Algorithmic Game Theory

Algorithmische Spieltheorie

Foundations of Noncooperative Game Theory

Wintersemester 2022/2023

Dozent: Prof. Dr. J. Rothe

Websites

- Vorlesungswebsite:

  https://ccc.cs.uni-duesseldorf.de/~rothe/games

- Anmeldung zur Vorlesung im LSF:

  https://lsf.uni-duesseldorf.de

- Anmeldung zur Prüfung über das Studierendenportal:

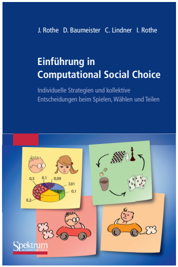
  https://studierende.uni-duesseldorf.de

- Weitere Informationen über ILIAS:

  https://ilias.uni-duesseldorf.de

Literature


Springer Texts in Business and Economics
Jörg Rothe
Editor
Economics and Computation
An Introduction to Algorithmic Game Theory, Computational Social Choice, and Fair Division
Springer


J. Rothe D. Baumeister C. Lindner I. Rothe
Einführung in Computational Social Choice
Individuelle Strategien und kollektive Entscheidungen beim Spielen, Wählen und Teilen
Spektrum

Literature: Email from Prof. Michael Wooldridge, Oxford

Dear Joerg,

I just received a copy of "Economics and Computation". It looks FANTASTIC! I already started reading some of it, and I think we will use it on a course we are giving here next year.

It was tremendously kind of you to think about sending me a copy – I'm very grateful!

Congratulations, and thanks again!

Mike

–

Professor Michael Wooldridge mailto:mjw@cs.ox.ac.uk
Department of Computer Science, University of Oxford.

http://www.cs.ox.ac.uk/people/michael.wooldridge/

Literature (Recommended for Additional Reading)

- **G. Chalkiadakis, E. Elkind, and M. Wooldridge: Computational Aspects of Cooperative Game Theory.** Morgan and Claypool Publishers, 2011
- **N. Nisan, T. Roughgarden, É. Tardos, and V. Vazirani: Algorithmic Game Theory.** Cambridge University Press, 2007
- **B. Peleg and P. Sudhölter: Introduction to the Theory of Cooperative Games.** Kluwer Academic Publishers, 2003
- **M. Osborne and A. Rubinstein: A Course in Game Theory.** MIT Press, 1994
- **J. von Neumann and O. Morgenstern: Theory of Games and Economic Behavior.** Princeton University Press, 1944

Two Quotes

"Der Mensch spielt nur, wo er in voller Bedeutung des Wortes Mensch ist, und er ist nur da ganz Mensch, wo er spielt."

Friedrich Schiller

Briefe über die ästhetische Erziehung des Menschen (1795)

"Blödem Volke unverständlich treiben wir des Lebens Spiel."

Christian Morgenstern

Galgenlieder (1905)

Foundations of Noncooperative Game Theory: Players

- Consider a *set $P = \{1, 2, \ldots, n\}$ of players*; occasionally, they will have names instead of numbers.
- Who these players are and how many of them there are depends on the game being played.
- Players can be
  - individual persons as well as
  - groups of individuals (as in a team sport),
  - they can be computer programs,
  - states (or their governments),
  - companies,
  - ethnic groups,
  - organizations,
  - etc.

Foundations of Noncooperative Game Theory: Rules

- A game is defined by its *rules*, which describe how the game is to be played and what each player is allowed or not allowed to do in which situation.
- It must also be always clear
  - what the single players can know about the current situation of the game,
  - when a game is over, and
  - who has won why and how much.

Foundations of Noncooperative Game Theory: Strategies

- Unlike the rules of a game, the players' *strategies* constitute complete and precise policies of how to act in each possible situation they might encounter during the game.
- Depending on the current game situation, a player can have several options for how to proceed, and so can have a choice between alternative *actions*, which all must be rule-consistent, of course.
- If a player has no further alternatives to choose from, this often (although not always) means that the game is over and ends with this player's defeat (as, for example, in the case of a "checkmate" in chess).

The Prisoners' Dilemma: Story

- Two most wanted criminals, in the underground milieu only known as "Smith & Wesson," have been arrested.
- They are being accused of a joint bank robbery, but incriminatory evidence is too thin to meet court standards, unfortunately.
- The detective superintendent responsible for this investigation interrogates both criminals separately, one after the other, and Smith is brought before him first.
- Despite intensive interrogation, Smith perseveringly remains silent, so the detective superintendent offers him a deal.

The Prisoners' Dilemma: Story

- "You do know the maximum penalty for bank robbery, Smith," he points out, "ten years behind bars!
- However, if you confess that you and Wesson have mugged the bank together, then I can guarantee you that the judge will
  - suspend your sentence on probation for good collaboration with the authorities, and
  - Wesson has to serve his ten years alone, if he goes on to be stubborn."

  Smith remains silent.
- "Think about it by tomorrow," the detective superintendent adds. "I will now offer Wesson the same deal."
- When Smith is led away, he turns around once more and asks: "What if Wesson makes a confession and incriminates me?"

The Prisoners' Dilemma: Story

- "It depends," the detective superintendent replies.
- "If only he confesses and you do not, then
  - his sentence will be suspended on probation and
  - you'll go to jail for ten years.
- If you both confess, then you both will have to serve four years, despite your collaboration with us, because the other confession is less valuable for us, as the first one would have been enough, and that applies to each of you."
- "And if none of us don't say nothing?"
- "You mean if you both remain silent?" the detective superintendent asks.

The Prisoners' Dilemma: Story

- "Let me be honest with you. Incriminatory evidence is too thin to meet court standards, so we won't be able to make you serve the maximum penalty.
- If you both refuse to confess, we'll get you only for possession of unregistered weapons and for obstructing our police officers in the course of their duty—one of them is still in hospital.
- That would then make two years of prison for each of you."

The Prisoners' Dilemma: Normal Form

Table: The prisoners' dilemma

		Wesson	
		Confession	Silence
Smith	Confession	$(-4,-4)$	$(0,-10)$
	Silence	$(-10,0)$	$(-2,-2)$

- An entry $(-k,-\ell)$ means that Smith is sentenced to serve k years in prison and Wesson is sentenced to serve ℓ years in prison.
- The players thus maximize their gains if they get away with a sentence of as few years as possible.

The Prisoners' Dilemma: Normal Form

- If one wants to avoid negative gains, one could scale all gains in equal measure, without causing the strategic aspects of the game to change.
- For example, dividing all entries by 2 and adding 5, one obtains the values in the table below, which strategically are equivalent.

Table: The prisoners' dilemma without negative entries

		Wesson	
		Confession	Silence
Smith	Confession	$(3, 3)$	$(5, 0)$
	Silence	$(0, 5)$	$(4, 4)$

Normal Form

- Noncooperative games with any number of players can be given in this *normal form* (a.k.a. the *strategic form*), which is attributed to Borel (1921) and von Neumann (1928). For more than two players, however, the gain vectors for all tuples of the players' strategies cannot be represented as a simple two-dimensional table as above.
- The normal form is best suitable for one-move games where all players make their moves simultaneously and without knowing the other players' moves and where randomness is not involved.
- Sequential games (with players taking turns) can better be represented by the so-called *extended form*.
- Games with one or more moves in which randomness plays a role are called *Bayesian games*.

Normal Form

Definition (normal form)

A game with n players is in *normal form* if for all i, $1 \leq i \leq n$, it holds that:

1. Player i can choose from a (finite or infinite) set S_i of (pure) strategies (or actions). The *set of profiles of (pure) strategies (or actions)* of all n players is represented as the Cartesian product
$$\mathscr{S} = S_1 \times S_2 \times \cdots \times S_n.$$
2. The gain function $g_i : \mathscr{S} \to \mathbb{R}$ gives the *gain $g_i(\vec{s})$ of player i* for the strategy profile $\vec{s} = (s_1, s_2, \ldots, s_n) \in \mathscr{S}$. Here, $\mathbb{R}$ denotes the set of real numbers and s_j, $1 \leq j \leq n$, is the strategy chosen by player j.

Dominant Strategy

Definition (dominant strategy)

Let $\mathscr{S} = S_1 \times S_2 \times \cdots \times S_n$ be the set of strategy profiles of the n players in a noncooperative game in normal form and let g_i be the gain function of player i, $1 \leq i \leq n$.

A strategy $s_i \in S_i$ of player i is said to be *dominant* (or *weakly dominant*) if

$$g_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq g_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n) \qquad (1)$$

for all strategies $s_i' \in S_i$ and all strategies $s_j \in S_j$ with $1 \leq j \leq n$ and $j \neq i$.

If inequality (1) is strict for all $s_i' \in S_i$ with $s_i' \neq s_i$ and all $s_j \in S_j$ with $1 \leq j \leq n$ and $j \neq i$, then s_i is a *strictly dominant strategy* for player i.

Strictly Dominant Strategies in the Prisoners' Dilemma

Fact

Making a confession is a strictly dominant strategy for both Smith and Wesson.

Remark

- *Having a (strictly) dominant strategy is of course very beneficial for a player: No matter what the other players do, this player has a "best" strategy independently of them.*
- *However, from a global point of view, it would be better if both players in the prisoners' dilemma deviated from their strictly dominant strategies, i.e., if they remained silent. Why?*

Pareto Dominance and Pareto Optimality

Definition (Pareto dominance and Pareto optimality)

Let $\mathscr{S} = S_1 \times S_2 \times \cdots \times S_n$ be the set of strategy profiles of the n players in a noncooperative game in normal form.

Let $\vec{s}$ and $\vec{t}$ be two strategy profiles from $\mathscr{S}$.

1. We say $\vec{s}$ *weakly Pareto-dominates* $\vec{t}$ if for all i, $1 \leq i \leq n$:

$$g_i(\vec{s}) \geq g_i(\vec{t}). \tag{2}$$

2. If inequality (2) holds true for all i, $1 \leq i \leq n$, and is strict for at least one j, $1 \leq j \leq n$, we say $\vec{s}$ *Pareto-dominates* $\vec{t}$.

3. If inequality (2) is strict for all i, $1 \leq i \leq n$, we say $\vec{t}$ is *strictly Pareto-dominated* by $\vec{s}$.

Pareto Dominance and Pareto Optimality

Definition (Pareto dominance and Pareto optimality—continued)

4. We say $\vec{t} \in \mathscr{S}$ is *Pareto-optimal* if for all $\vec{s} \in \mathscr{S}$: If $\vec{t}$ is weakly Pareto-dominated by $\vec{s}$, then $g_i(\vec{s}) = g_i(\vec{t})$ for all i, $1 \leq i \leq n$.

   That is, $\vec{t} \in \mathscr{S}$ is Pareto-optimal if there is no $\vec{s} \in \mathscr{S}$ that Pareto-dominates $\vec{t}$, i.e., if there is no $\vec{s}$ such that

   (a) $g_i(\vec{s}) \geq g_i(\vec{t})$ for all i, $1 \leq i \leq n$, and
   (b) $g_j(\vec{s}) > g_j(\vec{t})$ for at least one j, $1 \leq j \leq n$.

5. We say $\vec{t} \in \mathscr{S}$ is *weakly Pareto-optimal* if there is no $\vec{s} \in \mathscr{S}$ that strictly Pareto-dominates $\vec{t}$.

Pareto-Optimal Strategies in the Prisoners' Dilemma

- Intuitively, Pareto optimality of a strategy profile $\vec{t} = (t_1, t_2, \ldots, t_n)$ means that no other strategy profile gives all players at least as much profit as $\vec{t}$ and in addition at least one player a strictly larger profit.
- A *Pareto optimum* exists if and only if no player can increase her gains without making another player getting off worse at the same time.
- On the other hand, $\vec{t}$ is weakly Pareto-optimal if no other strategy profile gives *all* players a strictly larger profit.

  Thus, every Pareto optimum is also a weak Pareto optimum; conversely, weak Pareto optima are not necessarily Pareto-optimal.

Fact

The strategy profiles $(Silence, Silence)$, $(Confession, Silence)$, and $(Silence, Confession)$ are the Pareto optima in the prisoners' dilemma.

Stability of Solutions

- So far we have got to know two distinct concepts that can help to predict the outcome of a game:
  1. *dominant strategies*, such as the strategy profile (Confession, Confession) in the prisoners' dilemma,
  2. *Pareto optima*, such as (Silence, Silence), (Confession, Silence), and (Silence, Confession) in the prisoners' dilemma.
- A third concept that can help to predict the outcome of a game is the criterium of *stability of a solution*. Informally put, a solution (i.e., a profile of all players' strategies) is stable if no player has an incentive to deviate from her strategy in this profile, provided that also all other players choose their strategies according to this profile.
- Intuitively, this means that the strategies of this solution are *in equilibrium*.

Stability of Solutions: Nash Equilibria

Definition (Nash equilibrium in pure strategies)

Let $\mathscr{S} = S_1 \times S_2 \times \cdots \times S_n$ be the set of strategy profiles of the n players in a noncooperative game in normal form and let g_i be the gain function of player i, $1 \leq i \leq n$.

1. A strategy $s_i \in S_i$ of player i is said to be a *best response strategy to the profile* $\vec{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ *in* $\mathscr{S}_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$ *of the other players' strategies* if for all strategies $s_i' \in S_i$,

$$g_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq g_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n). \tag{3}$$

Stability of Solutions: Nash Equilibria

Definition (Nash equilibrium in pure strategies—continued)

2. If there is exactly one such strategy $s_i \in S_i$ of player i, then this is her *strictly best response strategy to the other players' strategy profile* $\vec{s}_{-i}$.

3. A strategy profile $\vec{s} = (s_1, s_2, \ldots, s_n) \in \mathscr{S}$ is in a *Nash equilibrium in pure strategies* if for all i, $1 \leq i \leq n$, $s_i \in S_i$ is a best response strategy of player i to the other players' strategy profile $\vec{s}_{-i}$.

4. If there is exactly one such strategy profile $\vec{s}$, then $\vec{s}$ is in a *strict Nash equilibrium in pure strategies*.

Stability of Solutions: Nash Equilibria

Remark

- *There exists a Nash equilibrium in pure strategies if every player chooses a best response strategy to the strategies she expects her opponents to choose (and all opponents meet that expectation).*
- *Thus, no player has an incentive to deviate from her chosen best response strategy, and the solution is stable.*
- *Since the inequalities (1) and (3) are identical, one might be tempted to think that a best response strategy were the same as a dominant strategy. However, there is a subtle, but decisive distinction: In the definition of best response strategy, (3) is true merely for all strategies $s'_i \in S_i$ of player i, while the context—the profile $\vec{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ of the other players' strategies—is fixed.*

Stability of Solutions: Nash Equilibria

Remark (continued)

- *By contrast, in the definition of dominant strategy, (1) holds true for all strategies $s'_i \in S_i$ of player i and for all of the other players' strategy profiles $\vec{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$.*
- *Hence, this is a sharper requirement: If a player has a dominant strategy, then this is also her best response strategy for every strategy profile of the other players.*
- *It follows that a profile of dominant strategies for all players, if it exists, is always in a Nash equilibrium in pure strategies; the converse, however, does not hold in general.*

Nash Equilibria in the Prisoners' Dilemma

Remark (continued)

- *Since (Confession, Confession) is a profile of dominant strategies of both players in the prisoners' dilemma, it is also a Nash equilibrium in pure strategies:*

  Neither Smith nor Wesson can improve his situation by "one-sided deviation" (that is, by deviating, assuming that the other player does not deviate from his strategy).

- *Therefore, they both can be expected (or predicted) to "stably keep to making a confession."*

- *This strategy profile even forms a strict Nash equilibrium, because there is no other one.*

Nash Equilibria in the Prisoners' Dilemma

Remark (continued)

- *This is due to the fact that both players even have strictly dominant strategies.*
- *That is generally true for every game in normal form:*

  If all players have strictly dominant strategies, then these form a strict Nash equilibrium in pure strategies.
- *Weakly dominant strategies, however, can occur in several Nash equilibria.*
- *Can you come up with a game in normal form that has no dominant strategies, even though there is a strict Nash equilibrium in pure strategies?*

Dominant Strategies, Pareto Optimality & Nash Equilibria

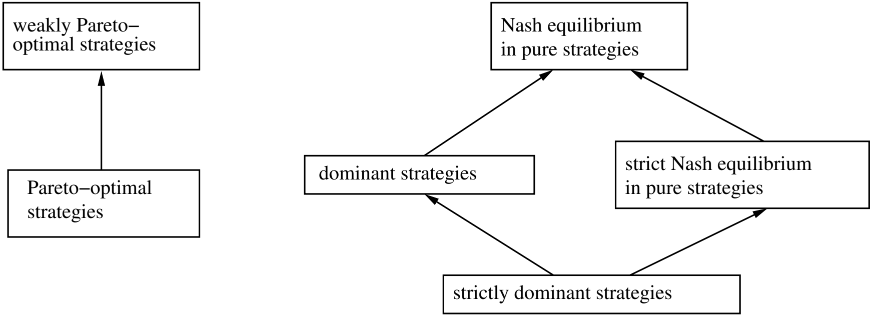


Figure: Solution concepts for noncooperative games in normal form: Relations

Dominant Strategies, Pareto Optimality & Nash Equilibria

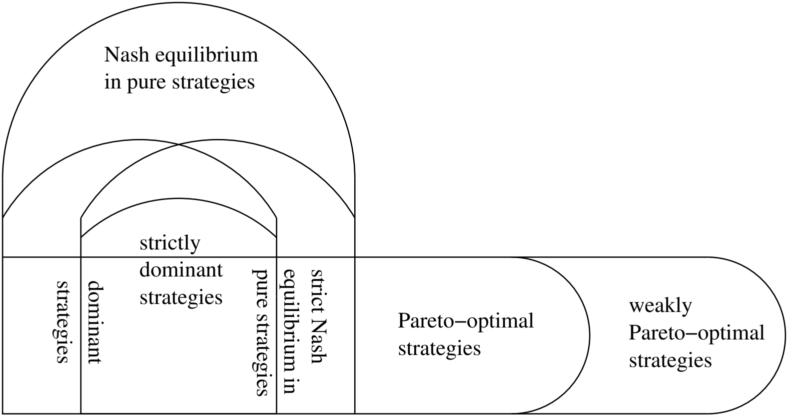


Figure: Solution concepts for noncooperative games in normal form

Battle of the Sexes: Story

- To celebrate their first anniversary, George and Helena are going to go out together. Unsurprisingly, they make quite different suggestions.
- "Let's just go to the stadium," George suggests. "By chance I got hold of two tickets for the soccer game tonight!" He proudly presents them to her.
- "Too bad!" Helena replies disappointedly and lifts a pair of tickets as well. "I was going to surprise you with those! Tori Amos is performing tonight, and so I figured . . ."
- "I'm really sorry!" George shouts. "Well, maybe you can sell them to somebody on the way to the stadium. But this is England against Germany, the classic! I surely can't miss that game!"

Battle of the Sexes: Story

- "Really not?" Helena snaps at him. "Then you just go to your classic! I'll certainly find someone who is interested in getting *your* Tori Amos ticket!"
- "I didn't mean it like this!" George wisely gives in. "That I want to spend this evening together with you, that is for sure, and for all I care we can also go to your concert. The main thing is we do something together! It's just that I would prefer going *with you together* to the soccer game ten times as much as going with you to the concert."

Battle of the Sexes: Normal Form

Table: The battle of the sexes

		Helena	
		Soccer	Concert
George	Soccer	$(\mathbf{10}, \mathbf{1})$	$(0, 0)$
	Concert	$(0, 0)$	$(\mathbf{1}, \mathbf{10})$

- George's gain is the left entry and Helena's gain is the right entry.
- Spending the night at different events is for none of them beneficial: The strategy profiles $(\text{Soccer}, \text{Concert})$ and $(\text{Concert}, \text{Soccer})$ are both rewarded by $(0, 0)$, since a separation from their partner kills all joy for both of them, even at their favorite events.

Battle of the Sexes: Equilibria and (Pareto) Dominance

Fact

In the battle of the sexes,

1. *the strategy profiles* $(Soccer, Soccer)$ *and* $(Concert, Concert)$ *with the boldfaced gain vectors* $(10, 1)$ *and* $(1, 10)$
   (a) *form two Nash equilibria in pure strategies and*
   (b) *are Pareto-optimal, and*
2. *there are no dominant strategies.*

Remark

The battle of the sexes is different from the prisoners' dilemma because:

- *it does not have a strict Nash equilibrium (but two);*
- *these coincide with the Pareto optima;*
- *it does not have dominant strategies.*

Chicken Game: Story

Figure: The chicken game

(The reader is strongly discouraged from playing this game.)

- David and Edgar, two ten year old boys, play the chicken game in their admittedly sexed up, but rather underperforming toy cars.

Chicken Game: Story

- By the rules of the game, they "race" with a maximum speed of 5 miles per hour, approaching each other, and whoever cowardly weasels out of driving on or swerves is the chicken and has lost.
- However, since he at least has been wise and has survived, he gets one gummy bear as a consolation prize, while the heroic winner rakes in the top prize of three gummy bears.
- If they both are wise and swerve just in time, then each of them gets a prize of two gummy bears.
- However, if both are boldly driving on to the bitter end, they are declared "dead" (only in play) after the inevitable crash and win no gummy bears.

Chicken Game: Normal Form

Table: The chicken game

		Edgar	
		Swerve	Drive on
David	Swerve	$(2,2)$	$(\mathbf{1},\mathbf{3})$
	Drive on	$(\mathbf{3},\mathbf{1})$	$(0,0)$

- The left entry gives David's gain, the right entry gives Edgar's gain.
- Chicken games are also referred to as *hawk-dove games*:
  - The *hawk strategy* corresponds to "Drive on,"
  - the *dove strategy* corresponds to "Swerve."

Chicken Game: Equilibria and (Pareto) Dominance

Fact

In the chicken game,

1. *the strategy profiles* $(\mathit{Drive\ on}, \mathit{Swerve})$ *and* $(\mathit{Swerve}, \mathit{Drive\ on})$ *with the boldfaced gain vectors* $(3, 1)$ *and* $(1, 3)$
   (a) *form two Nash equilibria in pure strategies and*
   (b) *are Pareto-optimal, and*
2. *in addition, also the strategy profile* $(\mathit{Swerve}, \mathit{Swerve})$ *with the gain vector* $(2, 2)$ *is Pareto-optimal.*

Question: Does any one of the two players, or perhaps each of them due to symmetry, have a dominant strategy?

Penalty Game: Story

"Football is a simple game; 22 men chase a ball for 90 minutes, and at the end the Germans always win."

Gary Lineker

FIFA World Cup, July 4, 1990

Penalty Game: Story

Figure: David as the kicker and Edgar as the goalkeeper at the penalty shoot-out

Penalty Game

- For simplification, we reduce the multitude of actually occurring strategies to only two for each player:
  - The kicker can kick the ball to the left or to the right of the goal, and
  - the goalkeeper can jump to the left or to the right to make a save.

  "Left" and "Right" are here meant always from the goalkeeper's point of view, even if it is the kicker's turn.

- Abstracting from reality, we also assume that
  - the goalkeeper is guaranteed to hold on to the ball when he jumps to the same side where the ball is being kicked, and
  - that the kicker never misses the goal.

  That is, if he kicks to the left (or to the right) and if the goalkeeper also jumps to the left (or to the right), the goalkeeper has definitely thwarted a goal; but if the goalkeeper jumps to the wrong side, the kicker has definitely converted his penalty.

Penalty Game: Normal Form

Table: The penalty game

		Goalkeeper	
		Left	Right
Kicker	Left	$(-1, 1)$	$(1, -1)$
	Right	$(1, -1)$	$(-1, 1)$

- The left entry refers to the kicker's gain and the right entry refers to the goalkeeper's gain.

Penalty Game: Equilibria and (Pareto) Dominance

Fact

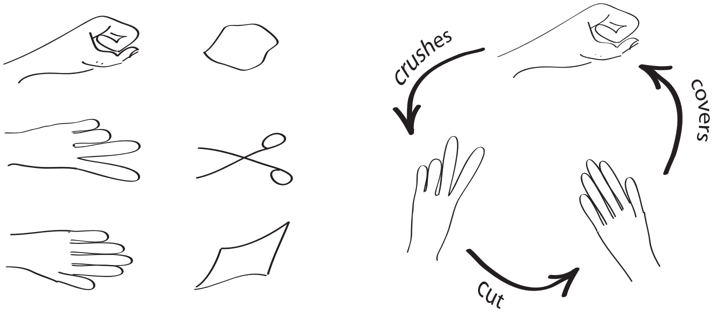
In the penalty game,

1. *there is no Nash equilibrium in pure strategies and*
2. *thus there can be no profile containing a dominant strategy for both the kicker and the goalkeeper.*
3. *In fact, no player can have a dominant strategy at all.*
4. *On the other hand, all four strategy profiles of this game are Pareto-optimal.*

Paper-Rock-Scissors Game: Story?

- *Rock* defeats *scissors*, because the rock can blunt the blades of the scissors.
- *Scissors* defeats *paper*, because the scissors can cut a sheet of paper.
- *Paper* defeats *rock*, because the paper can wrap the rock.
- If both players form distinct symbols, then the player whose symbol defeats the other player's symbol receives one point, and the other player loses one point.
- If both players form the same symbol, then none of them wins; so nobody receives a point.

Paper-Rock-Scissors Game: Story?



(a) Symbols for the strategies (b) Cycle of dominance relations

Figure: Strategies in the paper-rock-scissors game

Paper-Rock-Scissors Game: Normal Form

Table: The paper-rock-scissors game

		Edgar		
		Rock	Scissors	Paper
David	Rock	$(0,0)$	$(1,-1)$	$(-1,1)$
	Scissors	$(-1,1)$	$(0,0)$	$(1,-1)$
	Paper	$(1,-1)$	$(-1,1)$	$(0,0)$

- The left entry is David's gain and the right entry is Edgar's gain.

Paper-Rock-Scissors Game: Equilibria

Fact

In the paper-rock-scissors game, there does not exist a Nash equilibrium in pure strategies.

Questions:

- What about dominant strategies?
- What about Pareto optimality?

Paper-Rock-Scissors-Lizard-Spock Game

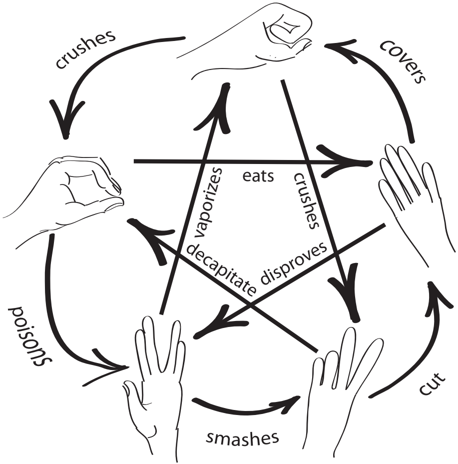


Figure: Strategies in the paper-rock-scissors-lizard-spock game

The Guessing Numbers Game: Rules

1. Any number of players can play this game.
2. Each of them guesses a real number between 0 and 100 (including these two values).
3. Take the average (i.e., the arithmetic mean) of the numbers chosen and let Z be exactly two thirds of this average.
4. Whoever comes closest to this number Z wins.

Remark

Thus, every player has infinitely many strategies to choose from, since there are infinitely (even uncountably) many real numbers in the interval $[0, 100]$*.*

The Guessing Numbers Game: Who Wins in 2022?

Your numbers:

0	0	0.003	0.0008	1	6.6	7
7.5	9	12.34	13.37	21	24	27
27.462	31	35	35	37	41.1	42
42	42.3141592653	42.5	53.17	59	66	

$\Rightarrow$ The average is: $\frac{682.359959265}{27} = 25.2725910839$.

$\Rightarrow Z = \frac{2}{3} \cdot 25.2725910839 = 16.8483940559$.

$\Rightarrow$ The winner is . . . **Eduard Bopp with 13.37!**

CONGRATULATIONS!!!

The Guessing Numbers Game: A First Thought

- If all players were to choose the largest possible number, 100, then

$$Z_{\text{max}} = 66.666\cdots,$$

  and no Z can be larger than Z_{max}.

- It would therefore be dull to choose a number greater than Z_{max}.
- In other words, *the strategy (or number) Z_{max} dominates all strategies (or numbers) exceeding it*.
- These can thus be safely eliminated.

The Guessing Numbers Game: A Second Thought

- *How will my opponents behave?*
- Suppose that all players randomly choose a number from $[0,100]$ under the uniform distribution.
- The average would then be 50 and two thirds of this is $33.333\cdots$.
- **But wait a minute!** Actually, why should all players choose an arbitrary number from $[0,100]$?
- If one assumes all players to behave *rationally*, they can be expected to *already have eliminated all numbers above $Z_{\text{max}} = 66.666\cdots$ as well*; but then the average of the remaining numbers (assuming the uniform distribution) would be $Z_{\text{rat}} = 33.333\cdots$ and two thirds of that would be $22.222\cdots$.

The Guessing Numbers Game: A Third Thought

- Every player behaves rationally and
  - every player knows that everybody behaves rationally, and
    - everybody knows that all players know that everybody behaves rationally, which again everybody is aware of, and so on.
- That is, *also the number* $Z_{\mathrm{rat}} = 33.333\cdots$ *dominates all numbers exceeding it*, which causes further numbers to be eliminated.
- Likewise, $22.222\cdots$ *dominates all greater numbers*, which implies their elimination as well, and so on.
- Why should one stop at any point with that argument?
- If you think this third thought consequently to the very end, only one number will remain that you should choose as your strategy:

the ZERO.

The Guessing Numbers Game: A Fourth Thought

- *Will all players in fact behave rationally?*
- This game—or variants thereof—has frequently been played in public, often with several thousands of players.
- In these games it has *never* been the case that all players have chosen their strategies according to the Nash equilibrium.
- However, as soon as some players deviate from the Nash equilibrium, the other players are no longer guaranteed to win when playing their Nash equilibrium strategy.
- Selten and Nagel (1998) give an overview of such game-theoretic experiments.