded as follows:*

$$0 \leq CoS(G) \leq n \cdot \max_{C \subseteq P} v(C)$$

and more generally:

$$\max_{\mathfrak{C} \in \mathscr{C}\mathscr{S}_P} (v(\mathfrak{C}) - v(P)) \leq CoS(G) \leq n \cdot \max_{C \subseteq P} v(C),$$

where $v(\mathfrak{C}) = \sum_{C_j \in \mathfrak{C}} v(C_j)$ *for* $\mathfrak{C} = \{C_1, C_2, \ldots, C_m\}$.

Cost of Stability for Superadditive Games

Remark (continued)

- *Just as the least core, the cost of stability can be computed by solving a linear program that has a constraint for each coalition:*

$$
\begin{aligned}
\min \ \Delta & & \\
\Delta &\geq 0 & \\
a_i &\geq 0 & \text{for all } i \in P \\
\textstyle\sum_{i \in P} a_i &= v(P) + \Delta & \\
\textstyle\sum_{i \in C} a_i &\geq v(C) & \text{for all } C \subseteq P.
\end{aligned}
$$

Cost of Stability for Superadditive Games

Remark (continued)

- *We have to impose the constraint* $\Delta \geq 0$*: Without it, if the game has a nonempty core, the value of this LP may be negative, which would correspond to imposing a fine on the grand coalition.*
- *This LP implicitly shows that the cost of stability for* G*,* $CoS(G)$*, is well-defined because the set* $\{\Delta \mid \Delta \geq 0 \text{ and } Core(G_\Delta) \neq \emptyset\}$ *contains its greatest lower bound* $CoS(G)$*, i.e., the game* $G_{CoS(G)}$ *has a nonempty core:*
  - *The optimal value of this LP is exactly* $CoS(G)$*.*
  - *Every optimal solution of this LP corresponds to an imputation in the core of* $G_{CoS(G)}$*.*

Cost of Stability for Superadditive Games

Theorem

For each superadditive game $G = (P, v)$ with $n = \|P\|$ players, we have

$$CoS(G) \leq (\sqrt{n} - 1)v(P),$$

and this bound is asymptotically tight. **without proof**

Definition (anonymous game)

A cooperative game $G = (P, v)$ is *anonymous* if $v(C) = v(C')$ for all $C, C' \subseteq P$ with $\|C\| = \|C'\|$.

Cost of Stability for Superadditive Games

Theorem

For each anonymous, superadditive game $G = (P, v)$, we have $CoS(G) \leq v(P)$, and this bound is asymptotically tight.

Proof: Fix an anonymous, superadditive game $G = (P, v)$ with $\|P\| = n$.

Consider a super-imputation $\vec{a} = (a_1, \ldots, a_n)$ given by $a_i = \frac{2v(P)}{n}$.

Clearly, we have $a(P) = 2v(P)$.

It remains to show that $\vec{a}$ is in the core of the adjusted game $G_{v(P)}$.

For any coalition $C \subset P$, there exists an integer k, $1 \leq k \leq n-1$, such that

$$\frac{n}{k+1} \leq \|C\| < \frac{n}{k}.$$

Cost of Stability for Superadditive Games

For this value of k, one can construct k pairwise disjoint coalitions $C_1, \ldots, C_k$ with $C_1 = C$ and $\|C_1\| = \cdots = \|C_k\|$.

Superadditivity then implies that $v(C) \leq \frac{v(P)}{k}$.

On the other hand, we have

$$a(C) = \|C\| \frac{2v(P)}{n} \geq \frac{n}{k+1} \cdot \frac{2v(P)}{n} = \frac{2v(P)}{k+1}.$$

Since $\frac{2v(P)}{k+1} \geq \frac{v(P)}{k}$ for any $k \geq 1$, it follows that $a(C) \geq v(C)$ for all $C \subset P$, so $\vec{a}$ is stable.

To see that this bound is tight, consider a game $G = (P, v)$ with $\|P\| = n = 2k+1$ given by $v(C) = 0$ if $\|C\| \leq k$, and $v(C) = 1$ if $\|C\| \geq k+1$.

Clearly, this game is anonymous.

Cost of Stability for Superadditive Games

Moreover, as any two winning coalitions intersect, this game is also superadditive.

Consider any stable super-imputation $\vec{a}$ for this game.

For any C with $\|C\| = k+1$, we have $\sum_{i \in C} a_i \geq 1$.

There are exactly $\binom{n}{k+1}$ coalitions of this size, and each agent participates in exactly $\binom{n-1}{k}$ such coalitions.

Thus, summing all these inequalities, we obtain $\binom{n-1}{k} a(P) \geq \binom{n}{k+1}$, or, canceling,

$$a(P) \geq \frac{n}{k+1} = 2 - \frac{1}{k+1}. \quad \square$$

Stable Sets in a Cooperative Game

- Another stability concept—and the first one ever—has been introduced by von Neumann and Morgenstern (1944).

Definition (dominance)

If $G = (P, v)$ is a superadditive game with more than two players and if $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{b} = (b_1, b_2, \ldots, b_n)$ are in $\mathscr{I}(G)$, we say *$\vec{a}$ dominates $\vec{b}$ via a coalition $C \neq \emptyset$* (denoted by $\vec{a} \succ_C \vec{b}$) if

- $a_i > b_i$ for all $i \in C$ and
- $a(C) \leq v(C)$.

We say that *$\vec{a}$ dominates $\vec{b}$* (and write $\vec{a} \succ \vec{b}$) if $\vec{a} \succ_C \vec{b}$ for some nonempty set $C \subseteq P$.

Stable Sets in a Cooperative Game

- Here, $a_i > b_i$ for all $i \in C$ means that the players in C would prefer $\vec{a}$ to $\vec{b}$ as their payoff vector, since each of them would strictly benefit from that, and
- $a(C) \leq v(C)$ means that the players in C can plausibly threaten to leave the grand coalition.
- This notion is reminiscent of that of dominant strategy in a noncooperative game.
- The dominance relation is not necessarily antisymmetric:
  It may be the case that $\vec{a} \succ_{C_1} \vec{b}$ and $\vec{b} \succ_{C_2} \vec{a}$ for distinct payoff vectors $\vec{a}$ and $\vec{b}$, as long as C_1 and C_2 are disjoint.

Stable Sets in a Cooperative Game

Mr. & Mrs. Smith – Shooting Scene

Stable Sets in a Cooperative Game

Definition (stable sets)

A *stable set of a superadditive game G* is a subset $S \subseteq \mathscr{I}(G)$ satisfying the following two conditions:

1. *Internal stability*: No vector $\vec{a} \in S$ is dominated by a vector $\vec{b} \in S$.
2. *External stability*: For all $\vec{b} \in \mathscr{I}(G) \smallsetminus S$, there is a vector $\vec{a} \in S$ such that $\vec{b}$ is dominated by $\vec{a}$.

Interpretation:

- Due to internal stability of S, there is no reason to remove a payoff vector from S.
- Due to external stability of S, there is no reason to add another payoff vector to S.

Stable Sets in a Cooperative Game

This interpretation is explained by von Neumann and Morgenstern (1944) as follows:

A stable set can be seen as a list of **"acceptable behaviors"** in a society.

- No behavior within this list is strictly superior to another behavior in the list.
- However, for each inacceptable behavior there is an acceptable behavior that is preferrable.

Remark

- *Stable sets exist in some, yet not in all cooperative games.*
- *If they exist, they usually are not unique, and also hard to find.*

Stable Sets in a Cooperative Game

Remark (continued)

- *These are some of the reasons why also other stability or solution concepts, such as the core, have been proposed.*
- *How is the core related to stable sets?*

Theorem

1. *If the core of G is nonempty, it is contained in all stable sets of G.*
2. *However, the core itself is not necessarily a stable set.*

Remark

In fact, if the core of a game G is a stable set, then it is its only stable set.

Stable Sets in a Cooperative Game

Proof:

1. Suppose that $\vec{a}$ is an imputation in the core of G.

   Then it cannot be dominated by any other imputation:

   - Suppose that it is dominated by some imputation $\vec{b}$, i.e., there exists a coalition C with $b_i > a_i$ for all $i \in C$ and $b(C) \leq v(C)$.
   - Then we have $a(C) = \sum_{i \in C} a_i < \sum_{i \in C} b_i = b(C) \leq v(C)$, a contradiction with $\vec{a}$ being in the core of G.

   This implies that $\vec{a}$ belongs to every stable set of G.

2. While the argument above implies that the core satisfies the condition of internal stability, it may fail external stability.

Stable Sets in a Cooperative Game

- A simple example is a game $G^1 = (P, v)$ with $P = \{1, 2, 3\}$ and

$$v(C) = \begin{cases} 1 & \text{if } 1 \in C \text{ and } \|C\| \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

- It can be shown that the only imputation in the core of this game is $\vec{a} = (1, 0, 0)$.

- Now, consider the imputation $\vec{b} = (0, 1/2, 1/2)$.
  It is easy to see that $\vec{a}$ does not dominate $\vec{b}$:
  The only coalition C such that $a_i > b_i$ for all $i \in C$ is $\{1\}$, and

$$a(\{1\}) = \sum_{i \in \{1\}} a_i = a_1 = 1 > 0 = v(\{1\}). \quad ❑$$

Some Other Stability/Solution Concepts

Remark

- *We have seen the following important stability/solution concepts:*
  - *the core,*
  - *the ε-core,*
  - *the least core,*
  - *the cost of stability, and*
  - *von Neumann and Morgenstern's stable sets.*
- *Some other stability/solution concepts (most of which will not be considered here) include:*
  - *the nucleolus,*
  - *the kernel,*
  - *the bargaining set, and*
  - *the Shapley value (to be introduced later on).*