Beyond Intractability: A Computational Complexity Analysis of Various Types of Influence and Stability in Cooperative Games

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Zusammenfassung

Diese Arbeit beschäftigt sich mit der Berechnungskomplexität unterschiedlicher Problemstellungen aus der kooperativen Spieltheorie. Zum einen betrachten wir Existenz und Verifizierung von Stabilitätskonzepten in kooperativen Spielen mit übertragbarem Nutzen und in hedonischen Spielen; zum anderen wenden wir uns der Einflussnahme auf den Ausgang eines Spiels durch Manipulation, Kontrolle und Bestechung zu. Wir widmen uns manipulativem Verhalten im Sinne von strategischem Zusammenschluss von Spielern oder unter Angabe falscher Identitäten sowie struktureller Kontrolle durch Hinzufügen oder Entfernen von Spielern in gewichteten Wahlspielen. Betrachtete Lösungskonzepte hier sind der probabilistische Penrose–Banzhaf-Index und der Shapley–Shubik-Index, die die Manipulatoren beabsichtigen, zu ihrem Vorteil zu verbessern. Wir zeigen unter anderem für das Problem, ob ein Zusammenschluss einer gegebenen Koalition in einem gegebenen Spiel vorteilhaft ist, Vollständigkeit für probabilistische Polynomialzeit. Außerdem verallgemeinern wir ein formales Modell für Manipulation auf beliebige Klassen von kooperativen Spielen und untersuchen allgemeine Eigenschaften sowie beispielhaft einzelne Klassen wie Einstimmigkeitsspiele.

Darüber hinaus betrachten wir Bestechung in Pfad-Unterbrechungs-Spielen, wobei ein Gegenspieler versucht, durch ein Netzwerk von einem Start- zu einem Zielknoten zu gelangen und dabei ausgewählte Agenten für die Freigabe ihrer Knoten zu bezahlen, sodass es sich für übrige Koalitionen nicht mehr lohnt oder unmöglich ist, alle Wege zu blockieren. Für mehrere Gegenspieler und Kosten zur Knotenblockierung zeigen wir, dass das Bestechungsproblem vollständig für die zweite Stufe der Polynomialzeithierarchie ist. Wir erweitern diese Spiele auf ein probabilistisches Modell, in dem das genaue Ziel des Gegenspielers unbekannt ist, und untersuchen es im Hinblick auf bekannte Stabilitätskonzepte wie den Kern. Hier kann beobachtet werden, dass sich der allgemeinere Fall bezüglich Komplexität nicht anders verhält als der speziellere.

Fragen der Stabilität analysieren wir ebenfalls in hedonischen Spielen. Dabei gehen wir zunächst auf wundervolle Stabilität in feind-orientierten hedonischen Spielen ein. Wir heben die untere Schranke des Problems, ob es im Graphen eines zugrunde liegenden gegebenen Spiels eine wundervoll stabile Aufteilung gibt, auf DP an, also auf die zweite Stufe der booleschen Hierarchie. Auf dem Weg zur exakten Komplexität zeigen wir, dass coDP-Härte ausreichen würde, um Vollständigkeit für parallelen Zugriff auf NP zu beweisen. Des Weiteren führen wir ein neues Modell hedonischer Spiele mit ordinalen Präferenzen und Schwellenwerten ein, in dem Spieler ihre Mitspieler in Freunde, Feinde und neutrale Spieler unterteilen und gleichzeitig eine schwache Ordnung über die ersten beiden Mengen angeben. Erweitert wird diese Relation auf eine Menge möglicher Präferenzen, sodass es sinnvoll ist, Begriffe der möglichen und notwendigen Stabilität zu definieren. Hierfür prüfen wir die Komplexität unterschiedlicher Konzepte wie Nash-Stabilität und ermitteln axiomatische Spieleigenschaften. Zuletzt stellen wir eine weitere Variante hedonischer Spiele mit altruistischen Einflüssen vor. Bisher bekannte Darstellungen hedonischer Spiele gehen von eigennützigen Agenten aus, deren Präferenzen nur von ihrer Meinung abhängen. Basierend auf freund-orientierten Erweiterungen, lassen wir nun Einflüsse von Freunden auf die Präferenzrelation eines Spielers zu, indem wir in drei Graden der Selbstlosigkeit die durchschnittliche Meinung der Freunde in einer Koalition miteinbeziehen. Auch hier behandeln wir neben sinnvoll hierfür modellierten Eigenschaften die Komplexität von Stabilitätskonzepten wie strikte Popularität beim direkten Vergleich von Koalitionsstrukturen.

Abstract

This thesis deals with the computational complexity of various problems from cooperative game theory. On the one hand we examine the existence and verification of stability concepts in cooperative games with transferable utility and in hedonic games; on the other hand we look into several forms of influence on the output of a game via manipulation, control and bribery. We turn to manipulative action in the sense of strategically merging players or splitting a player into false identities as well as structural control by adding or deleting players in weighted voting games. In this setting, considered solution concepts are the probabilistic Penrose–Banzhaf index and the Shapley–Shubik index which the manipulators intend to improve to their advantage. Amongst others, we show that the problem of whether merging a given coalition in a given game is beneficial, is complete for probabilistic polynomial time. Additionally, we generalize a framework for manipulation to arbitrary classes of cooperative games and reflect on properties in general and examplarily in classes like unanimity games.

Moreover, we consider bribery in path-disruption games, where an adversary tries to travel from a source vertex to a target vertex and pays selected agents in order for them to unblock their vertices. The corruption is successful if it is not profitable or impossible for the remaining coalitions to prevent the adversary from reaching the target via an open path. For several adversarial players and costs for blocking a vertex, we show that the bribery problem is complete for the second level of the polynomial hierarchy. We expand these games to a probabilistic model where the target of the adversary is uncertain, and inspect them with respect to common stability concepts such as the core. Here, it can be observed that the more general case does not behave differently from the more special case in terms of complexity.

Furthermore, we study questions of stability likewise in hedonic games. Firstly, we inquire into wonderful stability in enemy-oriented hedonic games. We raise the lower bound of the problem of whether there exists a wonderfully stable partition in the graph of an underlying game, to DP, that is, to the second level of the Boolean hierarchy. On the way towards its exact complexity, we show that coDP-hardness would be sufficient to prove completeness for parallel access to NP. Secondly, we introduce a new model of hedonic games with ordinal preferences and thresholds in which players partition their co-players into the sets of friends, enemies, and neutral players while at the same time they specify a weak order over the former two sets. This relation is extended to a set of possible preferences over coalitions such that it is reasonable to define the notions of possible and necessary stability. We analyse the complexity of various concepts such as Nash stability and establish axiomatic properties of these games. Finally, we propose a further variant of hedonic games with altruistic influences. In representations known in the literature so far, agents are assumed to be selfish and their preferences only depend on their own opinion. Based on friend-oriented extensions, we now allow influences of friends on a player's preference relation by incorporating the average opinion among friends within a coalition in three degrees of altruism. Besides properties reasonably modelled for this environment, we investigate the complexity of stability concepts such as strict popularity when comparing coalition structures.

Contents

1 Introduction

Theoretical computer science has been of interest long before there have been actual computers. Central ideas include those by Turing [Tur36] and Church [Chu36], which again are based on the theories by Gödel [Göd31]. They ask questions of how to formalize computability and of when a function is computable or a problem decidable. Further topics of interest are graph structures, algorithmic properties, and axiomatic analyses. The problems that constitute the studies in this thesis are founded on these formal concepts.

In complexity theory these kinds of questions are addressed. There are many textbooks about different aspects of computational complexity such as the book by Papadimitriou [Pap95], and Arora and Barak [AB09]. Important questions here include the classification of problems and the interrelation of complexity classes and hierarchies. It is not impossible that problems regarded as hard are solvable in polynomial time, although considered highly unlikely, see Section 2.1.

Moreover, the interrelation of topics has gained increasing interest in recent decades. In computational social choice (see [EL06, EG08, CR10, FB12, PW14] as well as [Rot16, BCE$^+$16, BCE13, RBLR11]), it has become common to study axiomatic and complexity theoretic questions of issues from different fields such as voting theory (see, e.g., [BR16]), judgement aggregation (see, e.g., [LP09, End16, BER16]), and resource allocation (see, e.g., [BT96, LR16]).

Equally, in algorithmic game theory (see, e.g., [NRTV07]), the disciplines of game theory, which already combines mathematical ideas with social sciences and economics, and computer science are intertwined. Classic game theory (see, e.g., [OR94]) was founded by von Neumann and Morgenstern [NM44] where the strategic behaviour of selfish players in a non-cooperative game is formally defined and analysed. The perhaps most famous and groundbreaking result to be mentioned in this context is by Nash [Nas50, Nas51] that there always exists an equilibrium in a normal form game. Next to non-cooperative games, cooperative games are studied where players are selfish but may gain advantage (for themselves or their environment) from working together in a coalition (see, e.g., [PS07]). Now, related disciplines extend to multiagent systems in artificial intelligence (see, e.g., [SL09, Woo02]), mechanism design [NR01], and logic for games [Ågo14].

Yet another sub-discipline of algorithmic game theory this thesis is settled in is the computational study of cooperative games. See, for instance, the textbook by Chalkiadakis et al. [CEW11], and the book chapters by Elkind et al. [ERJ13, ER16] for an overview. Main questions here are the algorithmic properties of coalition formation and payoff division. For more literature pointers in cooperative game theory and formal definitions relevant to this thesis, see Section 2.3.

Weighted voting games are a class of compactly representable, but not fully expressive simple cooperative games [TZ99] with a political background on parties with different impact and coalitions with majorities so as to win a decision [BFJL02]. From a complexity point of view they have been studied intensely, see, e.g., [EGGW09]. In Section 2.3.1 we also consider other representations of cooperative games with transferable utility.

In coalition formation games, the key question is which coalition structure will form in a decentralized manner. Drèze and Greenberg [DG80] originally proposed the idea of hedonic games in which a player's happiness only depends on the coalitions she is part of. Formally, these games have been modelled by Banerjee et al. [BKS01] and independently by Bogomolnaia and Jackson [BJ02]. These games combine ideas from cooperative games and voting inasmuch as players express their preferences over coalitions containing them. A central struggle is the trade-off between full expressiveness of arbitrary preferences over all 2^{n-1} coalitions for n players, and compact representation, see, e.g., the survey by Woeginger [Woe13a] and the book chapter by Aziz and Savani [AS16], and also Section 2.3.2.

Most commonly, for both, games with transferable and non-transferable utility, questions of stability are studied. These can refer to different stability concepts and representations of games. One of the most popular stability concepts is that of the core [Gil59, GW86], where a coalition structure is considered as stable if no coalition takes advantage from deviating from the coalition structure. Crucial questions are of how hard it is to verify whether a given coalition structure satisfies a certain concept in a given game and whether stable coalition structures for certain concepts always exist. If, for some concept and a game, such a coalition structure fails to exist, we are interested in the computational complexity of existence.

Especially in voting theory, problems of influence are studied with respect to computational complexity. There are three main types of negative influence: Bartholdi et al. [BTT89] introduced the notion of manipulation of an election, where a voter (or several voters [CSL07]) changes her true preference in order to make a distinguished candidate a winner. In contrast, in a bribery scenario, as presented by Faliszewski et al. [FHH06], an external player tries to pay voters in order to change their votes such that a certain candidate becomes a winner. In a third form, control, the chair of an election changes the structure, e.g., by adding, deleting, or partitioning voters or candidates, with the aim of letting one candidate win [BTT92]. Next to this constructive impact, preventing candidates from winning is also studied [HHR97a]. Many studies of such settings with regard to various voting systems are known up to now. Similar ideas have been adapted to other fields in computational social choice like manipulation in preference aggregation [End13] as well as bribery and control in judgement aggregation [EGP12, BEER15]. In algorithmic game theory influence has also been studied to some extent, e.g., manipulation via false names in weighted voting games has been introduced by Elkind et al. [BE08]. The concept here is the famous Shapley value [Sha53, SS54]. There are initial studies on sybil attacks in hedonic games [VBZB14].

Yielding hardness-results in terms of complexity for a problem of influence is considered as a shield against this kind of attack, whereas for stability tractability is desirable. For expressive games and concepts, we are interested in the exact complexity for higher classes, as natural problems therein are less explored than NP-complete problems.

In Chapter 2 we provide an overview of basic notions from computational complexity theory, graph theory, and cooperative game theory. The main part of the thesis consists of three parts. Chapter 3 deals with two types of influence in cooperative games, namely, manipulation and control. Manipulation takes place in form of beneficial merging of players and the closely related annexation of players as well as false-name manipulation where a player splits into several false identities. We study these types of manipulation in weighted voting games as well as in a general setting. Structural control in form of adding or deleting players is also studied for weighted voting games. The common goal in the scenarios is to increase certain players' significance in a game measured by power indices such as the Shapley–Shubik index. We prove the problem of beneficial merging to be PP-complete for both the probabilistic Penrose–Banzhaf and the Shapley–Shubik index. Moreover, for the same indices, we show that the problem of whether splitting is beneficial is PP-hard, and is even PP-complete whenever the new players' weights are given. Annexing a coalition of players is never disadvantageous; nevertheless we show that for both indices, it is NP-hard to decide whether it is advantageous, and NP-complete for a single player. An overview of the history of complexity results in this context can be found in Table 3.1. See Table 3.2 for the results of structural control. We propose a general framework for merging and splitting functions in classes of cooperative games with transferable utility such that reasonable properties are satisfied. For example, in unanimity games and for the probabilistic Penrose–Banzhaf index we show that splitting is always disadvantageous or neutral, whereas merging is neutral for size-two coalitions, yet advantageous for coalitions of size at least three.

In Chapter 4 we consider a third type of influence in cooperative games, namely bribery. The setting is a path-disruption game, where an external agent's goal is to travel from a certain source to a target in a graph. We generalize this model to a probabilistic one by allowing uncertainty about the targets. We study this model with respect to its game-theoretic properties as well as the complexity of problems related to common solution concepts. The computationally challenging aspect of these games lies within the case where costs for blocking coalitions occur. Table 4.1 summarizes the results for this setting.

In Chapter 5 we analyse various stability concepts in hedonic games. Firstly, we focus on the concept of wonderful stability in games with friend-and-enemy encoding and enemy-oriented preferences. For this concept, we show that verification is coNP-complete, and existence is DP-hard and Θ_2^p-complete if coDP-hardness holds. The developments of related results can be found in Table 5.1. Secondly, we study a new representation of hedonic games (games with ordinal preferences and thresholds), and an associated preference extension principle (generalized Bossong–Schweigert extensions). Since in this model there are several possible extensions to a hedonic game, we endue the problems of verification and existence with the notions of possibility and necessity and study their complexity. The results are specified in Table 5.2. Thirdly, we introduce a novel model for friend-oriented hedonic games that considers not only a player's own preferences but also her friends' preferences under three degrees of altruism. We examine these hedonic games with altruistic influences, each of which satisfy a number of desirable properties, under the aspect of stability and their computational complexity, see Table 5.3. We conclude with Chapter 6.

2 Preliminaries

To begin with, we provide the basic definitions required in this thesis. The foundations and employed tools from computational complexity theory can be found in Section 2.1, a short excursion to graph theory in Section 2.2, and the background of cooperative game theory in Section 2.3. Moreover, consider the following notions. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of non-negative integers, and call $\mathbb{N} \setminus \{0\}$ the set of positive integers; $\mathbb{Q}_{\geq 0}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}$ the sets of non-negative rational numbers, positive real numbers, non-negative real numbers, and real numbers, respectively. For a set S, let $\|S\|$ denote the cardinality of S, and let $\mathfrak{P}(S)$ be the power set of S.

By a *relation* we refer to a collection $\mathscr{R} \subseteq S \times S$ over a set S. $\mathscr{R}$ is called *reflexive* if $(a,a) \in \mathscr{R}$ for each $a \in S$; *transitive* if $(a,b) \in \mathscr{R}$ and $(b,c) \in \mathscr{R}$ imply $(a,c) \in \mathscr{R}$ for each $a,b,c \in S$; *antisymmetric* if $(a,b) \in \mathscr{R}$ and $(b,a) \in \mathscr{R}$ imply $a = b$ for each $a,b \in S$; and *total* if for each $a,b \in S$, $(a,b) \in \mathscr{R}$ or $(b,a) \in \mathscr{R}$. We consider *preference relations* $\succeq$ that are reflexive and transitive but not necessarily total. For two elements a and b, we write $a \succeq b$, if (a,b) fulfils the relation, and say a is *weakly preferred* to b. If $a \succeq b$ but not $b \succeq a$, a is *(strictly) preferred* to b, written $a \succ b$. If both, $a \succeq b$ and $b \succeq a$, we say there is an *indifference* between a and b, written $a \sim b$. Since a and b can be distinct, a preference relation does not have to be antisymmetric. A preference relation induces a *preference order* over the set. In case of a non-total preference relation we speak of a *partial order*. If the order is antisymmetric, it is called *strict*, otherwise *weak*.

In terms of propositional logic, we denote the negation of a Boolean variable x by $\bar{x}$ and of a set of Boolean variables X by $\bar{X}$. A Boolean formula is in *conjunctive normal form* if it is a conjunction of clauses which are a disjunction of literals each. It is in *disjunctive normal form* if it is a disjunction of implicants which are a conjunction of literals each. Throughout this thesis we consider predicate logic.

2.1 Computational Complexity Theory

The studies of computational complexity, as they can be found amongst others in the books by Garey and Johnson [GJ79], Papadimitriou [Pap95], and Rothe [Rot05, Rot08], are based on the computability concept by Turing [Tur36, Tur37, Tur50]. The *deterministic Turing machine* is a theoretical computer model consisting of an alphabet Σ, a tape alphabet $\Gamma \supseteq \Sigma$, a set of states Z where one is an initial state $z_0 \in Z$ and others are distinguished as final states $F \subseteq Z$, a blank symbol $\Box \notin \Gamma$, and a transition function $\delta : (Z \setminus F) \times \Gamma \to Z \times \Gamma \times \{L, N, R\}$. Informally, such a machine can be understood as an infinite tape on which an input word x

consisting of symbols in Σ (written as $x \in \Sigma^*$) is written, one symbol in one cell each. The rest of the tape is filled with blank symbols. The process of the machine is determined via a head in state z_0 that starts reading the first input symbol and changes the state, writes a new symbol, and moves a cell to the left (L), not at all (N), or to the right (R) according to δ. We call one such move between two *configurations* a *step*. If a configuration is reached which δ is not defined for, the machine halts. The input is accepted if the machine halts and is in a final state. Note that so far we have only considered deterministic machines. The transition function may also be defined in a way that there is a set of possible next steps from a configuration. In this case we call a Turing machine *nondeterministic*.

We distinguish decision problems and function problems. A *decision problem* consists of a *possible instance* (an element in a certain subset of Σ^*) and a question that can be answered with true or false. The set of instances with answer true is synonymous with the problem. A Turing machine *decides* a problem, if it accepts an input if and only if it is an element of the problem. A *functional problem* consists of a possible instance, and a question that asks for a certain output. A Turing machine *computes* a function, if it accepts an input in the domain of the function and the corresponding output is written on the tape from the head's position onwards. We speak of an *oracle machine* N to a machine M, if M has access to N as a black box and requires only one step to obtain the answer of N.

The running time of a machine for an input is described by the number of steps needed from the initial configuration to reach a halt. We measure the running time in dependence on the input length, denoted by $|x|$ for an input word x. If not indicated otherwise, input numbers and sizes are given in unary encoding, while input values are given in binary encoding with logarithmic input size. The worst-case running time f of a machine is categorized by an asymptotic running time for almost every input size:

$$f \in \mathcal{O}(g) \iff (\exists c \in \mathbb{R}_{>0})(\exists \text{ finite } N \subset \mathbb{N})(\forall n \in \mathbb{N} \setminus N)[f(n)+1 \leq c \cdot (g(n)+1)].$$

We say, for instance, that f is polynomial if $f \in \mathcal{O}(g)$ for some polynomial function g. The space required by a machine for an input is the maximum length of a word read or written on the tape at any configuration. In order to categorize problems into classes, we say that a problem is decidable in time g (or computable for functional problems) if there exists a machine that has a worst-case running time in $\mathcal{O}(g)$.

Complexity Classes P and NP The most prominent complexity classes are P and NP. P is the class of all problems that are decidable in polynomial time in the input size. The class NP consists of all problems that are nondeterministically decidable in polynomial time in the input size. Obviously, P is contained in NP. The question as to whether the opposite conclusion also holds is one of the most popular open questions.

The assignment of a decision problem to a class provides an upper bound for its computational complexity. In order to determine a lower bound, *hardness* with respect to a complexity class is defined via some reducibility. The most common notion is that of polynomial-time many-one reducibility. A problem $A \subseteq \Sigma^*$ is *polynomial-time many-one reducible* to a problem $B \subseteq \Sigma^*$ (we write $A \leq^{\mathrm{p}}_{\mathrm{m}} B$) if there exists a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ such that for each possible instance $x \in \Sigma^*$ for A,

$$x \in A \iff f(x) \in B.$$

A problem B is called *hard* for class $\mathfrak{C}$ if $A \leq^{\mathrm{p}}_{\mathrm{m}} B$ for each $A \in \mathfrak{C}$. Due to transitivity, it holds that B is hard for $\mathfrak{C}$ if there exists a problem A that is hard for $\mathfrak{C}$ such that $A \leq^{\mathrm{p}}_{\mathrm{m}} B$. In most cases, we use this properties as a means in the upcoming hardness proofs. Moreover, B is called *complete* for $\mathfrak{C}$ if B is hard for $\mathfrak{C}$ and contained in it. Completeness yields a classification of a problem's computability in comparison to other problems.

Initially, Cook [Coo71] proved the problem SATISFIABILITY (SAT for short) to be NP-complete. A likewise used variant is 3-SAT, a restriction of SAT to clauses of size three, which is NP-complete as well [Kar72].

3-SATISFIABILITY (3-SAT)	
Given:	A set $X = \{x_1, \dots, x_n\}$ of Boolean variables, a collection of clauses $\mathscr{C}$ consisting of three literals in $X \cup \bar{X}$ each.[1]
Question:	Is there an assignment to the variables in X such that in each $c \in \mathscr{C}$ at least one literal is true?

Many decision problems are known to be NP-complete. A substantial collection can be found, for example, in the book by Garey and Johnson [GJ79]. We will need further well-known problems as defined in the following. SUBSET SUM (which is a special variant of the KNAPSACK problem) and the even more restricted problem PARTITION are shown to be NP-complete by Karp [Kar72].

SUBSET SUM	
Given:	A set $A = \{1, \dots, n\}$, a value function $a : A \to \mathbb{N} \setminus \{0\}$, $i \mapsto a_i$, and a positive integer q.
Question:	Is there a subset $A' \subseteq A$ such that $\sum_{i \in A'} a_i = q$?

PARTITION	
Given:	A set $A = \{1, \dots, n\}$ and a value function $a : A \to \mathbb{N} \setminus \{0\}$, $i \mapsto a_i$, such that $\sum_{i=1}^{n} a_i$ is even.
Question:	Does a allow a partition into two subsets of equal weight, that is, is there a subset $A' \subseteq A$ such that $\sum_{i \in A'} a_i = \sum_{i \in A \setminus A'} a_i$?

Let $(a_1, \dots, a_n; q)$ and $(a_1, \dots, a_n)$ denote SUBSET SUM and PARTITION instances, respectively. EXACT COVER BY 3-SETS is also known to be NP-complete (see, e.g., [Pap95]).

[1] This input corresponds to a Boolean formula in 3-conjunctive normal form. The equivalent question then is of whether this formula is satisfiable.

Exact Cover By 3-Sets (XC_3)	
Given:	A set $B = \{1, \ldots, 3k\}$, $k > 0$, and a collection $\mathscr{S} = \{S_1, \ldots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$, $1 \leq i \leq n$.
Question:	Is there an exact cover of B in $\mathscr{S}$, that is, is there a subcollection $\mathscr{S}' \subseteq \mathscr{S}$ such that $\bigcup_{S \in \mathscr{S}'} S = B$ and $S_i \cap S_j = \emptyset$, for each $S_i, S_j \in \mathscr{S}'$, $i \neq j$?

Given an XC_3 instance $(B, \mathscr{S})$, we may assume that each element of B occurs at most three times in the sets in $\mathscr{S}$; in this case the problem remains NP-complete [GJ79].

Counting Classes Additionally to decision problems, we consider counting classes such as FP, the class of all polynomial-time computable functions, and #P. Valiant [Val79] introduced the latter as the class of functions that output the number of solutions for the instances of problems in NP. For a problem $A \in$ NP, we denote this function by #A. For example, the function #SAT maps any SAT-instance to the number of satisfying truth assignments.

There are several notions of reducibility for functional problems and, consequently, there are several types of hardness and completeness for complexity classes of functions. Let f and g be two functions mapping from Σ^* to $\mathbb{N}$. Analogously to polynomial-time many-one reducibility for decision problems, one notion of functional many-one reducibility is: f *many-one-reduces to* g if there exist two functions φ and ψ in FP such that for each $x \in \Sigma^*$, $f(x) = \psi(g(\varphi(x)))$ [Zan91]. The special case where ψ is the identity function yields parsimonious reducibility [Sim75], which preserves the number of solutions: We say f *parsimoniously reduces to* g if there exists a polynomial-time computable function φ such that for each input $x \in \Sigma^*$, $f(x) = g(\varphi(x))$. See [FH09] for a more detailed discussion on functional reducibilities, e.g., *metric reducibility* [Kre88]. A function g is called #P-*parsimonious-hard* (#P-*many-one-hard*, respectively) if every function $f \in$ #P parsimoniously (many-one, respectively) reduces to g. If g is both #P-parsimonious-hard (#P-many-one-hard, respectively) and in #P, then g is #P-*parsimonious-complete* (#P-*many-one-complete*, respectively). It is known that #XC_3 is #P-parsimonious-complete (see, e.g., [HMRS98] for parsimonious reductions from #3-SAT via various restrictions). Likewise, #SubsetSum is #P-parsimonious-complete, as the standard reduction from XC_3 to Subset Sum (see, e.g., [Pap95]) is parsimonious. #Partition is only known to be #P-many-one-complete (by the standard reduction [Kar72]). #P is closed under addition and multiplication by 2.

Higher Complexity Classes Above this level we consider a number of other complexity classes for decision problems between NP and the class of all problems decidable in polynomial space, PSPACE, which are illustrated in Figure 2.1 and defined in the following. None of the depicted inclusions are known to be strict.

For a class $\mathfrak{C}$, let co$\mathfrak{C}$ denote the class of all complements of problems in $\mathfrak{C}$, e.g., coNP contains all problems A whose complements $\bar{A} = \{x \in \Sigma^* \text{ possible input} \mid x \notin A\}$ are in NP.

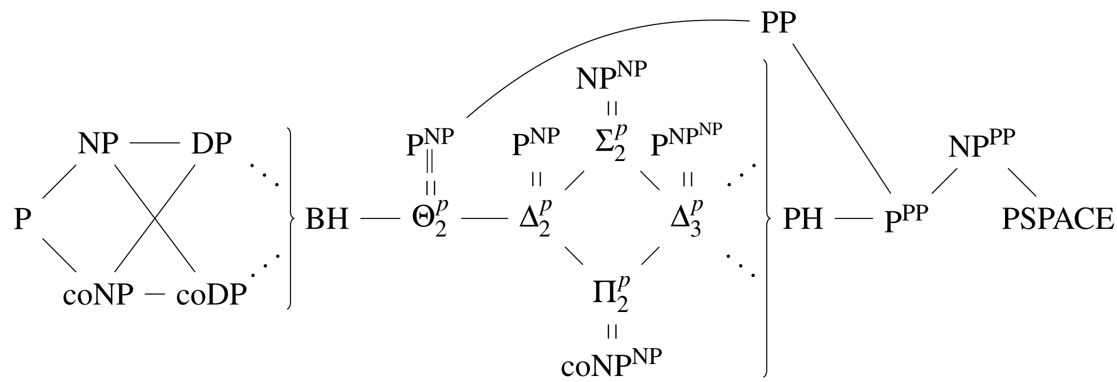


Figure 2.1: An overview of complexity classes. Edges illustrate inclusions from left to right. None of the inclusions are known to be strict.

Boolean Hierarchy The *Boolean hierarchy* over NP [CGH$^+$88, CGH$^+$89] comprises the classes $\mathrm{BH}_i(\mathrm{NP}) = \{A \cup B \mid A \in \mathrm{BH}_{i-2}(\mathrm{NP}) \text{ and } B \in \mathrm{BH}_2(\mathrm{NP})\}$, $i \geq 3$, where $\mathrm{BH}_2(\mathrm{NP}) = \{A \cap B \mid A \in \mathrm{NP} \text{ and } B \in \mathrm{coNP}\}$, $\mathrm{BH}_1(\mathrm{NP}) = \mathrm{NP}$, and $\mathrm{BH}_0(\mathrm{NP}) = \mathrm{P}$. The class $\mathrm{DP} = \mathrm{BH}_2(\mathrm{NP})$ was introduced by Papadimitriou and Yannakakis [PY84] as the class of differences of any two NP problems. They present the well-known DP-complete problem SAT-UNSAT.

SAT-UNSAT	
Given:	Two 3-SAT instances φ_1 and φ_2.
Question:	Is it true $\varphi_1 \in$ 3-SAT and $\varphi_2 \notin$ 3-SAT?

In this thesis, we study a DP-hard problem in Chapter 5 in the context of wonderful stability in hedonic games. For further natural complete problems in the levels of the boolean hierarchy, and especially in DP, see the survey by Riege and Rothe [RR06]. More recently and in the fields of voting and resource allocation, DP-hardness results have been discovered [RRS14, NNRR14]. The following lemma provides a sufficient condition for proving lower bounds for DP.

Lemma 2.1 (Wagner [Wag87]). *Let A be some* NP*-hard problem, and let B be any set. If there exists a function $f \in$* FP *such that, for any two instances x_1 and x_2 of A for which $x_2 \in A$ implies that $x_1 \in A$, we have $\|\{i \mid x_i \in A\}\|$ is odd $\iff f(x_1, x_2) \in B$, then B is* DP*-hard.*

Polynomial Hierarchy The *polynomial hierarchy* over NP [MS72, Sto76], PH, is the union of the classes $\Delta_i^p = \mathrm{P}^{\Sigma_{i-1}^p}$, $\Sigma_i^p = \mathrm{NP}^{\Sigma_{i-1}^p}$, and $\Pi_i^p = \mathrm{co}\Sigma_i^p$, $i \geq 1$, where $\Delta_0^p = \Sigma_0^p = \Pi_0^p = \mathrm{P}$. For instance, $\Sigma_2^p = \mathrm{NP}^{\mathrm{NP}}$ is the class of all problems that can be decided in nondeterministic polynomial time with access to an NP-oracle and $\Pi_2^p = \mathrm{coNP}^{\mathrm{NP}}$ is the class of all problems whose complements can be decided in Σ_2^p. Natural complete problems in the levels of the polynomial hierarchy are the quantified variants of SAT; for instance, the second-level quantified Boolean formula problem is Σ_2^p-complete.

QUANTIFIED BOOLEAN FORMULA 2 (QBF_2)	
Given:	Two sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ of Boolean variables, a collection of implicants $\mathscr{C}$ consisting of three literals in $X \cup \bar{X} \cup Y \cup \bar{Y}$ each.[2]
Question:	Is there an assignment to the variables in X such that for all assignments to the variables in Y it holds that in each $c \in \mathscr{C}$ at least one literal is true?

In this thesis, we study a Σ_2^p-complete problem in Chapter 4 in the context of bribery. The survey by Schaefer and Umans [SU02a, SU02b] provides an extensive collection of natural complete problems in the levels of the polynomial hierarchy, and especially in Σ_2^p. Recent Σ_2^p-completeness results on core stability in additively separable hedonic games (see also Section 2.3.2 and Chapter 5) are due to Woeginger [Woe13b] and Peters [Pet15]. Meyer and Stockmeyer[MS72, Wra77] show a quantifier characterization of the polynomial hierarchy. For instance, for Σ_2^p the following characterization holds.

Lemma 2.2 ([MS72]). *A problem A is in Σ_2^p if and only if there exists a set $B \in \mathrm{P}$ and a polynomial p such that for each possible input x for A,*

$$x \in A \iff (\exists y \in \Sigma^*)(\forall z \in \Sigma^*)[|y| \leq p(|x|) \text{ and } |z| \leq p(|x|) \implies (x,y,z) \in B].$$

Parallel Access to NP In between the Boolean hierarchy and the second level of the polynomial hierarchy lies Θ_2^p. It is equivalent to $\mathrm{P}^{\mathrm{NP}[\log]}$ [PZ83], the class of problems that can be decided in polynomial time by asking $\mathscr{O}(\log n)$ sequential Turing queries to an NP oracle. Moreover, it is known as $\mathrm{P}_{||}^{\mathrm{NP}}$, where the access to an NP oracle is restricted to polynomially many queries asked in parallel. Independently, Hemachandra [Hem89] and Köbler et al. [KSW87] have shown that $\mathrm{P}^{\mathrm{NP}[\log]}$ and $\mathrm{P}_{||}^{\mathrm{NP}}$ are equal. We turn to Θ_2^p-hardness in a challenge in Chapter 5 in the context of wonderful stability in hedonic games. Natural problems occur in different fields, for instance, for graph and satisfiability problems [Wag87], winner determination problems [HHR97a, HHR97b, RSV03, HSV05], and, more recently, covering sets problems [BBF$^+$13]. Similarly to Lemma 2.1, the following lemma provides a useful tool for proving lower bounds for Θ_2^p.

Lemma 2.3 ([Wag87]). *Let A be some* NP*-hard problem, and let B be any set. If there exists a function $f \in \mathrm{FP}$ such that, for all $k \geq 1$ and any $2k$ instances $x_1, \dots, x_{2k}$ of A for which $x_j \in A$ implies that $x_i \in A$ for $i < j$, we have $\|\{i \mid x_i \in A\}\|$ is odd $\iff f(x_1, x_2, \dots, x_{2k}) \in B$, then B is Θ_2^p-hard.*

[2] This input corresponds to a Boolean formula in 3-disjunctive normal form.

Propabilistic Polynomial Time The class PP has been introduced by Gill [Gil74, Gil77] as the class of all decision problems X for which there exist a function $f \in \#\mathrm{P}$ and a polynomial p such that for all instances x, $x \in X$ if and only if $f(x) \geq 2^{p(|x|)-1}$. PP is considered to be a class by far larger than NP due to Toda's theorem [Tod91] saying that $\mathrm{PH} \subseteq \mathrm{P}^{\mathrm{PP}}$, i.e., PP is at least as hard (in terms of polynomial-time Turing reductions) as any problem in the polynomial hierarchy. Also, Θ_2^p is contained in PP [BHW89]. PP is closed under complement and union. A typical PP-complete problem is the majority variant of SAT.

MAJORITY SATISFIABILITY (MAJSAT)	
Given:	A Boolean formula φ dependent on n variables in a set X.
Question:	Are there at least 2^{n-1} satisfying assignments for the variables in X?

Note that the related problem of asking whether more than 2^{n-1} assignments satisfy the formula is also PP-complete, inasmuch as PP can be equally characterized by the conditions above and $f(x) > 2^{p(|x|)-1}$ [Ogi93]. In this thesis, we study PP-complete problems in Chapter 3 in the context of manipulation in weighted voting games. Faliszewski and Hemaspaandra [FH09] prove PP-completeness of the problem of comparing a player's power in two weighted voting games with respect to the probabilistic Penrose–Banzhaf and the Shapley–Shubik index (see also Section 2.3.1 and Chapter 3).

The following result is due to Faliszewski and Hemaspaandra [FH09, Lemma 2.3].

Lemma 2.4 ([FH09]). *Let F be a #P-parsimonious-complete function. The problem* COMPARE-$F = \{(x,y) \mid F(x) > F(y)\}$ *is* PP-*complete.*

The following corollaries hold, for instance, since $\#\mathrm{XC}_3$ and #SUBSET SUM are #P-parsimonious-complete.

Corollary 2.5. COMPARE-$\#\mathrm{XC}_3$ *is* PP-*complete.*

Corollary 2.6. COMPARE-#SUBSETSUM *is* PP-*complete.*

Counting Hierarchy $\mathrm{NP}^{\mathrm{PP}}$ is the class on the second level of Wagner's counting hierarchy [Wag86] containing all problems solvable by an NP machine with access to a PP oracle. Littman et al. [LGM98] define an $\mathrm{NP}^{\mathrm{PP}}$-complete variant of SAT.

EXISTENTIAL MAJORITY SATISFIABILITY ($\exists$-MAJSAT)	
Given:	A Boolean formula φ dependent on variables $x_1, \dots x_n$ and a positive integer $k \leq n$.
Question:	Does there exist an assignment of $x_1, \dots x_k$ such that more than 2^{n-k-1} assignments for $x_{k+1}, \dots x_n$ satisfy φ?

We turn to $\mathrm{NP}^{\mathrm{PP}}$ in a challenge in Chapter 3 in the context of false-name manipulation in weighted voting games. Other natural $\mathrm{NP}^{\mathrm{PP}}$-complete problems have been identified by Mundhenk et al. [MGLA00] related to finite-horizon Markov decision processes and Littman et al. related to probabilistic planning.

2.2 Graph Theory

A *graph* is a pair $G = (V, E)$, where V is the set of vertices[3] and E the set of either directed ($E \subseteq V \times V$) or undirected ($E \subseteq \{e \subseteq V \mid \|e\| = 2\}$) edges. In a directed graph an edge is denoted by (u, v) and is illustrated by a link from vertex u to vertex v. In an undirected graph an edge is denoted by $\{u, v\}$ and is illustrated by an edge between vertices u and v. Vertices and edges can be weighted; in this case a function $w : V \to \mathbb{R}$, or $w : E \to \mathbb{R}$, respectively, is given for a graph $G = (V, E)$.

Given an undirected graph $G = (V, E)$, we denote an *induced subgraph restricted to a subset of edges* $E' \subseteq E$ by $G|_{E'} = (V, E')$ and an *induced subgraph restricted to a subset of vertices* $V' \subseteq V$ by $G|_{V'} = (V', \{\{u, v\} \in E \mid u \in V' \text{ and } v \in V'\})$. The set of vertices $V'(v) = \{u \in V \mid \{u, v\} \in E\}$ attached to a vertex $v \in V$ is called *(open) neighbourhood* of v, $V'(v) \cup \{v\}$ is called *closed neighbourhood* of v.

Properties A *clique* in an undirected graph $G = (V, E)$ is a subset $C \subseteq V$ such that for each two distinct vertices $u, v \in C$, $u \neq v$, $e = \{u, v\}$ is contained in E. For a vertex $v \in V$, let $\omega_G(v)$ denote the *clique number of v in G*, which is the size of a largest clique in G that contains v. A *maximal clique* is one that is not contained in a larger clique. A partition of the vertex set V is a division of the vertices into subsets $\Pi = \{P_1, \dots, P_k\}$ such that $\bigcup_{i=1}^{k} P_i = V$ and $P_i \cap P_j = \emptyset$ for $i \neq j$. We denote the set in Π containing a vertex v by $\Pi(v)$. An undirected graph $G = (V, E)$ consists of k *independent components* $G_1 = (V_1, E_1), \dots, G_k = (V_k, E_k)$ if $\{V_1, \dots, V_k\}$ is a partition of V, that is, $G_i = G|_{V_i}$, $1 \leq i \leq k$, and $E = \bigcup_{i=1}^{k} E_i$.

Definition 2.7 ([Woe13a])**.** *Given a graph $G = (V, E)$, a partition Π of V is called* wonderfully stable *if each $P \in \Pi$ is a clique and $\|\Pi(v)\| = \omega_G(v)$ for each vertex $v \in V$.*

A clique $P \subseteq V$ *blocks* a partition Π into cliques if there exists a vertex[4] $v \in P$ with $\|P\| > \|\Pi(v)\|$. Consequently, $\omega_G(v) > \|\Pi(v)\|$. By definition of clique number, $\omega_G(v) \geq \|\Pi(v)\|$ holds for each $v \in V$, since $\Pi(v)$ is a clique that contains v. These notations are used in the context of hedonic games, see Sections 2.3.2 and 5.1.

Decision Problems We refer to several well-known problems on graphs. GAP is known to be in P.[5]

GRAPH ACCESSIBILITY PROBLEM (GAP)	
Given:	A directed graph G and two distinguished vertices s (source) and t (target).
Question:	Can t be reached from s via the edges in G?

[3] We consider graphs with a finite vertex set.

[4] Note that it is actually the vertex that blocks the partition, not the whole clique; nevertheless we use this notion due to parallels to group deviations in hedonic games.

[5] Indeed, it is even considered to be decidable much faster than in polynomial time, namely in nondeterministic logarithmic space [Sav73, Jon75].

Related well-studied graph problems concern maximal connected subgraphs and generalized connectivity, see, e.g., [AAA+06, LM14].

CLIQUE and MAXCUT are further well-known NP-complete problems [Kar72].

CLIQUE	
Given:	An undirected graph G and a positive integer k.
Question:	Is there a clique of size k in G?

Note that the problem of whether there exists a partition into a limited number of cliques in a graph is NP-hard (see, e.g., [GJ79]). If, however, the number of cliques is not limited, a partition into cliques can easily be found.

MAX CUT	
Given:	An undirected graph $G = (V,E)$, edge weights $w : E \to \mathbb{N} \setminus \{0\}$ and a positive integer K.
Question:	Is there a partition of V into two vertex sets V_1 and V_2 such that $\sum_{\{v_1,v_2\} \in E, v_1 \in V_1, v_2 \in V_2} w(\{v_1,v_2\}) \geq K$?

MULTIPAIRCUT WITH VERTEX COSTS is a decision problem mentioned by Bachrach and Porat [BP10].

MULTIPAIRCUT WITH VERTEX COSTS (MCVC)	
Given:	A graph $G = (V,E)$, m vertex pairs (s_j,t_j), $1 \leq j \leq m$, a weight function $w : V \to \mathbb{Q}_{\geq 0}$, and a bound $K \in \mathbb{Q}_{\geq 0}$.
Question:	Does there exist a subset $V' \subseteq V$ such that $\sum_{v \in V'} w(v) \leq K$ and the induced subgraph $G\|_{V \setminus V'}$ contains no path linking a pair (s_j,t_j), $1 \leq j \leq m$?

It is known that MCVC belongs to P for problem instances with $m < 3$, yet is NP-complete for problem instances with $m \geq 3$. The related optimization problem for $m < 3$ can be solved in polynomial time using the same algorithm as the decision problem with a corresponding output [DJP+94]. Without loss of generality, we can assume that the bound K and the vertex weights $w(v)$, $v \in V$, in an MCVC instance are natural numbers, since in the reduction from MAXCUT to MULTITERMINALCUT by Dahlhaus et al. [DJP+94, Theorem 3] weights and bounds are also natural numbers.

We turn to domains on graphs in Chapter 4 in the context of path-disruption games and in Chapter 5 in the context of network of friends representations of hedonic games. Further graph theoretic background can be found amongst others in the textbooks [Dis05, GRRW10].

2.3 Cooperative Game Theory

The basic domain the problems studied in this thesis are settled in is that of cooperative games. Basic concepts can be found, e.g., in textbooks by Chalkiadakis et al. [CEW11], Shoham and Leyton-Brown [SL09], and Peleg and Sudhölter [PS07], and the book chapters by Elkind et al. [ERJ13], and Elkind and Rothe [ER16]. We differentiate between cooperative games with transferable utility where a coalition of players is assigned a certain value dividable among the players in the coalition and coalition formation games without such transferable utility among which we focus on hedonic games.

2.3.1 Cooperative Games with Transferable Utility

A *cooperative game with transferable utility* $\mathscr{G} = (N, v)$ consists of a finite set $N = \{1, \dots, n\}$ of *players* or *agents*[6], and a *coalitional function* $v : \mathfrak{P}(N) \to \mathbb{R}$ that assigns a value $v(C) \in \mathbb{R}$ to each subset of players C, called a *coalition*. It is common to assume that $v(\emptyset) = 0$.

A cooperative game is called *monotonic* if for any two coalitions $B \subseteq C \subseteq N$ it holds that $v(B) \leq v(C)$. It is *simple* if it is monotonic and the coalitional function $v : \mathfrak{P}(N) \to \{0, 1\}$ is the *characteristic function* for success, mapping each coalition $C \subseteq N$ to a value that indicates whether C is *successful* or not. We say that C *wins* if $v(C) = 1$, and *loses* if $v(C) = 0$. The coalition N is called the *grand coalition*.

A *constant-sum* game $\mathscr{G} = (N, v)$ is a cooperative game with transferable utility which satisfies $v(C) + v(N \smallsetminus C) = v(N)$ for each coalition $C \subseteq N$. A cooperative game $\mathscr{G} = (N, v)$ is *convex* if $v(C \cup D) \geq v(C) + v(D) - v(C \cap D)$ holds for all coalitions $C, D \subseteq N$. This implies *superadditivity* which is satisfied if for all coalitions $C, D \subseteq N$ with $C \cap D = \emptyset$ it holds that $v(C \cup D) \geq v(C) + v(D)$. Two cooperative games $\mathscr{G} = (N, v)$ and $\mathscr{G}' = (N, v')$ are called *strategically equivalent* if there exist $\alpha > 0$ and $\beta : N \to \mathbb{R}$ such that $v'(C) = \alpha v(C) + \sum_{i \in C} \beta(i)$ holds for each $C \subseteq N$.

Player Properties The following solution concepts and related problems are commonly defined for simple games. Different players may have different significance in a game. A player of high importance is a *veto player*. Player i has the *veto property* if no coalition wins without her, in a simple game $\mathscr{G} = (N, v)$ that is, due to monotonicity, $v(N \smallsetminus \{i\}) = 0$. For a cooperative game $\mathscr{G} = (N, v)$, let

$$v(C \cup \{i\}) - v(C)$$

be the *marginal contribution* of a player $i \in N$ to a coalition $C \subseteq N \smallsetminus \{i\}$. A player of little significance in $\mathscr{G}$ is a *dummy player*. There are two different interpretations of what that means and, thus, two main definitions. Firstly, we call a player $i \in N$ a *null player* if adding her does not change the value of any coalition at all that is, $v(C \cup \{i\}) - v(C) = 0$ for each $C \subseteq N$ [PS07]. Such a player is also known as a dummy player (see, e.g., [DS79]). Secondly,

[6] We use these terms analogously due to applications in both game theory and multiagent systems.

we call a player j a *dummy player* if adding him changes the value of each coalition only to his own value, that is, $v(C \cup \{j\}) - v(C) = v(\{j\})$ for each $C \subseteq N \setminus \{j\}$ (see, e.g., [SL09] and [Sha53] in the context of superadditive games). The two notions coincide whenever the coalition consisting of the player considered has value 0. By definition, a player in a simple game can also be a dummy player if he has value 1, then, he is also a veto player. For simple games, we focus on the notion of null players. Two players i and j are called *symmetric* if they are interchangeable, that is $v(C \cup \{i\}) - v(C) = v(C \cup \{j\}) - v(C)$ for every coalition $C \subseteq N \setminus \{i, j\}$. A more precise measurement of a player's significance is provided by so called power indices, see below. Before we define them, we have a look at other concepts at first.

Payoff and Group Deviations One of the key goals in cooperative game theory is the question of how to distribute the game's total payoff among the players. An outcome in a cooperative game with transferable utility is a coalition structure, that is, a partition of the player set into coalitions, together with a payoff vector. A *payoff vector* $\vec{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n_{\geq 0}$ is a distribution of the value of a coalition to the players within the coalition. In this thesis, in the context of transferable utility, we focus on questions of stability of the grand coalition, thus, a payoff vector corresponds to the grand coalition and satisfies $\sum_{i=1}^{n} q_i \leq v(N)$. A *pre-imputation* is a payoff vector $\vec{q} = (q_1, \ldots, q_n)$ satisfying *efficiency*, i.e.,

$$\sum_{i=1}^{n} q_i = v(N).$$

An *imputation* is a pre-imputation additionally satisfying *individual rationality*, i.e.,

$$q_i \geq v(\{i\}) \quad \text{for each } i \in N.$$

Let $q(C) = \sum_{i \in C} q_i$ denote the total payoff of the players in coalition $C \subseteq N$ with respect to $\vec{q}$.

The *core* [Gil59] of a game $\mathscr{G} = (N, v)$ is the set of all payoff vectors that stabilize the game with respect to deviating coalitions. A coalition $C \subseteq N$ has an incentive to deviate from the grand coalition if its value is greater than the payoff of the players in C. The core of $\mathscr{G}$ consists of all payoff vectors $\vec{q}$ such that

$$q(C) \geq v(C) \quad \text{for each } C \subseteq N.$$

Due to its restrictive nature, it might be the case that the core does not contain any elements. It is known that the core of a simple game is non-empty if and only if there is a veto player in the game. A weaker form of the core is the ε*-core* of $\mathscr{G}$, where a deficit not exceeding a bound ε is allowed. Let $v(C) - \vec{q}(C)$ denote the deficit of a coalition. The ε-core of $\mathscr{G}$ consists of all payoff vectors[7] $\vec{q}$ such that

$$q(C) \geq v(C) - \varepsilon \quad \text{for each } C \subseteq N.$$

[7] Note that sometimes only imputations are allowed in the ε-core, e.g., Bachrach and Porat [BP10]. However, then, there might be trouble with successful singleton coalitions, and there might not exist a finite ε such that the ε-core is non-empty.

Maschler et al. [MPS79] introduce the *least core of a game* as its minimal non-empty ε-core. Note that the least core of a cooperative game is never empty.

Further concepts include the cost of stability due to Bachrach et al. [BEM$^+$09] or the stable sets due to von Neumann and Morgenstern [NM44].

Power Indices Other important solution concepts, so called *power indices*, measure the significance of a player in a game in different ways. It is common to study these indices in simple games. Nevertheless, note that they can easily be redefined for a more general setting. A player i in a simple game $\mathcal{G} = (N, v)$ is called *pivotal* (or *crucial* or *critical*) for a coalition $C \subseteq N \smallsetminus \{i\}$ if her marginal contribution to C is 1, that is, if $C \cup \{i\}$ is successful, but C is not.

In this thesis we study two popular indices, the Penrose–Banzhaf index and the Shapley–Shubik index. For other indices, see, e.g., an early overview by Felsenthal and Machover [FM05]. Banzhaf [Ban65] rediscovered a notion originally introduced by Penrose [Pen46]. The *raw Penrose–Banzhaf power index* of player i in $\mathcal{G}$ is defined by

$$\text{PenroseBanzhaf}^*(\mathcal{G}, i) \;=\; \sum_{C \subseteq N \smallsetminus \{i\}} (v(C \cup \{i\}) - v(C)).$$

This indicates the number of coalitions a player is pivotal for. It is useful to normalize this value; in fact, two different ways of normalization have been proposed for the Penrose–Banzhaf index. In the original definition of the *normalized Penrose–Banzhaf power index* [Ban65], the raw Penrose–Banzhaf index of a given player is divided by the sum of all players' raw indices:

$$\overline{\text{PenroseBanzhaf}}(\mathcal{G}, i) \;=\; \frac{\text{PenroseBanzhaf}^*(\mathcal{G}, i)}{\sum_{j=1}^{n} \text{PenroseBanzhaf}^*(\mathcal{G}, j)},$$

so that all players' normalized indices add up to one.[8] This index was analysed in detail by Dubey and Shapley [DS79], who introduced an alternative normalization, which divides the raw Penrose–Banzhaf index of a given player by the total number of coalitions without that player, obtaining the *probabilistic Penrose–Banzhaf power index*:

$$\text{PenroseBanzhaf}(\mathcal{G}, i) \;=\; \frac{\text{PenroseBanzhaf}^*(\mathcal{G}, i)}{2^{n-1}}.$$

Intuitively, this index measures the probability that a player is pivotal for any possible coalition. Here, normalization is done with respect to the number of coalitions, which makes games with different numbers of players better comparable. Both indices have different advantages resulting from their axiomatic properties. In a nutshell, while the normalized Penrose–Banzhaf index yields an efficient payoff vector and is therefore studied by Aziz et al. [ABEP11], the probabilistic Penrose–Banzhaf index shows other mathematical advantages. Dubey and Shapley [DS79] comprehensively analyse these two indices comparing various mathematical properties. In particular, for a vector $\vec{q}(N, v) =$

[8] Note that for this definition it is necessary to assume that the grand coalition has as positive value.

$(q_1(N,v),\dots,q_n(N,v)) \in \mathbb{R}^n_{\geq 0}$ of power indices for n players in a game (N,v), they study four fundamental axioms:

symmetry: *Whenever two players i and j are symmetric in a game $\mathscr{G}$, it holds that $q_i(N,v) = q_j(N,v)$;*

dummy player: *If player i is a null player in a game (N,v), then $q_i(N,v) = v(\{i\})$;*

additivity: *For any two games (N,v_1) and (N,v_2), it holds that $q_i(N,v_1+v_2) = q_i(N,v_1) + q_i(N,v_2)$ for all players $i \in N$, where v_1+v_2 is defined via $(v_1+v_2)(C) = v_1(C) + v_2(C)$ for all coalitions $C \subseteq N$;*

valuation: *For any two simple games (N,v_1) and (N,v_2), it holds that $q_i(N,v_1 \vee v_2) + q_i(N,v_1 \wedge v_2) = q_i(N,v_1) + q_i(N,v_2)$ for each $i \in N$, where $v_1 \vee v_2$ and $v_1 \wedge v_2$ are defined via $(v_1 \vee v_2)(C) = \max\{v_1(C), v_2(C)\}$, and $(v_1 \wedge v_2)(C) = \min\{v_1(C), v_2(C)\}$ for all coalitions $C \subseteq N$.*

Amongst others they discover the fact that the probabilistic Penrose–Banzhaf index satisfies all four axioms while the normalized Penrose–Banzhaf index lacks the latter two. Another evidence in advantage to the probabilistic Penrose–Banzhaf index is that, in contrast to the normalized index, it is not subject to the so-called *bloc paradox* (see [FM95]), that is, a player can lose power by taking over another player. See Chapter 3, Section 3.1 for the computational complexity of the annexation problem, the problem of whether a player gains power by annexing a player. We study the probabilistic index due to its comparability for two games in the context of manipulation (see, e.g., Remark 3.8). We refer to the work of Dubey and Shapley [DS79] as well as the work of Felsenthal and Machover [FM05, FM95] for a careful, detailed discussion on the differences between the two normalizations.

There is a unique pre-imputation satisfying all four axioms mentioned above. Shapley [Sha53] defines the underlying concept for superadditive games as a value satisfying symmetry, efficiency and additivity. He states that "it is remarkable that no further conditions are required to determine the value uniquely". This value, known as the *Shapley value* is one of the most popular concepts studied in cooperative games, see, e.g., the textbook by Peleg and Sudhölter [PS07] for further axiomatic characterizations. For a simple game $\mathscr{G} = (N,v)$ the *Shapley–Shubik power index* [SS54] describes the marginal contributions of a player to all possible coalitions with respect to the order in which players enter the coalitions. Let the raw Shapley–Shubik index be characterized[9] by

$$\text{ShapleyShubik}^*(\mathscr{G},i) \;=\; \sum_{C \subseteq N \smallsetminus \{i\}} \|C\|! \cdot (n-1-\|C\|)! \cdot (v(C \cup \{i\}) - v(C)),$$

and normalized by

$$\text{ShapleyShubik}(\mathscr{G},i) \;=\; \frac{\text{ShapleyShubik}^*(\mathscr{G},i)}{n!}.$$

[9] The original definition considers the sum over all possible permutations and is known to be equivalent to this one.

Representations Having defined these concepts, we have to narrow down the setting we want to study them in. Since the number of coalitions is exponential in the number of players, representing a cooperative game by listing the values of its coalitional function, requires exponential space. Hence, for algorithmic purposes, it is essential to find succinct representations. Succinct representations, however, often enough restrict expressiveness. In the following we consider weighted voting games as one way of representing simple games succinctly. For further representations and their succinctness and expressiveness, see, e.g., the books and book chapters mentioned at the beginning of this section. We assume a neutral environment, that is, although we may write players' weights etc. as lists, there is no order over players with an influence on a coalitional function.

Weighted Voting Games Weighted voting games (also known as weighted threshold games) [NM44] are an important class of simple cooperative games that are compactly representable but not fully expressive.[10] Formally, a *weighted voting game*

$$\mathcal{G} \leftrightsquigarrow (w_1, \ldots, w_n; q)$$

is represented by weights $w_i \in \mathbb{N}$, $1 \leq i \leq n$, where w_i is the i^{th} player's weight, and a quota $q \in \mathbb{N}$.[11] A coalition $C \subseteq N$ wins if and only if the sum of the players' weights involved in the coalition reaches or exceeds the quota. That is, for each coalition $C \subseteq N$, letting $w(C)$ denote $\sum_{i \in C} w_i$, C wins if $w(C) \geq q$, and it loses otherwise. Requiring the quota to satisfy $0 < q \leq w(N)$ ensures that the empty coalition loses and the grand coalition wins. Weighted voting games have been intensely studied from a computational complexity point of view, see, e.g., the work by Elkind et al. [EGGW09, EGGW07] and the book by Chalkiadakis et al. [CEW11, Chapter 4] for an overview.

Prasad and Kelly [PK90, Theorem 5] show that for a given weighted voting game, the computation of the Penrose–Banzhaf index is #P-many-one-complete. For the raw version of the index their proof implies that its computation in a given weighted voting game for a given player, is #P-parsimonious-complete [FH09]. Deng and Papadimitriou [DP94, Theorem 9] show that computing the raw Shapley–Shubik power index in a given weighted voting game of a given player is #P-many-one-complete; Faliszewski and Hemaspaandra [FH09] show that it is not #P-parsimonious-complete.

A *weighted majority game* is defined similarly to weighted voting games, except that no fixed quota q is given; instead, a coalition is successful if and only if the sum of the players' weights within this coalition is greater than half of the total sum of all players' weights, that is, the quota is, dependent on the players' weights, set to $q = \lfloor w(N)/2 \rfloor + 1$.

[10] Full expressiveness for one-dimensional simple games can be gained by so-called vector weighted voting games, a multi-dimensional game represented by the intersection of several weighted voting games (see, e.g., [CEW11]).

[11] See [CEW11, Theorem 4.2] for why non-negative integer weights and quotas may be assumed.

Path-Disruption Games Path-disruption games are cooperative games introduced by Bachrach and Porat [BP10]. Given a graph[12] $G = (V,E)$, the set of agents $N = \{1,\dots,n\}$ corresponds to the set of vertices $V = \{v_1,\dots,v_n\}$. Moreover, there are m adversarial players who each want to travel from a source vertex s_j to a target vertex t_j in V, $1 \leq j \leq m$. We say a coalition $C \subseteq N$ *blocks a path from* s_j *to* t_j if there is no path from s_j to t_j in the induced subgraph $G|_{V \smallsetminus \{v_i | i \in C\}}$ or if s_j or t_j are not even in $V \smallsetminus \{v_i \mid i \in C\}$. Four types of path-disruption games are distinguished: those with a single adversary and with multiple adversaries, and for both with and without costs. The most general game is the model with several adversarial players and costs for each vertex to be blocked. Letting $c : V \to \mathbb{R}_{\geq 0}$ be a function defining a cost for each vertex to be blocked, $c(C) = \sum_{i \in C} c(v_i)$ denotes a coalition C's cost.

Definition 2.8. *We are given an undirected graph $G = (V,E)$ with $n = \|V\|$ vertices and with an adversary associated with each of the pairs $(s_1,t_1),\dots,(s_m,t_m)$, $s_j,t_j \in V$, $1 \leq j \leq m$, a cost function $c : V \to \mathbb{R}_{\geq 0}$, a reward $R \in \mathbb{R}_{\geq 0}$. A* path-disruption game with costs and multiple adversaries *is defined by players $N = \{1,\dots,n\}$, where i is represented by v_i, $1 \leq i \leq n$, and the coalitional function*

$$v(C) = \begin{cases} R - \mu(C) & \text{if } \mu(C) < \infty \\ 0 & \text{otherwise} \end{cases}$$

with

$$\mu(C) = \begin{cases} \min\{c(B) \mid B \subseteq C \text{ and } \tilde{v}(B) = 1\} & \text{if } \tilde{v}(C) = 1 \\ \infty & \text{otherwise,} \end{cases}$$

where

$$\tilde{v}(C) = \begin{cases} 1 & \text{if } C \text{ blocks each path from } s_j \text{ to } t_j, \text{ for each } j,\ 1 \leq j \leq m \\ 0 & \text{otherwise.} \end{cases}$$

Letting $m = 1$, we have a restriction to a single adversary. Letting $c(v_i) = 0$ for all i, $1 \leq i \leq n$, $R = 1$, and, thus, $v(C) = \tilde{v}(C)$, the simple games without costs are defined. We say a coalition $C \subseteq N$ *wins* the game if $\tilde{v}(C) = 1$, and *loses* it otherwise.

In Definition 2.8 weights and bounds are real numbers; however, to make the problems for these games suitable for computer processing (and to define their complexity in a reasonable way), we will henceforth assume that all weights and bounds are rational numbers.

In the simple form of path-disruption games, the value of a coalition can be computed in polynomial time, however, when costs are included, polynomial-time computability of a value is only known for a single adversary [BP10]. Path-disruption games with costs are not monotonic. In Chapter 4 we will generalize path-disruption games to a model with uncertainty about the targets.

[12] In this thesis we consider path-disruption games on undirected graphs, see Chapter 4.

2.3.2 Hedonic Games

Hedonic games are cooperative games, however not structured as a game with transferable utility. Here, the central ideas of, on the one hand, cooperative game theory where players form coalitions in order to cooperate managing certain tasks as teams, and, on the other hand, voting scenarios (see, e.g., [BF02, BCE13]) where players cast their preferences over alternatives in order to elect a solution in mutual agreement, are combined. In hedonic games the players vote on coalitions they are contained in by expressing weak preference orders. This hedonic model, where a coalition's happiness only depends on the players involved in it, is introduced by Drèze and Greenberg [DG80] and formalized by Banerjee et al. [BKS01] and independently by Bogomolnaia and Jackson [BJ02].

Formally, a hedonic game is a tuple $\mathscr{H} = (N, \succeq)$, where $N = \{1, \dots, n\}$, again, denotes the finite set of players, and $\succeq = (\succeq_1, \dots, \succeq_n)$ is a profile of preferences, where $\succeq_i$ is a reflexive, transitive, and total relation over $\mathscr{N}_i = \{C \subseteq N \mid i \in C\}$. This relation induces a total weak preference order for player i. For two coalitions $A, B \in \mathscr{N}_i$, we say that player i *weakly prefers* A to B if $A \succeq_i B$, i *(strictly) prefers* A to B if $A \succ_i B$, and i *is indifferent between* A and B if $A \sim_i B$. A *coalition structure* in $\mathscr{H}$ is a partition $\Gamma = \{C_1, \dots, C_k\}$ of the players into $k \geq 1$ coalitions $C_1, \dots, C_k \subseteq N$ (i.e., $\bigcup_{r=1}^{k} C_r = N$ and $C_r \cap C_s = \emptyset$ for $r \neq s$, $C_r \neq \emptyset$, $1 \leq r \leq k$). For a coalition structure Γ, let $\Gamma(i)$ denote the coalition containing player i.

Stability Concepts The following solution concepts, or *stability concepts*, are commonly studied for hedonic games [BKS01, BJ02, ABS13, ABH13, Woe13a]. A coalition structure Γ is called

perfect *if for each player $i \in N$, $\Gamma(i)$ is one of her favourite coalitions, that is, i weakly prefers $\Gamma(i)$ to every coalition in $\mathscr{N}_i$,*

uniquely perfect *if Γ is perfect and no other coalition structure is perfect,*

individually rational *if $\Gamma(i)$ is* acceptable*, for each player $i \in N$, that is, i weakly prefers $\Gamma(i)$ to being alone in $\{i\}$.*

While perfection is rather rare, individual rationality is guaranteed by $\{\{i\} \mid i \in N\}$. More demanding concepts consider deviations of a single player to another (possibly empty) existing coalition. We say that a coalition structure Γ is called

Nash-stable *if for each player $i \in N$, $\Gamma(i) \succeq_i C \cup \{i\}$ holds for each coalition $C \in \Gamma \cup \{\emptyset\}$, that is, no player wants to move to another existing or empty coalition,*

individually stable *if for each player $i \in N$ and for each coalition $C \in \Gamma \cup \{\emptyset\}$, it holds that $\Gamma(i) \succeq_i C \cup \{i\}$ or there exists a player $j \in C$ such that $C \succ_j C \cup \{i\}$, that is, no player can move to another preferred coalition without making a player in the new coalition worse off,*

contractually individually stable *if for each player $i \in N$ and for each coalition $C \in \Gamma \cup \{\emptyset\}$, it holds that $\Gamma(i) \succeq_i C \cup \{i\}$, or there exists a player $j \in C$ such that $C \succ_j C \cup \{i\}$, or there exists a player $j' \in \Gamma(i) \setminus \{i\}$ such that $\Gamma(i) \succ_{j'} \Gamma(i) \setminus \{i\}$, that is, no player can move to another preferred coalition without making a player in the new coalition or in the old coalition worse off.*

Note that Nash stability implies individual stability, which, in turn, implies contractually individual stability, as the requirements increase: Firstly, no player wants to deviate from her coalition; then, a player might want to deviate but is not welcome in the new coalition; and finally, she might be welcome but is contractually bounded to the former coalition. Moreover, the next two commonly studied concepts deal with group deviation. A coalition structure Γ is called

core-stable *if for each non-empty coalition $C \subseteq N$, there exists a player $i \in C$ such that $\Gamma(i) \succeq_i C$, that is, no coalition* blocks Γ,

strictly core-stable *if for each coalition $C \subseteq N$, there exists a player $i \in C$ such that $\Gamma(i) \succ_i C$ or for each player $i \in C$, $\Gamma(i) \sim_i C$, that is, no coalition* weakly blocks Γ.

Alternatively, other concepts are based on a relation comparing different coalition structures. A coalition structure Γ is called

Pareto-optimal *if for each coalition structure Δ, there exists a player $i \in N$ such that $\Gamma(i) \succ_i \Delta(i)$ or for each player $j \in N$, $\Gamma(j) \sim_j \Delta(j)$, that is, no other coalition structure* Pareto-dominates Γ,

popular *if for each coalition structure Δ, the number of players i with $\Gamma(i) \succ_i \Delta(i)$ is at least as large as the number of players j with $\Delta(j) \succ_j \Gamma(j)$.*

Popularity implies Pareto optimality, since a popular coalition structure cannot be Pareto-dominated. We furthermore introduce the notion of *strict popularity*. A coalition structure Γ is called *strictly popular* if it *strictly beats* each coalition structure $\Delta \neq \Gamma$ *in pairwise comparison*,[13] that is,

$$\|\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}\| > \|\{j \in N \mid \Delta(j) \succ_j \Gamma(j)\}\|.$$

We study these stability concepts in Chapter 5 regarding their guaranteed existence in a game and their complexity of existence and verification.

The interrelations of these solution concepts follow from their definitions (see, e.g., [ABS13]), and are depicted in Figure 2.2. As regarding strict popularity, it can be integrated as follows.

[13] This notion is adapted from the voting-theoretic term *Condorcet winner* where a candidate wins an election if she beats each other candidate in pairwise comparison.

Proposition 2.9. *Let $\mathscr{H} = (N, \succeq)$ be a hedonic game. For a coalition structure Γ in $(N, \succeq)$ it holds that*

1. *Γ strictly popular $\implies$ Γ popular,*
2. *Γ uniquely perfect $\implies$ Γ strictly popular,*
3. *Γ non-uniquely perfect $\implies$ Γ not strictly popular,*
4. *If Γ is strictly popular, it is not always individually rational,*
5. *If Γ is not strictly popular, even if it is not perfect, it might be strictly core-stable or Nash-stable.*

Proof. 1. By definition, $\|\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}\| > \|\{j \in N \mid \Delta(j) \succ_j \Gamma(j)\}\| \implies \|\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}\| \geq \|\{j \in N \mid \Delta(j) \succ_j \Gamma(j)\}\|$.

2. If Γ is uniquely perfect, for each other coalition structure Δ it holds that $\|\{j \in N \mid \Delta(j) \succ_j \Gamma(j)\}\| = 0$, and $\|\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}\| > 0$.
3. If there exist two perfect coalition structures Γ and Δ, it holds that $\|\{j \in N \mid \Delta(j) \succ_j \Gamma(j)\}\| = \|\{i \in N \mid \Gamma(i) \succ_i \Delta(i)\}\| = 0$.
4. Consider the game with three players $N = \{1,2,3\}$ and preferences as follows. Players 1 and 2 prefer the grand coalition to every other coalition, player 3 prefers $\{3\}$ to $\{1,2,3\}$ and this to the rest. Coalition structure $\Gamma = \{\{1,2,3\}\}$ is not individually rational because of player 3, but is strictly popular since at least two players prefer their position in Γ to any other.
5. If Γ is not strictly popular, but (non-uniquely) perfect, it is, of course, Nash- and strictly core-stable. If Γ is neither strictly popular nor perfect, it might still be Nash-stable or strictly core-stable. Consider, for instance, the game with $N = \{1,2,3\}$ and preferences as follows.

$$\begin{aligned}
&\{1,2\} \succ_1 \{1,2,3\} \sim_1 \{1\} \succ_1 \{1,3\},\\
&\{2,3\} \succ_2 \{1,2,3\} \sim_2 \{2\} \succ_2 \{1,2\},\\
&\{1,3\} \succ_3 \{1,2,3\} \sim_3 \{3\} \succ_3 \{2,3\}.
\end{aligned}$$

   Coalition structure $\Gamma = \{\{1,2,3\}\}$ is obviously Nash-stable. It is strictly core-stable, since it is individually rational, and since size-two coalitions do not block because of one player each. On the other hand, Γ is not perfect, since $\{1,2,3\}$ is no one's favourite coalition. It is not strictly popular, since the coalition structure consisting of singletons is equally popular, as well as any coalition structure consisting of a size-two coalition and a singleton, with a comparison of 1 to 1 each. ◻

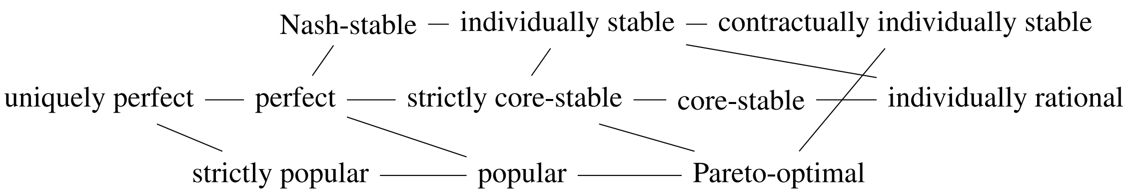


Figure 2.2: Interrelations of stability concepts for hedonic games. Edges illustrate implications from left to right.

Representations Similarly to cooperative games with transferable utility, an entire list of the players' preferences of a hedonic game requires n comparisons of 2^{n-1} coalitions each, that is, exponential space in the number of players n. Therefore, different representations for hedonic games are studied. Again, there is a trade-off between succinctness and expressiveness. Fully expressive representations include *hedonic coalition nets* [EW09]: Each agent specifies her utility function over the set of all coalitions via a set of weighted logical formulae. In case of *individually rational encoding* [Bal04] each agent ranks only the coalitions she prefers to being alone. This model is in many cases more compact, but there are cases that cannot be represented polynomially in the number of players; and it is not fully expressive, but it expresses all relevant sets in terms of core and individual stability. Other approaches allow compact representation but cannot describe all possible hedonic games, although a reasonable selection of natural games that fulfil desirable properties are represented. Those representations include the following: In the *anonymous encoding* [Bal04, DEK$^+$12] each agent specifies a preference relation over the number of agents in her coalition and coalitions with the same cardinality are, independently from the identities of the agents, considered as indifferent.

In other representations the game is given by an encoding of a certain form of a profile of opinions on the players instead of coalitions in a game. Then, a preference extension maps a player's opinion[14] on the other players to a preference relation over the coalitions in $\mathscr{N}_i$. We do not only want the encoding of a hedonic game to be compact but also the preference extension to be computable succinctly: Since we cannot write down the whole preference order of a player in polynomial time in the number of players, the desired property here is that for two coalitions their relation can be computable in polynomial time in the number of players.

If the encoding is a list of functions $u_i : N \to \mathbb{Q}$, $i \in N$, a preference extension $\succeq_i$ is called *additive* if $C \succeq_i D$ holds for two coalitions C and D and a player i, if $\sum_{j \in C} u_i(j) \geq \sum_{k \in D} u_i(k)$. A hedonic game $\mathscr{H} = (N, \succeq)$ is called *additively separable* [BJ02] if for each player $i \in N$, there exists a function $u_i : N \to \mathbb{Q}$ such that $u_i(i) = 0$ and $\succeq_i$ is the additive extension of these functions. The following properties are an excerpt of the concepts we define in [NRR$^+$16] relevant for this thesis. These properties are inspired by various related topics such as voting theory [Tid06, End13], fair division [LR16], and ranking sets of objects in general [BBP04].

[14] This could have any imaginable form such as the encodings presented in the following.

A profile of preference extensions $\succeq$ is called *anonymous/neutral*[15] if switching the names of two players in the encoding does not have an effect on $\succeq$ but a renaming of these two players in the profile. Let i, j and k be three distinct players. We define *symmetry* for $\succeq$ by the following property: If swapping j and k in the encoding does not change the game, then $\left(\forall C \in \mathcal{N}_i \setminus (\mathcal{N}_j \cup \mathcal{N}_k)\right) [C \cup \{j\} \sim_i C \cup \{k\}]$. An even stronger property would be the opposite implication. In some cases it might be useful, if $\succeq$ satisfies *independence*, that is, $A \cup C \succeq_i B \cup C$ implies $A \succeq_i B$ for each $A, B \in \mathcal{N}_i$ and $C \subseteq N \setminus (A \cup B)$ and each $i \in N$. Let $i \neq j$ be two players and $A, B \in \mathcal{N}_i$ two coalitions with $j \in A \setminus B$. Consider a preference relation $\succeq_i'$ that results from j ascending in i's opinion. We say that $\succeq$ is *monotonic* if it holds that

(1) if $A \succ_i B$, then $A \succ_i' B$, and

(2) if $A \sim_i B$, then $A \succeq_i' B$.

In the context of other players' influences on a player's preference order and dependent on the underlying model, we define an additional type of monotonicity and a notion of *unanimity*, see Section 5.3.1. *Additively separable hedonic games* are studied, e.g., in [SD07, SD10, ABS13, Woe13b, Pet15]. They comprise all games that can be encoded with an additive valuation function: Each agent gives a rational valuation of each agent and preferences satisfy additive separability, that is, are derived from the values that are, from the point of view of one player, extended to coalitions by summing up the values of the agents in the coalition. A recent variant considers subset-additive hedonic games [Suk15]. In *fractional hedonic games* [ABH14, BBS15, AGG$^+$15] each agent assigns a value to each other agent (and 0 to herself) and an agent's utility of a coalition is the average value she assigns to the members of the coalition. Then, a coalition A is preferred to B if the utility of A is greater than that of B. In Chapter 5 we consider the following succinct representations:

Network of Friends Dimitrov et al. [DBHS06, SD07] introduce this encoding, a special variant of additively separable hedonic games. Each agent $i \in N$ partitions the set of other agents into two sets: her friends $N_i^+ \subseteq N \setminus \{i\}$ and her enemies $N_i^- = N \setminus (N_i^+ \cup \{i\})$.[16] Visually, let the players $N = \{1, \dots, n\}$ be represented by vertices $V = \{v_1, \dots, v_n\}$ in a graph $G = (V, E)$ and let a directed edge $(v_i, v_j) \in E$ denote that j is i's friend, that is, the open neighbourhood of v_i represents the set of i's friends $N_i^+ = \{j \mid (v_i, v_j) \in E\}$. We understand such a network of players, for example, as a social network, where two friends are able to communicate with each other but might not like or, more importantly, not know the other players. Either way, their means of communication with unconnected players is restricted. In the context of stability it is reasonable to consider undirected edges, that is, symmetric friendship relations, only [Woe13a]. We will focus on this case.

[15] We use both terms here to emphasize that in comparison to an election players here have both, the role of the *voters* for which anonymity is required, and of the *candidates* for which neutrality is essential.

[16] Note that in the literature is is also common to denote the set of friends by F_i and the set of enemies by E_i.

Dimitrov et al. suggest two ways of deriving the players' preferences from such a network, appreciation of friends and aversion to enemies. Under the *friend-oriented preference extension*, $\succeq_i^+$, coalition A is preferred to coalition B by a player $i \in N$ if A contains more friends than B, or as many friends as B but fewer enemies than B:

$$A \succeq_i^+ B \iff \|A \cap N_i^+\| > \|B \cap N_i^+\| \text{ or } (\|A \cap N_i^+\| = \|B \cap N_i^+\| \text{ and } \|A \cap N_i^-\| \leq \|B \cap N_i^-\|).$$

Note that friend-oriented preferences can be represented additively, by assigning a value of $n = \|N\|$ to a friend and a value of -1 to an enemy [DBHS06]. For any player $i \in N$ and for any coalition $A \in \mathscr{N}_i$, let $u_i(A) = n\|A \cap N_i^+\| - \|A \cap N_i^-\|$. Then, for $A, B \in \mathscr{N}_i$, it holds that $A \succeq_i^+ B \iff u_i(A) \geq u_i(B)$. Under the *enemy-oriented preference extension*, $\succeq_i^-$, for player i, A is preferred to B if A contains fewer enemies than B, or as many enemies as B and more friends than B:

$$A \succeq_i^- B \iff \|A \cap N_i^-\| < \|B \cap N_i^-\| \text{ or } (\|A \cap N_i^-\| = \|B \cap N_i^-\| \text{ and } \|A \cap N_i^+\| \geq \|B \cap N_i^+\|).$$

Analogously, enemy-oriented preferences are additively separable, with a value of coalition A of

$$u_i(A) = \|A \cap N_i^+\| - n\|A \cap N_i^-\|.$$

Note that under enemy-oriented preferences, the graph-theoretic concept of wonderful stability (Definition 2.7) translates to a stability concept that is even more demanding than strict core stability: A coalition structure is wonderfully stable if each player is part of her favourite coalition amongst those containing no mutual enemies.

Lemma 2.10. *Let $G = (V, E)$ be the graph representing an enemy-oriented hedonic game $\mathscr{H}$. Let Π be a partition of V and let Γ be the corresponding coalition structure in $\mathscr{H}$.*

1. *If Π is wonderfully stable for G, then Γ is strictly core-stable for $\mathscr{H}$.*
2. *If there exists an integer $c \in \mathbb{N} \setminus \{0\}$ such that $\omega_G(v) = c$ for all vertices $v \in V$ and Γ is strictly core-stable for $\mathscr{H}$, then Π is wonderfully stable.*

Singleton Encoding The *singleton encoding* as introduced by Cechlárová and Romero-Medina [CR01] and studied by Cechlárová and Hajduková [CH03, CH04] and Aziz et al. [AHP12, ABH13] provides for each agent a ranking $\trianglerighteq_i$, (originally a strict ranking $\triangleright_i$), of all single agents, and allows two extensions. Under the optimistic extension ($\mathscr{B}$-preferences), a player prefers coalition A to coalition B if the best agent in A is preferred to the best agent in B (or in case of indifference, the smaller coalition is preferred). Reversely, under the pessimistic extension ($\mathscr{W}$-preferences), A is preferred to B if the worst agent in A is preferred to the worst agent in B.

Note that for the representations needed later on, namely additively separable, friend-oriented, and enemy-oriented hedonic games and $\mathscr{B}$- and $\mathscr{W}$-preferences, anonymity/neutrality, symmetry and the following properties hold.

Proposition 2.11. *Additively separable, friend-oriented, and enemy-oriented hedonic games satisfy independence, while $\mathscr{B}$- and $\mathscr{W}$-preferences don't. All five extension principles are monotonic.*

Proof. Let $i \in N$ be a player, $A, B \in \mathscr{N}_i$ and $C \in N \setminus (A \cup B)$. An additive preference extension $\succeq$ obviously adds the same value of C to A and B alike, such that $A \succeq_i B$ implies $A \cup C \succeq_i A \cup B$. Also, if the value of a player $j \in A \setminus B$ in increased in $\succeq'$, the value of A is increased, but that of B remains the same.

Since both, the friend-oriented and the enemy-oriented extensions, are additive, independence and monotonicity are implied.

Let $b_i(D)$ denote the best and $w_i(D)$ denote the worst player in a coalition $D \in \mathscr{N}_i$. For $\mathscr{W}$-preferences $\succeq_i^{\mathscr{W}}$, consider the following case. Let

$$w_i(A) \rhd_i w_i(B) \rhd_i w_i(C).$$

Thus, $A \succ_i^{\mathscr{W}} B$, but $w_i(A \cup C) = w_i(C) = w_i(B \cup C)$ which means that $A \cup C \succ_i^{\mathscr{W}} B \cup C$ does not hold. For $\mathscr{B}$-preferences $\succeq_i^{\mathscr{B}}$, additionally assume that $\|B\| < \|A\|$ and let

$$b_i(C) \rhd_i b_i(A) \rhd_i b_i(B).$$

Then, $A \succ_i^{\mathscr{B}} B$, but $b_i(A \cup C) = b_i(C) = b_i(B \cup C)$. In the case of equality, the cardinality is compared. Hence, $\|B \cup C\| < \|A \cup C\|$ implies $B \cup C \succ_i^{\mathscr{B}} A \cup C$, a contradiction to independence.

Monotonicity, however, holds in both cases: If $w_i(A) = j$, then either j remain the worst player and is still better than the worst player in B, or j overtakes a new worst player in A which cannot be worse than $w_i(B)$, otherwise this would have been the worst player in A before. If $j \rhd_i w_i(A)$, then the relation remains the same. Similarly, if $b_i(B) = j$, then it is even better in the new relation. If $b_i(A) \neq j$, it may be the case that the best player remains the same or j becomes an even better best player, then the relation remains the same. It may also be the case that $b_i(A) = b_i(B)$, then the cardinality inequality remains the same, or j becomes the best player in A and therefore, again A is preferred. ◻

In Chapter 5 we will introduce a new representation that combines the singleton and the network of friends encodings. Moreover, we will define an extension to a network of friends considering altruistic influences.

3 Weighted Voting Games: Manipulation and Control

This chapter deals with influence in cooperative games. In Section 3.1 three types of manipulation in weighted voting games are analysed and extended to a generalized framework for different classes of games in Section 3.2. Section 3.3 contains studies of structural control scenarios in weighted voting games. We conclude with ideas for future work in Section 3.4. If not indicated otherwise, the results of Sections 3.1 and 3.2 can be found in the article [RR14a], and the conference contribution [RR14b]. The paper [RR16] is based on the contents of Section 3.3

As a running example consider the following weighted voting game.

Example 3.1. *Let $\mathcal{G} = (N, v)$ be a weighted voting game with six players in $N = \{1,2,3,4,5,6\}$ represented by*

$$(1,2,2,3,4,5;\,10).$$

The players' probabilistic Penrose–Banzhaf and Shapley–Shubik power indices are distributed as follows:

player	1	2	3	4	5	6
PenroseBanzhaf	$\frac{4}{32}$	$\frac{6}{32}$	$\frac{6}{32}$	$\frac{10}{32}$	$\frac{12}{32}$	$\frac{18}{32}$
ShapleyShubik	$\frac{4}{60}$	$\frac{6}{60}$	$\frac{6}{60}$	$\frac{11}{60}$	$\frac{13}{60}$	$\frac{20}{60}$

with no player being a dummy player and power weakly increasing with the player's weights.

False-name manipulation describes the strategic simulation of false identities in order to increase a player's power. More concretely, we study the question as to whether a player in a cooperative game can increase her significance measured by a power index by splitting up into several new players. Reversely, beneficial merging is the problem of whether a coalition of players can pretend to be one single player and thereby gain a higher index in sum. Relatedly, we consider the problem of beneficial annexation where a single player takes over other players' weights and achieves an advantage for herself. The latter problem is related to the bloc-paradox as described in Section 2.3.1 and studied by, for example, Felsenthal and Machover [FM95]. Bachrach and Elkind [BE08] were the first to study false-name manipulation from a computational complexity perspective in weighted voting games as defined in Section 2.3.

Table 3.1 provides an overview of the development of the complexity results of beneficial merging, splitting, and annexation for the two power indices studied here, the probabilistic Penrose–Banzhaf index and the Shapley–Shubik index.

PI-BENEFICIALMERGE	PI-BENEFICIALSPLIT	PI-BENEFICIALANNEXATION
• open question [*] • NP-hard *(Shapley–Shubik index)* [†] [††] • NP-hard *(probab. Penrose–Banzhaf index)* [‖]	• NP-hard *(Shapley–Shubik index, $m = 2$)* [*] [††] • NP-hard *(probab. Penrose–Banzhaf index)* [‖]	• never disadv. *(Shapley–Shubik index)* [‡] [††] • never disadv. *(probab. Penrose–Banzhaf index)* (Eq. (3.3)) [¶]
• in PP *(Shapley–Shubik index, $\|S\| = 2$)* [§] • in P *(probab. Penrose–Banzhaf index, $\|S\| = 2$)* (Prop. 3.4) [‖] • in PP *(probab. Penrose–Banzhaf index)* [‖]	• in P *(probab. Penrose–Banzhaf index, $m = 2$)* (Prop. 3.4) [‖]	• NP-complete *(player)* (Thm. 3.12, 3.16) [¶]
• PP-complete (Thm. 3.7, 3.14) [¶]	• PP-hard (Thm. 3.10, 3.15) [¶]	• NP-hard *(coalition)* (Rem. 3.13, 3.17) [¶]

[*] [BE08]
[†] [AP09]
[††] [ABEP11]
[‡] [FM95]
[§] [FH09]
[‖] [RR14a, RR10a, RR10b]
[¶] this thesis ([RR14a, RR14b])

Table 3.1: Overview of the history of complexity results of beneficial merging, splitting, and annexation for the probabilistic Penrose–Banzhaf index and the Shapley–Shubik index. Chronologically, the first row describes initial results and lower bounds; the second row contains the subsequent upper bounds and special cases; and the third row reports the latest results from this thesis. Key: $\|S\|$ denotes the size of a merging coalition and m is the number of players a given player splits into.

In an assembled and extended article of the results by Bachrach and Elkind [BE08] and Aziz and Paterson [AP09], Aziz et al. [ABEP11] study the problems of beneficial merging, splitting, and annexation in weighted voting games in terms of the Shapley–Shubik and the normalized Penrose–Banzhaf index. In [RR10a] this study for the merging and splitting problems is extended for the probabilistic Penrose–Banzhaf index. These results, however, provide merely NP-hardness lower bounds (row 1). Aziz et al. note that the problem might not be NP-complete. Faliszewski and Hemaspaandra [FH09] provide the best known upper bound for the beneficial merging problem for two players with respect to the Shapley–Shubik index: It is contained in the class PP, and they conjecture that this problem is PP-complete. We observe, by the same arguments, the same upper bound for beneficial merging in terms of the probabilistic Penrose–Banzhaf index. In contrast to the normalized Penrose–Banzhaf index and the Shapley–Shubik index, for the probabilistic Penrose–Banzhaf index, the problems of increasing power by merging or splitting are in P for coalitions of size two and a split into two players, respectively (row 2).

For the beneficial merging problem, we bridge the gap between the NP-hardness lower bound and the PP upper bound, for both the Shapley–Shubik and the probabilistic Penrose–Banzhaf index, resolving the conjecture of PP-completeness in the affirmative. For false-name manipulation, we also raise the lower bound to PP-hardness. The upper bound, however, depends on the particular definition of the problem. PP-hardness holds in the case in which the new players' weights are given in the problem instance (row 3).

While these problems deal with voluntary actions by a group of players, [FM95] study the question of whether it is possible for a player to change her power by taking another player's weight over without this player's consent (column 3). Similar to their *bloc paradox*, stating that for the normalized Penrose–Banzhaf index it is possible to lose power by annexing another player's weight, Aziz et al. [ABEP11] discuss the *annexation non-monotonicity paradox*, which says that it sometimes can be more useful for a player to annex another player of small weight than to annex another player of large weight. Nevertheless, they show that with respect to the normalized Penrose–Banzhaf index it is always beneficial for a player to annex another player with a larger weight than her own weight. As annexation can be disadvantageous for this index, Aziz et al. study the complexity of beneficial annexation and show that it is NP-hard to decide whether a player can benefit from annexing a coalition of players. For the Shapley–Shubik index, Felsenthal and Machover show that annexation is never disadvantageous. Nonetheless, one can still ask the question of whether it is in fact *advantageous*. We show that it is NP-complete to decide whether annexing another player is advantageous for the Shapley–Shubik index, as well as for the probabilistic Penrose–Banzhaf index. We furthermore show that annexation can never be disadvantageous for the probabilistic Penrose–Banzhaf index either, which thus behaves like the Shapley–Shubik index in this regard.

Moreover, we propose a general framework for merging and splitting that can be applied to different classes and representations of games. Introducing new properties of merging and splitting functions, *consistency* and *independence*, which in particular are satisfied by the standard merging and splitting functions for weighted voting games, we can generalize the P results for the probabilistic Penrose–Banzhaf index. As an example of applying this more general framework to a concrete class of games, we consider threshold network flow games on hypergraphs, a model adapted here from the threshold network flow games introduced by Bachrach and Rosenschein [BR09]. In unanimity games and with respect to the probabilistic Penrose–Banzhaf index, we show that splitting is always disadvantageous or neutral, whereas merging is neutral for size-two coalitions, yet advantageous for coalitions with at least three players. This strongly contrasts with the results by Aziz et al. [AP09, ABEP11] showing that merging is always disadvantageous and splitting is always advantageous for the normalized Penrose–Banzhaf index in unanimity weighted voting games. These are only two examples of how different properties of a game or restrictions caused by a certain representation can lead to different behaviour when considering merging and splitting. Lasisi and Allan [LA10] study related problems in unanimity weighted voting games as well.

As has been mentioned in the introduction, a major field in computational social choice is the complexity analysis of the question whether a certain form of influence is possible in an election under a certain voting rule. Besides merging, splitting, and annexation other forms of manipulation have been studied in weighted voting games. Zuckerman et al. [ZFBE12, ZFBE08] study manipulation of the quota in weighted voting games. Relatedly, Zick et al. [Zic13, ZSE11] study algorithmic properties of the quota. In dynamic weighted voting games, as presented by Elkind et al. [EPZ13], the quota is changed as well, this time over time.

Inspired by the notion of control, where the chair of an election changes the structure of the election in order to achieve a desired goal [BTT92, HHR07], we consider control scenarios in weighted voting games. We define problems of whether a chair, or referee of a game, or supervisor in a real world application of the game can change the structure of a game in order to achieve a certain goal. Structural changes include adding or deleting a certain number of players. This could also be viewed as a static change of the players' participation over time. Goals include increasing or decreasing the power of a distinguished player, in relation to the player's power in the original game. Power, again, refers to power indices like the Penrose–Banzhaf or Shapley-Shubik index. We show, as summed up in Table 3.2, that the complexity depends on the control type and the goal alike as well as on whether the parameter of how many players can be added or deleted is fixed or given in the problem instance.

Goal \ Control Type	ADDING PLAYERS	DELETING PLAYERS
increase	• PP-complete (k fixed) (Thm. 3.27)¶ • PP-hard (k given) (Thm. 3.25)¶	• NP-hard *(Shapley–Shubik index, $k = 1$)* (Thm. 3.29)¶
non-decrease	• PP-complete (k fixed) (Thm. 3.27)¶ • PP-hard (k given) (Thm. 3.25)¶	
decrease	• PP-complete (k fixed) (Thm. 3.27)¶ • PP-hard (k given) (Thm. 3.25)¶	
non-increase	• PP-complete (k fixed) (Thm. 3.27)¶ • PP-hard (k given) (Thm. 3.25)¶	• coNP-hard *(probab. Penrose–Banzhaf index, $k = 1$)* (Thm. 3.30)¶
maintain	• coNP-hard, PP (k fixed) (Thm. 3.28)¶ • PP-hard (k given) (Thm. 3.25)¶	• coNP-hard *(probab. Penrose–Banzhaf index, $k = 1$)* (Thm. 3.30)¶

¶ this thesis ([RR16])

Table 3.2: Overview of complexity results of structural control problems in weighted voting games for the probabilistic Penrose–Banzhaf index and the Shapley–Shubik index. Key: k denotes the number of players to be added or deleted, respectively.

3.1 Beneficial Merging, Splitting, and Annexation

We use the following notation for merging and splitting operations for weighted voting games as introduced by Aziz et al. [ABEP11]. Given a weighted voting game $\mathscr{G} \leftrightsquigarrow (w_1, \ldots, w_n; q)$ and a non-empty[1] coalition $S \subseteq \{1, \ldots, n\}$, let $\mathscr{G}_{\&S} \leftrightsquigarrow (w(S), w_{j_1}, \ldots, w_{j_{n-\|S\|}}; q)$ with $\{j_1, \ldots, j_{n-\|S\|}\} = N \smallsetminus S$ denote the new weighted voting game in which the players in S have been merged into one new player of weight $w(S)$. Note that the players' order does not matter, since we are in a neutral environment (see Section 2.3.1). For a power index PI, the beneficial merging problem is defined as follows.

PI-BENEFICIALMERGE	
Given:	A weighted voting game $\mathscr{G} \leftrightsquigarrow (w_1, \ldots, w_n; q)$ and a non-empty coalition $S \subseteq \{1, \ldots, n\}$.
Question:	Is merging of S beneficial, that is, does $\mathrm{PI}(\mathscr{G}_{\&S}, 1) > \sum_{i \in S} \mathrm{PI}(\mathscr{G}, i)$ hold?

Similarly, given a weighted voting game $\mathscr{G} \leftrightsquigarrow (w_1, \ldots, w_n; q)$, a player i, and an integer $m \geq 2$, define the set of weighted voting games $\mathscr{G}_{i \div m} \leftrightsquigarrow (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n, w_{n+1}, \ldots, w_{n+m}; q)$ in which i with weight w_i is split into m new players $n+1, \ldots, n+m$ with weights $w_{n+1}, \ldots, w_{n+m}$ such that $\sum_{j=1}^{m} w_{n+j} = w_i$. Note that there is a *set* of such weighted voting games $\mathscr{G}_{i \div m}$, since there might be several possibilities of distributing i's weight w_i to the new players $n+1, \ldots, n+m$ satisfying $\sum_{j=1}^{m} w_{n+j} = w_i$.

We distinguish between two different splitting problems.[2] Firstly, for a power index PI, consider the problem where a weighted voting game, a player i, and the number m of false identities i splits into are given in the problem instance, but not the weights of the new players:

PI-BENEFICIALSPLIT	
Given:	A weighted voting game $\mathscr{G} \leftrightsquigarrow (w_1, \ldots, w_n; q)$, a player i, and an integer $m \geq 2$.
Question:	Is it possible to split i into m new players $n+1, \ldots, n+m$ with weights $w_{n+1}, \ldots, w_{n+m}$ satisfying $\sum_{j=1}^{m} w_{n+j} = w_i$ such that in this new weighted voting game, call it $\mathscr{G}_{i \div m}$, it holds that $\sum_{j=1}^{m} \mathrm{PI}(\mathscr{G}_{i \div m}, n+j) > \mathrm{PI}(\mathscr{G}, i)$?

As mentioned above, for an instance $(\mathscr{G}, i, m)$ of PI-BENEFICIALSPLIT, there might be various ways of distributing i's weight to her false identities, giving rise to various new games $\mathscr{G}_{i \div m}$. In the second (more special) variant of the problem we consider, the new players' weights are given explicitly in the problem instance and the number of false identities

[1] We omit the empty coalition, since this would slightly change the idea of the problem. We deal with structural changes in Section 3.3.

[2] This distinction would not make sense for beneficial merging or annexation.

(which is given implicitly) is polynomially bounded by the number of original players. In this case, there is only one unique new game $\mathscr{G}_{i \div m}$, and splitting is the inverse function to merging.

PI-BENEFICIALSPLIT INTO GIVEN WEIGHTS	
Given:	A weighted voting game $\mathscr{G} \leftrightsquigarrow (w_1, \dots, w_n; q)$, a player i, and integer weights $w_{n+1}, \dots, w_{n+m}$ such that $\sum_{j=1}^{m} w_{n+j} = w_i$ and m is polynomially bounded by n.
Question:	Does it hold for the split i into m new players $n+1, \dots, n+m$ with weights $w_{n+1}, \dots, w_{n+m}$ such that in the new weighted voting game $\mathscr{G}_{i \div m}$, that $\sum_{j=1}^{m} \text{PI}(\mathscr{G}_{i \div m}, n+j) > \text{PI}(\mathscr{G}, i)$?

We will explicitly mention it whenever we speak of the latter more restricted variant.

We say that merging (or splitting, respectively) is *advantageous* if it is beneficial; it is *disadvantageous* if it is not beneficial and in the related inequation $<$ holds; and it is *neutral* if there is no change in power, that is, in the related inequation equality holds.

Involuntary participation in a manipulative action has been studied by Aziz et al. [ABEP11] for coalitions instead of a single annexed player. Here, we focus on a single annexed player. Let PI be a power index.

PI-BENEFICIALANNEXATION	
Given:	A weighted voting game $\mathscr{G} \leftrightsquigarrow (w_1, \dots, w_n; q)$ and two players $i, j \in N$, $i \neq j$.
Question:	Does player i benefit from annexing player j, that is, is it true that $\text{PI}(\mathscr{G}_{\&\{i,j\}}, 1) > \text{PI}(\mathscr{G}, i)$?

Example 3.2. *Consider the weighted voting game $\mathscr{G}$ in Example 3.1. Let $S = \{2,3\}$ be a coalition that wants to merge together. The resulting weighted voting game is*

$$\mathscr{G}_{\&S} \leftrightsquigarrow (4, 1, 3, 4, 5; 10).$$

It holds that

$$\begin{aligned} \text{PenroseBanzhaf}(\mathscr{G}_{\&\{2,3\}}, 1) &= \frac{6}{16} \\ = \frac{6}{32} + \frac{6}{32} &= \text{PenroseBanzhaf}(\mathscr{G}, 2) + \text{PenroseBanzhaf}(\mathscr{G}, 3), \end{aligned}$$

that is, no increase of the combined probabilistic Penrose–Banzhaf power index. If player 3 *had annexed player* 2*, there would have been an increase. Regarding the Shapley–Shubik index, merging is beneficial for players* 2 *and* 3*, since*

$$\text{ShapleyShubik}(\mathscr{G}_{\&\{2,3\}}, 1) = \frac{14}{60} > \frac{6}{60} = \text{ShapleyShubik}(\mathscr{G}, 2) + \text{ShapleyShubik}(\mathscr{G}, 3).$$

As an example for splitting, consider $\mathscr{G}$ again and let $i = 5$ and $m = 2$. There are two (up to the order of players which is neglected due to neutrality) possible new games

$$\mathscr{G}_{5\div 2} = (1,2,2,4,5,2,2;\, 10) \quad or \quad \mathscr{G}_{5\div 2} = (1,2,2,4,5,1,3;\, 10).$$

In neither case splitting is beneficial for the probabilistic Penrose–Banzhaf index, since

$$\begin{aligned}\text{PenroseBanzhaf}(\mathscr{G}_{5\div 2},6) + \text{PenroseBanzhaf}(\mathscr{G}_{5\div 2},7) &= \frac{12}{64} + \frac{12}{64} \\ = \frac{12}{32} = \text{PenroseBanzhaf}(\mathscr{G},5)\end{aligned}$$

in the first case and $5/64 + 19/64 = 12/32$ *in the second case. For the Shapley-Shubik index there is even a decrease of power in both cases:*

$$\begin{aligned}\text{ShapleyShubik}(\mathscr{G}_{5\div 2},6) + \text{ShapleyShubik}(\mathscr{G}_{5\div 2},7) &= \frac{41}{420} + \frac{41}{420} \\ < \frac{91}{420} = \text{ShapleyShubik}(\mathscr{G},5)\end{aligned}$$

and, respectively, $17/420 + 73/420 < 91/420$.

In this thesis we focus on the complexity classification of these merging, splitting and annexation problems for both the Shapley–Shubik and the probabilistic Penrose–Banzhaf index.

Before doing so, since we allow players with zero weight, we state another simple fact required for the analysis of the beneficial splitting problem (see the proofs of Theorems 3.10 and 3.15).

Lemma 3.3. *For both the probabilistic Penrose–Banzhaf index and the Shapley–Shubik index, given a weighted voting game, adding a player with weight zero does not change the original players' power indices, and the new player's power index is zero.*

Proof. Let $\mathscr{G}_1 \leftrightsquigarrow (w_1,\dots,w_n;\, q)$ be a weighted voting game. The new player $n+1$ in the game $\mathscr{G}_2 \leftrightsquigarrow (w_1,\dots,w_n,0;\, q)$ does not change the total weight of any coalition by joining it, that is, $v(C \cup \{n+1\}) = v(C)$ for each $C \subseteq N$. Therefore, $\text{PenroseBanzhaf}(\mathscr{G}_2,i) = 1/2^n \sum_{C\subseteq N\smallsetminus\{i\}} 2(v(C\cup\{i\}) - v(C)) = \text{PenroseBanzhaf}(\mathscr{G}_1,i)$, for each i, $1 \le i \le n$. For the same reason, player $n+1$ is not pivotal for any coalition, thus, it holds that $\text{PenroseBanzhaf}(\mathscr{G}_2,n+1) = 0$. ∎

In this section we prove that beneficial merging and splitting is PP-hard, and we provide matching upper bounds for beneficial merging and splitting in the variant where the new players' weights are given, both for the Shapley–Shubik and the probabilistic Penrose–Banzhaf index. We start with the latter.

3.1.1 Complexity Results for the Probabilistic Penrose–Banzhaf Power Index

Both the beneficial merging problem for a coalition S of size 2 and the beneficial splitting problem for $m = 2$ false identities can trivially be decided in polynomial time for the probabilistic Penrose–Banzhaf index, since the sum of power (in terms of this index) of two players is always equal to the power of the player that is obtained by merging them.

Proposition 3.4 ([RR10a, RR10b, RR14a]). *Let $\mathscr{G}$ be a weighted voting game and $S \subseteq \{1,\dots,n\}$ be a coalition of its players.*

1. PenroseBanzhaf-BENEFICIALMERGE *is in* P *for instances* $(\mathscr{G}, S)$ *with* $\|S\| = 2$.
2. PenroseBanzhaf-BENEFICIALSPLIT *is in* P *for instances* $(\mathscr{G}, i, 2)$.

Proof. Let $\mathscr{G} \leftrightsquigarrow (w_1,\dots,w_n;\, q)$ be a weighted voting game. Without loss of generality, let $S = \{1,n\}$. We obtain a new game $\mathscr{G}_{\&S} \leftrightsquigarrow (w_1 + w_n, w_2, \dots, w_{n-1};\, q)$, where the first player is the new player merging S. Letting $v_{\mathscr{G}}$ and $v_{\mathscr{G}_{\&S}}$ denote the corresponding coalitional functions, it holds that

$$\text{PenroseBanzhaf}(\mathscr{G}_{\&S}, 1) - (\text{PenroseBanzhaf}(\mathscr{G}, 1) + \text{PenroseBanzhaf}(\mathscr{G}, n)) \;=\; 0.$$

In the case of splitting, it similarly holds that

$$\text{PenroseBanzhaf}(\mathscr{G}_{n\div 2}, n+1) + \text{PenroseBanzhaf}(\mathscr{G}_{n\div 2}, n+2) - \text{PenroseBanzhaf}(\mathscr{G}, n) \;=\; 0$$

for a weighted voting game $\mathscr{G} = (N, v)$, $m = 2$, and, without loss of generality, player n in $\mathscr{G}$ splitting into players $n+1$ and $n+2$ in a new game $\mathscr{G}_{n\div 2}$. ☐

Although it may seem as if Proposition 3.4 implied that merging (and splitting) were never beneficial regarding this index, this cannot be generalized to merging (or splitting into) more than two players, by repeatedly applying the above result to pairs of players step by step. For example, as soon as two players merge, a third player's probabilistic Penrose–Banzhaf index might have already changed in the new game, before merging her with another player in a subsequent step. Suppose three players in $\{1,2,3\}$ want to merge in a game $\mathscr{G}$. Let $\beta_i = \text{PenroseBanzhaf}(\mathscr{G}, i)$, $1 \leq i \leq 3$, be their original probabilistic Penrose–Banzhaf indices. Let β be their common Penrose–Banzhaf index after the merge. After merging the first two players, let β_1' and β_3' be the indices of the new player replacing $\{1,2\}$ and of 3, respectively. Then, due to Proposition 3.4, $\beta = \beta_1' + \beta_3' = \beta_1 + \beta_2 + \beta_3'$. Hence, $\beta > \beta_1 + \beta_2 + \beta_3$ if and only if $\beta_3' > \beta_3$. That is, for the probabilistic Penrose–Banzhaf index, beneficial merging of three players boils down to comparing the index of one player in two games—the original game and the one where the other two players have merged. If these were two arbitrary games, the result for the comparison of power indices by Faliszewski and Hemaspaandra [FH09] would have applied. Here, however, the indices need to be compared in two closely related games; this requires a different proof. Indeed, next we show that it is by far harder than for two players (unless the polynomial hierarchy collapses to its first level) to decide whether merging three players is beneficial in terms of the probabilistic Penrose–Banzhaf index.

Our goal is to provide a $\leq_m^p$-reduction from the PP-complete problem COMPARE-#SUBSETSUM (see Corollary 2.6) to PenroseBanzhaf-BENEFICIALMERGE. In order to make this reduction work, it will be useful to consider two restricted variants of COMPARE-#SUBSETSUM, which we denote by COMPARE-#SUBSETSUM-R and COMPARE-#SUBSETSUM-RR, show their PP-hardness, and then reduce COMPARE-#SUBSETSUM-RR to PenroseBanzhaf-BENEFICIALMERGE. This will be performed in Lemmas 3.5 and 3.6 and in Theorem 3.7. In all restricted variants of COMPARE-#SUBSETSUM we may assume, without loss of generality, that the target value q in a related #SUBSETSUM instance $(a_1,\dots,a_n;q)$ satisfies $1 \leq q \leq \alpha - 1$, where $\alpha = \sum_{i=1}^{n} a_i$.

COMPARE-#SUBSETSUM-R	
Given:	A set $A = \{1,\dots,n\}$, a value function $a : A \to \mathbb{N} \setminus \{0\}$, $i \mapsto a_i$, and two positive integers q_1 and q_2 with $1 \leq q_1, q_2 \leq \alpha - 1$, where $\alpha = \sum_{i=1}^{n} a_i$.
Question:	Is the number of subsets of A with values summing up to q_1 greater than the number of subsets of A with values summing up to q_2, that is, does it hold that #SUBSET SUM$((a_1,\dots,a_n;q_1))$ > #SUBSET SUM$((a_1,\dots,a_n;q_2))$?

Similar to SUBSET SUM, let $(a_1,\dots,a_n;q_1,q_2)$ denote an instance of COMPARE-#SUBSETSUM-R.

Lemma 3.5. COMPARE-#SUBSETSUM $\leq_m^p$ COMPARE-#SUBSETSUM-R.

Proof. Given an instance (X,Y) of COMPARE-#SUBSETSUM, $X = (x_1,\dots,x_m;q_x)$ and $Y = (y_1,\dots,y_n;q_y)$, construct the COMPARE-#SUBSETSUM-R instance $(x_1,\dots,x_m, 2\alpha y_1,\dots,2\alpha y_n; q_x, 2\alpha q_y)$, where $\alpha = \sum_{i=1}^{m} x_i$. This construction can obviously be achieved in polynomial time.

It holds that the constructed values can only sum up to $q_x \leq \alpha - 1$ if they do not contain multiples of 2α, thus #SUBSET SUM$((x_1,\dots,x_m,2\alpha y_1,\dots,2\alpha y_n;q_1))$ = #SUBSET SUM(X). On the other hand, q_2 cannot be obtained by adding any of the x_i, since this would yield a non-zero remainder modulo 2α, because $\sum_{i=1}^{m} x_i = \alpha$ is too small. Thus, it holds that #SUBSET SUM$((x_1,\dots,x_m,2\alpha y_1,\dots,2\alpha y_n;q_2))$ = #SUBSET SUM(Y). It follows that (X,Y) belongs to COMPARE-#SUBSETSUM if and only if the constructed instance is in COMPARE-#SUBSETSUM-R. ∎

In order to perform the next step, we need to ensure that all integers in a COMPARE-#SUBSETSUM-R instance are divisible by 8. This can easily be achieved, by multiplying each integer in an instance $(a_1,\dots,a_n;q_1,q_2)$ by 8, obtaining $(8a_1,\dots,8a_n;8q_1,8q_2)$ without changing the number of solutions for both related SUBSETSUM instances. Thus, from now on, without loss of generality, we assume that for a given COMPARE-#SUBSETSUM-R instance $(a_1,\ldots,a_n;q_1,q_2)$, it holds that $a_i, q_j \equiv 0 \bmod 8$ for $1 \leq i \leq n$ and $j \in \{1,2\}$.

Now, we consider our even more restricted variant of this problem.

COMPARE-#SUBSETSUM-RR	
Given:	A set $A = \{1,\ldots,n\}$ and a value function $a : A \to \mathbb{N} \smallsetminus \{0\}$, $i \mapsto a_i$.
Question:	Is the number of subsets of A with values summing up to $(\alpha/2) - 2$, where $\alpha = \sum_{i=1}^{n} a_i$, greater than the number of subsets of A with values summing up to $(\alpha/2) - 1$, i.e., is it true that #SUBSET SUM$((a_1,\ldots,a_n;(\alpha/2)-2)) >$ #SUBSET SUM$((a_1,\ldots,a_n;(\alpha/2)-1))$?

Again, let $(a_1,\ldots,a_n)$ denote an instance of COMPARE-#SUBSETSUM-RR.

Lemma 3.6. COMPARE-#SUBSETSUM-R $\leq_{\mathrm{m}}^{\mathrm{p}}$ COMPARE-#SUBSETSUM-RR.

Proof. Given an instance $X = (a_1,\ldots,a_n;q_1,q_2)$ of COMPARE-#SUBSETSUM-R, where we assume that $a_i, q_j \equiv 0 \bmod 8$ for $1 \leq i \leq n$ and $j \in \{1,2\}$, we construct an instance B of COMPARE-#SUBSETSUM-RR as follows. This reduction is inspired by the standard reduction from SUBSET SUM to PARTITION due to Karp [Kar72]. Letting $\alpha = \sum_{i=1}^{n} a_i$, define

$$Y = (a_1,\ldots,a_n, 2\alpha - q_1, 2\alpha + 1 - q_2, 2\alpha + 3 + q_1 + q_2, 3\alpha).$$

This instance can obviously be constructed in polynomial time. Observe that

$$T = \left(\sum_{i=1}^{n} a_i\right) + (2\alpha - q_1) + (2\alpha + 1 - q_2) + (2\alpha + 3 + q_1 + q_2) + 3\alpha = 10\alpha + 4,$$

and therefore, $(T/2) - 2 = 5\alpha$ and $(T/2) - 1 = 5\alpha + 1$. We show that X belongs to COMPARE-#SUBSETSUM-R if and only if Y is in COMPARE-#SUBSETSUM-RR.

Firstly, we examine which values of Y sum up to 5α. Consider two cases.

Case 1: If 3α is added, $2\alpha + 3 + q_1 + q_2$ cannot be added, as it would be too large. Also, $2\alpha + 1 - q_2$ cannot be added, leading to an odd sum. So, $2\alpha - q_1$ has to be added, as the remaining α are too small. Since $3\alpha + 2\alpha - q_1 = 5\alpha - q_1$, 5α can be achieved by adding some integers a_i if and only if there exists a subset $A' \subseteq \{1,\ldots,n\}$ such that $\sum_{i \in A'} a_i = q_1$ (i.e., A' is a solution of the SUBSETSUM instance $(a_1,\ldots,a_n;q_1)$).

Case 2: If 3α is not added, but (a) $2\alpha + 3 + q_1 + q_2$, an even number can only be achieved by adding $2\alpha + 1 - q_2$, thus, $\alpha - 4 - q_1$ remains. Hence, $2\alpha - q_1$ is too large, while no subset of $\{1,\ldots,n\}$ has values summing up to $\alpha - 4 - q_1$, because of the assumption of divisibility by 8. If (b) neither 3α nor $2\alpha + 3 + q_1 + q_2$ are added, the remaining $5\alpha + 1 - q_1 - q_2$ are too small.

Thus, the only possibility to obtain 5α is to find a subsequence of $\{1,\ldots,n\}$ with values adding up to q_1. Therefore, #SUBSET SUM$((a_1,\ldots,a_n;q_1))$ = #SUBSET SUM$((a_1,\ldots,a_n,2\alpha-q_1,2\alpha+1-q_2,2\alpha+3+q_1+q_2,3\alpha;5\alpha))$.

Secondly, for similar reasons, a sum of $5\alpha+1$ can only be achieved by adding $3\alpha+(2\alpha+1-q_2)$ and a term $\sum_{i\in A'} a_i$, where A' is a subset of $\{1,\ldots,n\}$ such that $\sum_{i\in A'} a_i = q_2$. Hence, #SUBSET SUM$((a_1,\ldots,a_n;q_2))$ = #SUBSET SUM$((a_1,\ldots,a_n,2\alpha-q_1,2\alpha+1-q_2,2\alpha+3+q_1+q_2,3\alpha;5\alpha+1))$.

Thus, the relation #SUBSET SUM$((a_1,\ldots,a_n;q_1))$ > #SUBSET SUM$((a_1,\ldots,a_n;q_2))$ holds if and only if #SUBSET SUM$((a_1,\ldots,a_n,2\alpha-q_1,2\alpha+1-q_2,2\alpha+3+q_1+q_2,3\alpha;5\alpha))$ > #SUBSET SUM$((a_1,\ldots,a_n,2\alpha-q_1,2\alpha+1-q_2,2\alpha+3+q_1+q_2,3\alpha;5\alpha+1))$, which completes the proof. ∎

We now are ready to prove the main theorem of this section.

Theorem 3.7. PenroseBanzhaf-BENEFICIALMERGE *is* PP*-complete, even if only three players of equal weight merge.*

Proof. Membership of PenroseBanzhaf-BENEFICIALMERGE in PP has already been observed in [RR10a, Theorem 3]. It follows from the fact that the raw Penrose–Banzhaf index is in #P and that #P is closed under addition and multiplication by two, and, furthermore, since comparing the values of two #P functions on two (possibly different) inputs reduces to a PP-complete problem. This technique (which was proposed by Faliszewski and Hemaspaandra [FH09] and applies their Lemma 2.10) works, since PP is closed under $\leq_m^p$-reducibility.

We show PP-hardness of PenroseBanzhaf-BENEFICIALMERGE by means of a $\leq_m^p$-reduction from COMPARE-#SUBSETSUM-RR, which is PP-hard by Corollary 2.6 via Lemmas 3.5 and 3.6.

Given an instance $(a_1,\ldots,a_n)$ of COMPARE-#SUBSETSUM-RR, construct the following instance for PenroseBanzhaf-BENEFICIALMERGE. Let $\alpha = \sum_{i=1}^n a_i$. Define the weighted voting game

$$\mathscr{G} \leftrightsquigarrow (2a_1,\ldots,2a_n,1,1,1,1;\alpha)$$

with $n+4$ players, and let the merging coalition be $S=\{n+2,n+3,n+4\}$. Letting $A = \{1,\ldots,n\}$, it holds that PenroseBanzhaf$(\mathscr{G},n+2)$ =

$$\frac{1}{2^{n+3}}\left\|\left\{C\subseteq\{1,\ldots,n+1,n+3,n+4\}\,\middle|\,\sum_{i\in C} w_i = \alpha-1\right\}\right\|$$

$$= \frac{1}{2^{n+3}}\left(\left\|\left\{A'\subseteq A\,\middle|\,\sum_{i\in A'}2a_i=\alpha-1\right\}\right\| + 3\cdot\left\|\left\{A'\subseteq A\,\middle|\,1+\sum_{i\in A'}2a_i=\alpha-1\right\}\right\|\right. \tag{3.1}$$

$$\left.+3\cdot\left\|\left\{A'\subseteq A\,\middle|\,2+\sum_{i\in A'}2a_i=\alpha-1\right\}\right\| + \left\|\left\{A'\subseteq A\,\middle|\,3+\sum_{i\in A'}2a_i=\alpha-1\right\}\right\|\right) \tag{3.2}$$

$$= \frac{1}{2^{n+3}} \left(3 \cdot \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} 2a_i = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} 2a_i = \alpha - 4 \right\} \right\| \right),$$

since the $2a_i$ can only add up to an even number. The first of the four sets in (3.1) and (3.2) refers to those coalitions that do not contain any of the players $n+1$, $n+3$, and $n+4$; the second, third, and fourth set in (3.1) and (3.2) refer to those coalitions containing either one, two, or three of them, respectively. Since the players in S have the same weight, players $n+3$ and $n+4$ have the same probabilistic Penrose–Banzhaf index as player $n+2$.

Furthermore, the new game after merging is $\mathscr{G}_{\&\{n+2,n+3,n+4\}} \leftrightsquigarrow (3, 2a_1, \dots 2a_n, 1; \alpha)$ with $n+2$ players, and similarly as above the Penrose–Banzhaf index of the first player is calculated as follows:

$$\begin{aligned}
&\text{PenroseBanzhaf}\left(\mathscr{G}_{\&\{n+2,n+3,n+4\}}, 1\right) \\
&\quad = \frac{1}{2^{n+1}} \left\| \left\{ C \subseteq \{2, \dots, n+2\} \,\middle|\, \sum_{i \in C} w_i \in \{\alpha-3, \alpha-2, \alpha-1\} \right\} \right\| \\
&\quad = \frac{1}{2^{n+1}} \left(\left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} 2a_i \in \{\alpha-3, \alpha-2, \alpha-1\} \right\} \right\| \right. \\
&\qquad \left. + \left\| \left\{ A' \subseteq A \,\middle|\, 1 + \sum_{i \in A'} 2a_i \in \{\alpha-3, \alpha-2, \alpha-1\} \right\} \right\| \right) \\
&\quad = \frac{1}{2^{n+1}} \left(2 \cdot \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} 2a_i = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} 2a_i = \alpha - 4 \right\} \right\| \right).
\end{aligned}$$

Altogether, it holds that

$$\begin{aligned}
&\text{PenroseBanzhaf}\left(\mathscr{G}_{\&\{n+2,n+3,n+4\}}, 1\right) - \sum_{i \in \{n+2,n+3,n+4\}} \text{PenroseBanzhaf}(\mathscr{G}, i) \\
&= \frac{1}{2^{n+1}} \left(2 \cdot \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} 2a_i = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} 2a_i = \alpha - 4 \right\} \right\| \right) \\
&\quad - \frac{3}{2^{n+3}} \left(3 \cdot \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} 2a_i = \alpha - 2 \right\} \right\| + \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} 2a_i = \alpha - 4 \right\} \right\| \right) \\
&= \left(\frac{1}{2^{n+1}} \cdot 2 - \frac{3}{2^{n+3}} \cdot 3 \right) \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} 2a_i = \alpha - 2 \right\} \right\| \\
&\quad + \left(\frac{1}{2^{n+1}} - \frac{3}{2^{n+3}} \right) \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} 2a_i = \alpha - 4 \right\} \right\| \\
&= -\frac{1}{2^{n+3}} \cdot \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} a_i = \frac{\alpha}{2} - 1 \right\} \right\| + \frac{1}{2^{n+3}} \cdot \left\| \left\{ A' \subseteq A \,\middle|\, \sum_{i \in A'} a_i = \frac{\alpha}{2} - 2 \right\} \right\|,
\end{aligned}$$

which is greater than zero if and only if

$$\left\| \left\{ A' \subseteq A \;\middle|\; \sum_{i \in A'} a_i = \frac{\alpha}{2} - 2 \right\} \right\| > \left\| \left\{ A' \subseteq A \;\middle|\; \sum_{i \in A'} a_i = \frac{\alpha}{2} - 1 \right\} \right\|,$$

which, in turn, is the case if and only if the original instance $(a_1, \ldots, a_n)$ is in COMPARE-#SUBSETSUM-RR. ∎

Remark 3.8. *Note that the proof cannot be transferred straightforwardly to the normalized Penrose–Banzhaf index, since in different games the indices have possibly different denominators, not only different by a factor of some power of two, as is the case for the probabilistic Penrose–Banzhaf index.*

Analogously to the proof of Theorem 3.7, it can be shown that PenroseBanzhaf-BENEFICIALSPLIT INTO GIVEN WEIGHTS for at least three false identities is PP-complete. Note that there is no direct reduction from beneficial merging by an identity function, since we ask for $<$ instead of $\leq$. Nevertheless, the instances are comparable as weights of players in S correspond to the new weights. The same arguments hold for the upper bound; recall that we can characterize PP with $>$ and $\geq$ alike.

Corollary 3.9. PenroseBanzhaf-BENEFICIALSPLIT INTO GIVEN WEIGHTS *is* PP-*complete, even if the given player splits into three players, and the given weights are equal.*

However, for the more general beneficial splitting problem where the new players' weights are not given, a PP upper bound cannot be shown straightforwardly. Yet, it can be shown that this problem is PP-hard, even for three false identities.

Theorem 3.10. PenroseBanzhaf-BENEFICIALSPLIT *is* PP-*hard, even if the given player can only split into three players of equal weight.*

Proof. In order to show PP-hardness for PenroseBanzhaf-BENEFICIALSPLIT, we use the same techniques as in Theorem 3.7, appropriately modified. In fact, we will now show PP-hardness for $m = 3$ false identities.[3]

Firstly, we slightly change the definition of COMPARE-#SUBSETSUM-RR by switching $(\alpha/2) - 2$ and $(\alpha/2) - 1$. The problem, called COMPARE-#SUBSETSUM-ЯЯ, of whether the number of subsequences of a given sequence A of positive integers summing up to $(\alpha/2) - 1$ is greater than the number of subsequences of A summing up to $(\alpha/2) - 2$, is PP-hard by the same proof as in Lemma 3.6 with the roles of q_1 and q_2 exchanged.

[3] This result can be expanded to all fixed $m \geq 3$ by splitting into additional players with weight 0. More precisely, if $m > 3$, we consider the same game $\mathcal{G}$ as below and split into three players of weight 1 each and $m - 3$ players of weight 0 each. By Lemma 3.3, the sum of all m new players' Penrose–Banzhaf power indices is equal to the combined Penrose–Banzhaf power index of the three players. Thus, PP-hardness will hold for splitting into fixed $m > 3$ players by essentially the same arguments as given below for splitting into three players.

Now, we reduce this problem to PenroseBanzhaf-BENEFICIALSPLIT by constructing the following instance of the beneficial splitting problem from an instance $(a_1,\dots,a_n)$ of COMPARE-#SUBSETSUM-ЯЯ. Let $\mathscr{G} \leftrightsquigarrow (2a_1,\dots,2a_n,1,3;\alpha)$, where $\alpha = \sum_{j=1}^{n} a_j$, and let $i = n+2$ be the player to be split. $\mathscr{G}$ is (apart from the order of players) equivalent to the game obtained by merging in the proof of Theorem 3.7. Thus, letting $A = \{1,\dots,n\}$, PenroseBanzhaf$(\mathscr{G},n+2)$ equals

$$\frac{1}{2^{n+1}}\left(2\cdot\left\|\left\{A'\subseteq A \,\middle|\, \sum_{j\in A'} 2a_j = \alpha-2\right\}\right\| + \left\|\left\{A'\subseteq A \,\middle|\, \sum_{j\in A'} 2a_j = \alpha-4\right\}\right\|\right).$$

Allowing players with weight zero, there are different possibilities to split player $n+2$ into three players. By Lemma 3.3, splitting $n+2$ into one player with weight 3 and two others with weight 0 is not beneficial. Likewise, splitting $n+2$ into two players with weights 1 and 2 and one player with weight 0 is not beneficial, by Lemma 3.3 and since splitting into two players is not beneficial (by Theorem 3.20.1). Thus, the only possibility left is splitting $n+2$ into three players of weight 1 each. This corresponds to the original game in the proof of Theorem 3.7, $\mathscr{G}_{i\div 3} \leftrightsquigarrow (2a_1,\dots,2a_n,1,1,1,1;\alpha)$. Therefore,

$$\begin{aligned}
&\text{PenroseBanzhaf}(\mathscr{G}_{i\div 3},n+2)\\
&= \text{PenroseBanzhaf}(\mathscr{G}_{i\div 3},n+3) = \text{PenroseBanzhaf}(\mathscr{G}_{i\div 3},n+4)\\
&= \frac{1}{2^{n+3}}\left(3\cdot\left\|\left\{A'\subseteq A \,\middle|\, \sum_{j\in A'} 2a_j = \alpha-2\right\}\right\| + \left\|\left\{A'\subseteq A \,\middle|\, \sum_{j\in A'} 2a_j = \alpha-4\right\}\right\|\right).
\end{aligned}$$

Altogether, as in the proof of Theorem 3.7, the sum of the three new players' probabilistic Penrose–Banzhaf indices minus the probabilistic Penrose–Banzhaf index of the original player is greater than zero if and only if

$$\left\|\left\{A'\subseteq A \,\middle|\, \sum_{j\in A'} a_j = \frac{\alpha}{2}-1\right\}\right\| > \left\|\left\{A'\subseteq A \,\middle|\, \sum_{j\in A'} a_j = \frac{\alpha}{2}-2\right\}\right\|,$$

which is true if and only if $(a_1,\dots,a_n)$ is in COMPARE-#SUBSETSUM-ЯЯ. □

Remark 3.11. *As an upper bound for the general beneficial splitting problem, we can only show membership in* $\mathrm{NP}^{\mathrm{PP}}$*, whenever the number of false identities is given in unary, and we conjecture that this problem is even complete for this class. When the number m of false identities but not their weights are given in unary, there are exponentially many possibilities to distribute the split player's weight among her false identities. Non-deterministically guessing such a distribution and then, for each distribution guessed, asking an appropriate* PP *oracle to check in polynomial time whether their combined Penrose–Banzhaf power in the new game is greater than the original player's Penrose–Banzhaf power in the original game, shows that* PenroseBanzhaf-BENEFICIALSPLIT *is in* $\mathrm{NP}^{\mathrm{PP}}$.

Whenever the number of false identities is given in the binary input format, even this upper bound might no longer be valid.

For a given weighted voting game $\mathscr{G}$ and two players i and j in $\mathscr{G}$, the proof of Proposition 3.4 implies that

$$\text{PenroseBanzhaf}(\mathscr{G}_{\&\{i,j\}},1) - \text{PenroseBanzhaf}(\mathscr{G},i) \;=\; \text{PenroseBanzhaf}(\mathscr{G},j) \geq 0. \quad (3.3)$$

Therefore, it is never disadvantageous for player i to annex player j. Furthermore, we have the following result on the complexity of beneficial annexation for the probabilistic Penrose–Banzhaf index.

Theorem 3.12. PenroseBanzhaf-BENEFICIALANNEXATION *is* NP-*complete.*

Proof. By Equation (3.3) above, the question of whether the new player's probabilistic Penrose–Banzhaf index is greater than the original player's probabilistic Penrose–Banzhaf index is equivalent to the question of whether the annexed player has a positive value in the original game. This property can be decided in nondeterministic polynomial time and is NP-hard due to a result by Prasad and Kelly [PK90] (see Section 2.3.1). ◻

Remark 3.13. *For the probabilistic Penrose–Banzhaf index, it holds that while the beneficial annexation problem for a coalition of annexed players inherits* NP-*hardness from the special case in Theorem 3.12, the problem's* NP *upper bound does not generalize straightforwardly.*

3.1.2 Complexity Results for the Shapley–Shubik Power Index

In order to prove PP-hardness for the merging and splitting problems with respect to the Shapley–Shubik index, we need to take a further step back.

Theorem 3.14. ShapleyShubik-BENEFICIALMERGE *is* PP-*complete, even if only two players of equal weight merge.*

Proof. The PP upper bound, which has already been observed for two players by Faliszewski and Hemaspaandra [FH09], can be shown analogously to the upper bound in Theorem 3.7.

For proving the lower bound, observe that the size of a coalition a player is pivotal for is crucial for determining the player's Shapley–Shubik index. Pursuing the techniques by Faliszewski and Hemaspaandra, we examine the problem COMPARE-#XC_3, which is PP-complete by Corollary 2.5. We will apply the following useful properties of XC_3 instances shown by Faliszewski and Hemaspaandra [FH09, Lemma 2.7]: Every XC_3 instance $(B',\mathscr{S}')$ can be transformed into an XC_3 instance $(B,\mathscr{S})$, where $\|B\| = 3k$ and $\|\mathscr{S}\| = n$, that satisfies $k/n = 2/3$ without changing the number of solutions, i.e., $\#XC_3(B,\mathscr{S}) = \#XC_3(B',\mathscr{S}')$. Now, observe, that the parsimonious standard reduction from XC_3 to SUBSETSUM (see, e.g., [Pap95]) does not only preserve the number of solutions, but also the input size n and the size of each solution k. Hence, we can assume that in a given COMPARE-#SUBSET SUM instance each subsequence summing up to the given integer q is of size $2n/3$. Following the track of the reductions from COMPARE-#SUBSET SUM via COMPARE-#SUBSETSUM-R to COMPARE-#SUBSETSUM-RR in Lemmas 3.5 and 3.6, a solution $A' \subseteq \{1,\dots,n\}$ to a given instance $(a_1,\dots,a_n)$ of the latter problem (A' satisfying either $\sum_{i\in A'} a_i = (\alpha/2) - 2$ or $\sum_{i\in A'} a_i = (\alpha/2) - 1$, where $\alpha = \sum_{i=1}^{n} a_i$) can be assumed to satisfy $\|A'\| = k = (n+2)/3$. Under this assumption, we show PP-hardness of ShapleyShubik-BENEFICIALMERGE via a reduction from COMPARE-#SUBSET SUM-RR. Given such an instance, we construct the weighted voting game $\mathcal{G} \leftrightsquigarrow (a_1,\dots,a_n,1,1;\alpha/2)$ and consider coalition $S = \{n+1,n+2\}$. Let $N = \{1,\dots,n\}$ and define $\xi = \#\text{SUBSET SUM}((a_1,\dots,a_n;(\alpha/2)-1))$ and $\upsilon = \#\text{SUBSET SUM}((a_1,\dots,a_n;(\alpha/2)-2))$. Then,

$$\begin{aligned}
\text{ShapleyShubik}(\mathcal{G},n+1) &= \text{ShapleyShubik}(\mathcal{G},n+2)\\
&= \frac{1}{(n+2)!}\left(\left(\sum_{\substack{C\subseteq N,\\ \sum_{i\in C} a_i=\frac{\alpha}{2}-1}} \|C\|!(n+1-\|C\|)!\right)+\left(\sum_{\substack{C\subseteq N,\\ \sum_{i\in C} a_i=\frac{\alpha}{2}-2}} (\|C\|+1)!(n-\|C\|)!\right)\right)\\
&= \frac{1}{(n+2)!}\left(\xi\cdot k!(n+1-k)!+\upsilon\cdot(k+1)!(n-k)!\right).
\end{aligned}$$

Merging the players in S, we obtain $\mathcal{G}_{\&S} \leftrightsquigarrow (2,a_1,\dots,a_n;\alpha/2)$. The Shapley–Shubik index of the new player in $\mathcal{G}_{\&S}$ is

$$\begin{aligned}
\text{ShapleyShubik}(\mathcal{G}_{\&S},1) &= \frac{1}{(n+1)!}\sum_{\substack{C\subseteq N,\\ \sum_{i\in C} a_i\in\left\{\frac{\alpha}{2}-1,\frac{\alpha}{2}-2\right\}}} \|C\|!(n-\|C\|)!\\
&= \frac{1}{(n+1)!}(\xi+\upsilon)\cdot k!(n-k)!.
\end{aligned}$$

All in all,

$$\begin{aligned}
&\text{ShapleyShubik}(\mathcal{G}_{\&S},1)-(\text{ShapleyShubik}(\mathcal{G},n+1)+\text{ShapleyShubik}(\mathcal{G},n+2))\\
&= \frac{(\xi+\upsilon)\cdot k!(n-k)!}{(n+1)!}-\frac{2\left(\xi\cdot k!(n+1-k)!+\upsilon\cdot(k+1)!(n-k)!\right)}{(n+2)!}\\
&= \frac{k!(n-k)!}{(n+2)!}(n-2k)(-\xi+\upsilon). \qquad (3.4)
\end{aligned}$$

Since we assumed that $k = (n+2)/3$ and since we can also assume that $n > 4$ (because we added four integers in the construction in the proof of Lemma 3.6), it holds that

$$n-2k = \frac{n-4}{3} > 0.$$

Thus the term (3.4) is greater than zero if and only if υ is greater than ξ, which is true if and only if $(a_1, \ldots, a_n)$ is in COMPARE-#SUBSETSUM-RR. ◻

Analogously to the probabilistic Penrose–Banzhaf index, we can also show for the Shapley–Shubik index that it is PP-complete to decide if splitting a player into players with given weights is beneficial. For the more general case where the number of false identities but no actual weights are given, we can as well raise the previously known lower bound to PP-hardness. Again, the upper bound of PP cannot be transferred straightforwardly.

Theorem 3.15. ShapleyShubik-BENEFICIALSPLIT *is* PP*-hard, even if the given player can only split into two players of equal weight.*

Proof. PP-hardness can be shown analogously to the proof of Theorem 3.10, appropriately modified to use the arguments from the proof of Theorem 3.14 instead of those from the proof of Theorem 3.7. ◻

An upper bound of $\mathrm{NP}^{\mathrm{PP}}$ holds due to analogous arguments as in the proof of Remark 3.11, whenever m is given in unary.

Felsenthal and Machover [FM95] have shown that annexation is never disadvantageous for the Shapley–Shubik index. Still, the question of whether it is advantageous is hard to decide.

Theorem 3.16. ShapleyShubik-BENEFICIALANNEXATION *is* NP*-complete.*

Proof. Let $\mathscr{G} \leftrightsquigarrow (w_1, \ldots, w_n; q)$ be a weighted voting game and, without loss of generality, let player 1 annex player n. It holds that

$$\begin{aligned} &\text{ShapleyShubik}(\mathscr{G}_{\&\{1,n\}}, 1) - \text{ShapleyShubik}(\mathscr{G}, 1) \\ &\quad = \frac{1}{n!} \sum_{C \subseteq \{2,\ldots,n-1\}} \big((v(C \cup \{1,n\} - v(C \cup \{1\})) \cdot \|C\|!(n-1-\|C\|)! \\ &\qquad\qquad + (v(C \cup \{n\} - v(C)) \cdot (\|C\|+1)!(n-2-\|C\|)!\big). \end{aligned}$$

Unlike for the probabilistic Penrose–Banzhaf index, this term is in general not equal to ShapleyShubik$(\mathscr{G}, n)$, but is greater than zero if and only if player n is pivotal for at least one coalition $C \subseteq \{1, \ldots, n-1\}$ in the original game. So, analogously to Theorem 3.12, this property can be decided in nondeterministic polynomial time and is NP-hard by a result due to Prasad and Kelly [PK90] (see also [DP94]; Section 2.3.1). ◻

Remark 3.17. *Analogously to annexation with respect to the probabilistic Penrose–Banzhaf index, it holds for the Shapley–Shubik index, that while the beneficial annexation problem for an annexed coalition immediately inherits* NP*-hardness from the special case in Theorem 3.16, that problem's* NP *upper bound does not generalize straightforwardly.*

3.2 Generalizing Merging and Splitting Functions

We extend the definition of merging and splitting functions from weighted voting games to general classes $\mathfrak{G}$ of cooperative games. A *class* is any set of cooperative games; one may think of $\mathfrak{G}$ as being the class of simple games or the family of games that can be represented as weighted voting games or any representation of simple games such as the vector weighted voting games [CEW11], or the threshold network flow games due to Bachrach and Rosenschein [BR09], or even the class of all cooperative games.

A *merging function on* $\mathfrak{G}$,

$$\mu_{\mathfrak{G}} : \{\mathscr{G} = (N,v) \mid \mathscr{G} \in \mathfrak{G}\} \times (\mathfrak{P}(N) \setminus \emptyset) \to \mathfrak{G},$$

turns a given cooperative game $\mathscr{G} = (N,v)$ in suitable representation and a given non-empty coalition $S \subseteq N$ into a new game $\mu_{\mathfrak{G}}(\mathscr{G},S) = (N',v')$. The set $N' = \{i_{\&S}\} \cup (N \setminus S)$ contains a new player $i_{\&S}$ merging S. The function $v' : \mathfrak{P}(N') \to \mathbb{R}$ is the new coalitional function whose values are to be specified according to the type of games in class $\mathfrak{G}$; that is, every class $\mathfrak{G}$ is closed under $\mu_{\mathfrak{G}}$. For example, for weighted voting games a possible v' has been specified in Section 3.1.

Similarly, a *splitting function on* $\mathfrak{G}$,

$$\sigma_{\mathfrak{G}} : \{\mathscr{G} = (N,v) \mid \mathscr{G} \in \mathfrak{G}\} \times N \times (\mathbb{N} \setminus \{0,1\}) \to \mathfrak{P}(\mathfrak{G}),$$

turns a given cooperative game $\mathscr{G} = (N,v)$, a given player $i \in N$, and a given integer $m \geq 2$ into a set of new games of the form (N',v'), where player i is split into m players such that $N' = \{n+1,\dots,n+m\} \cup (N \setminus \{i\})$ and $v' : \mathfrak{P}(N') \to \mathbb{R}$ is the new coalitional function whose values are to be specified according to the type of games in class $\mathfrak{G}$. Again, for weighted voting games v' has been specified in Section 3.1, and for other classes of cooperative games, v' needs to be suitably defined.

For example, if $\mathfrak{G}$ is the class of monotonic cooperative games, v' must be defined such that monotonicity is maintained, and since there are various possibilities of doing so, various distinct splitting functions can be defined for this class of games. As a second example, let μ_{wvg} and σ_{wvg} denote the merging and splitting functions for weighted voting games as defined in Section 3.1. That is, for a weighted voting game $\mathscr{G} \leftrightsquigarrow (w_1,\dots,w_n;q)$ and a coalition $S \subseteq N = \{1,\dots,n\}$, define $\mu_{\text{wvg}}(\mathscr{G},S) = \mathscr{G}_{\&S}$, and given a weighted voting game $\mathscr{G} \leftrightsquigarrow (w_1,\dots,w_n;q)$, a player $i \in N$, and an integer $m \geq 2$, define $\sigma_{\text{wvg}}(\mathscr{G},i,m)$ to be the set of weighted voting games $\mathscr{G}_{i\div m}$.

We define the following properties of merging and splitting functions.

Definition 3.18. *Let $\mathfrak{G}$ be a class of cooperative games and let $\mu_{\mathfrak{G}}$ be a merging function on $\mathfrak{G}$ and $\sigma_{\mathfrak{G}}$ be a splitting function on $\mathfrak{G}$.*

1. *We say $\mu_{\mathfrak{G}}$ satisfies* consistency *if for each $\mathscr{G} = (N,v) \in \mathfrak{G}$ and for each coalition $S \subseteq N$, if $\mu_{\mathfrak{G}}(\mathscr{G},S) = (N',v')$ then $v(C \cup S) = v'(C \cup \{i_{\&S}\})$ holds for each coalition $C \subseteq N \setminus S$.*

2. *We say $\mu_{\mathfrak{G}}$ satisfies* independence *if for each $\mathcal{G} = (N, v) \in \mathfrak{G}$ and for each coalition $S \subseteq N$, if $\mu_{\mathfrak{G}}(\mathcal{G}, S) = (N', v')$ then $v(C) = v'(C)$ holds for each coalition $C \subseteq N \setminus S$.*

3. *We say $\sigma_{\mathfrak{G}}$ satisfies* consistency *if for each $\mathcal{G} = (N, v) \in \mathfrak{G}$, for each player $i \in N$, and for each integer $m \geq 2$, if $(N', v') \in \sigma_{\mathfrak{G}}(\mathcal{G}, i, m)$ then $v(C \cup \{i\}) = v'(C \cup \{n+1, \ldots, n+m\})$ for each coalition $C \subseteq N \setminus \{i\}$.*

4. *We say $\sigma_{\mathfrak{G}}$ satisfies* independence *if for each $\mathcal{G} = (N, v) \in \mathfrak{G}$, for each player $i \in N$, and for each integer $m \geq 2$, if $(N', v') \in \sigma_{\mathfrak{G}}(\mathcal{G}, i, m)$ then $v(C) = v'(C)$ for each coalition $C \subseteq N \setminus \{i\}$.*

Intuitively, consistency means that the value of a coalition subject to merging or splitting should be the same before and after these operations. Independence means that the value of a coalition not affected by merging or splitting should remain the same in the new game, i.e., it depends only on the players in this coalition. In weighted voting games, both μ_{wvg} and σ_{wvg} satisfy consistency and independence, since the weight of the new player in $\mu_{\mathfrak{G}}(\mathcal{G}, S)$ equals $\sum_{i \in S} w_i$ for merging, and since $\sum_{j=1}^{m} w_{n+j} = w_i$ for splitting.

The following example presents a merging function for the class of weighted majority games such that neither consistency nor independence is satisfied.

Example 3.19. *Let μ_{wmg} be the merging function that maps a given weighted majority game $\mathcal{G} = (w_1, \ldots, w_n)$ and a given coalition $S \subseteq N$ to a new weighted majority game, where each player not in S keeps her weight, and the new player $i_{\&S}$ merging S receives weight $w_{i_{\&S}} = \prod_{i \in S} w_i$.*

Consider the game $\mathcal{G} = (2, 3, 4, 4)$ and the coalition $S = \{1, 3\}$. Then, the game $\mu_{\mathrm{wmg}}(\mathcal{G}, S) = (8, 3, 4)$ is formed. The value of the merged player in the new game is $v'(\{i_{\&S}\}) = 1$, whereas the value of S in the original game is $v(S) = 0$. Thus, μ_{wmg} is not consistent. On the other hand, the value of the coalition of the other players ($\{2, 4\}$ in $\mathcal{G}$ and $\{2, 3\}$ in $\mu_{\mathrm{wmg}}(\mathcal{G}, S)$) decreases from 1 to 0. Thus, μ_{wmg} is not independent.

A similar example is obtained by using, e.g., the maximum or minimum weight of the coalition's players instead of the product of their weights, or any other function that is not additive. One could consider any class of cooperative games with transferable utility, that have, for instance, a certain property in common. An important property of cooperative games is being a constant-sum game (see Section 2.3.1). Any merging function μ_{csg} on the class of constant-sum games is neither consistent nor independent whenever for $\mathcal{G} = (N, v)$, some coalition $S \subseteq N$, and $\mu_{\mathrm{csg}}(\mathcal{G}, S) = (N', v')$, it holds that $v(S) \neq v'(\{i_{\&S}\})$ and $v(N) = v'(N')$, since then $v(N \setminus S) \neq v'(N \setminus \{i_{\&S}\})$. See Section 3.4 for further applications.

We can now define the beneficial merging and splitting problems in general. Let $\mu_{\mathfrak{G}}$ be a merging function and $\sigma_{\mathfrak{G}}$ be a splitting function on a class $\mathfrak{G}$ of cooperative games and let PI be a power index.

$\mu_{\mathfrak{G}}$-PI-BENEFICIALMERGE	
Given:	A game $\mathscr{G} = (N, v)$ in $\mathfrak{G}$ and a non-empty coalition $S \subseteq N$.
Question:	Is merging of S beneficial, that is, does $\text{PI}(\mu_{\mathfrak{G}}(\mathscr{G}, S), i_{\&S}) > \sum_{i \in S} \text{PI}(\mathscr{G}, i)$ hold, where $\mu_{\mathfrak{G}}(\mathscr{G}, S) = (N', v')$ with $N' = \{i_{\&S}\} \cup (N \smallsetminus S)$?

Intuitively, $\mu_{\mathfrak{G}}$-PI-BENEFICIALMERGE is the problem of whether a coalition of players can benefit from merging via $\mu_{\mathfrak{G}}$ by raising their power in terms of PI. Similarly, $\sigma_{\mathfrak{G}}$-PI-BENEFICIALSPLIT is the problem of whether a player can benefit from splitting into a number of new players via $\sigma_{\mathfrak{G}}$ by raising her power in terms of PI.

$\sigma_{\mathfrak{G}}$-PI-BENEFICIALSPLIT	
Given:	A game $\mathscr{G} = (N, v)$ in $\mathfrak{G}$, $N = \{1, \dots, n\}$, a player $i \in N$, and an integer $m \geq 2$.
Question:	Is a beneficial split possible, that is, is there a game $\mathscr{G}' = (N', v') \in \sigma_{\mathfrak{G}}(\mathscr{G}, i, m)$ with $N' = \{n+1, \dots, n+m\} \cup (N \smallsetminus \{i\})$ such that $\sum_{j=1}^{m} \text{PI}(\mathscr{G}', n+j) > \text{PI}(\mathscr{G}, i)$?

Generalizing Proposition 3.4, if consistency and independence are satisfied by the merging function, a coalition of two players cannot benefit from merging nor can a player benefit from splitting into two players considering the probabilistic Penrose–Banzhaf index.

Theorem 3.20. *Let $\mu_{\mathfrak{G}}$ be a merging function and let $\sigma_{\mathfrak{G}}$ be a splitting function, both satisfying consistency and independence.*

1. $\mu_{\mathfrak{G}}$-PenroseBanzhaf-BENEFICIALMERGE *is in* P *for instances* $(\mathscr{G}, S)$ *with* $\|S\| = 2$.
2. $\sigma_{\mathfrak{G}}$-PenroseBanzhaf-BENEFICIALSPLIT *is in* P *for instances* $(\mathscr{G}, i, 2)$.

Proof. Let $\mathscr{G} = (N, v)$ be a cooperative game and let $\mu_{\mathfrak{G}}$ be a consistent and independent merging function. Without loss of generality (see Section 2.3.1 for the assumption of neutrality), let $S = \{n-1, n\}$. We obtain a new game $\mu_{\mathfrak{G}}(\mathscr{G}, S) = (\{1, \dots, n-1\}, v')$, where n' is the new $(n+1)^{\text{st}}$ player merging S in $\mathscr{G}$. It holds that

$$\text{PenroseBanzhaf}(\mu_{\mathfrak{G}}(\mathscr{G}, S), n') - (\text{PenroseBanzhaf}(\mathscr{G}, n-1) + \text{PenroseBanzhaf}(\mathscr{G}, n))$$

$$= \frac{1}{2^{n-1}} \left(\sum_{C \subseteq \{1, \dots, n-2\}} 2(v'(C \cup \{n'\}) - v'(C)) \right.$$
$$- \sum_{C \subseteq N \smallsetminus \{n-1, n\}} (v(C \cup \{n-1\}) - v(C)) \quad - \sum_{\substack{C \subseteq N \smallsetminus \{n-1\}, \\ n \in C}} (v(C \cup \{n-1\}) - v(C))$$
$$\left. - \sum_{C \subseteq N \smallsetminus \{n, n-1\}} (v(C \cup \{n\}) - v(C)) \quad - \sum_{\substack{C \subseteq N \smallsetminus \{n\}, \\ n-1 \in C}} (v(C \cup \{n\}) - v(C)) \right)$$

$$= \frac{1}{2^{n-1}} \left(\sum_{C \subseteq \{1,\dots,n-2\}} \left(\underbrace{2v'(C \cup \{n'\}) - 2v(C \cup \{n-1,n\})}_{=\,0\ \text{(by consistency)}} + \underbrace{2v(C) - 2v'(C)}_{=\,0\ \text{(by independence)}} \right) \right) = 0.$$

In the case of splitting, consider a game $\mathscr{G} = (N,v)$ with n players, a consistent and independent splitting function $\sigma_{\mathfrak{G}}$, and, without loss of generality, player n in $\mathscr{G}$ splitting into players $n+1$ and $n+2$, which results in a new game $\mathscr{G}' \in \sigma_{\mathfrak{G}}(\mathscr{G},n,2)$. Now, it similarly holds that

$$\text{PenroseBanzhaf}(\mathscr{G}',n+1) + \text{PenroseBanzhaf}(\mathscr{G}',n+2) - \text{PenroseBanzhaf}(\mathscr{G},n) = 0,$$

as claimed. ∎

Note that this immediately implies Proposition 3.4 for μ_{wvg} and σ_{wvg}.

Threshold Network Flow Games on Hypergraphs As another example, we briefly consider threshold network flow games on hypergraphs, a class of compactly representable simple cooperative games. Bachrach and Rosenschein [BR09] (see also the earlier work of Kalai and Zemel [KZ82a, KZ82b]) analysed threshold network flow games on graphs. A *threshold network flow game* is defined on an edge-weighted graph with n agents that each control one edge, a source vertex $s \in V$ and a target vertex $t \in V$, and a threshold $k \in \mathbb{R}$. The coalitional function is the characteristic function, where a coalition of agents is *successful* if and only if data of size k can be sent from s to t over paths on edges represented by the agents in the coalition. Here, a flow of data of size k means that data can be split up on paths each of which allows a flow of its smallest edge weight k', while the sum of those k' over all paths cannot be exceeded by k.

How can merging and splitting be defined in this setting? The approach to assign a merging coalition to the union of sets of edges controlled by players in the coalition of the original game, does not apply here, since agents control single edges: Merging two or more agents would yield one new agent who controls more than one edge and so would be qualitatively different from the remaining agents; Similarly, splitting an agent into several subagents would mean to "split" the original agent's edge, and it is unclear how to do that. Our approach for solving this issue is to consider threshold network flow games on hypergraphs rather than on graphs. A hyperedge in a hypergraph is any subset of the vertex set (so a graph is the special case of a hypergraph with hyperedges of size two only). Of course, agents in a hypergraph can have control over hyperedges of different sizes, but that is merely a quantitative difference. Kalai and Zemel's model is different from ours, because agents control sets in the first place and cannot overlap. Deng and Papadimitriou [DP94] study a representation of games by weighted hypergraphs and extend the Shapley value to this model.

Definition 3.21. *A* threshold hypergraph network flow game $\mathscr{G} = (N,v)$ *is set on a weighted hypergraph $H = (V,E)$ with vertex set V and a set $E = \{e_1,\dots,e_n\}$ of n weighted hyperedges (where agent i represents hyperedge e_i), a weight function $w : E \to \mathbb{N}$ (represented as a list*

$(w_1,\dots,w_n)$ with $w_i = w(e_i)$), a source vertex $s \in V$ and a target vertex $t \in V$, and a threshold $k \in \mathbb{R}$. The coalitional function $v : \mathfrak{P}(N) \to \{0,1\}$ is defined by $v(C) = 1$ if a data flow of size k from s to t is possible in $H_{|C}$, the subhypergraph of H induced by the hyperedge set $\{e_i \mid i \in C\}$, and $v(C) = 0$ otherwise.

For threshold network flow games, determining the raw Penrose–Banzhaf index is #P-many-one-complete, while the Shapley–Shubik index is only known to be at least as hard as problems in NP [BR09]. For threshold network flow games on hypergraphs the problem simply reduces from the corresponding problem for weighted voting games by mapping a given weighted voting game $\mathscr{G} \leftrightsquigarrow (w_1,\dots,w_n,q)$ to the game

$$\mathscr{G}' \leftrightsquigarrow ((\{v_0,v_1,\dots,v_{n+1}\},\{e_1,\dots e_n\}),v_0,v_{n+1},(w_1,\dots,w_n),q),$$

where $e_i = \{v_0,v_i,v_{n+1}\}$ for each i, $1 \le i \le n$, with the same weights and threshold. Since the value of each coalition C in $\mathscr{G}$ equals the value of C in $\mathscr{G}'$, we have $\text{PenroseBanzhaf}^*(\mathscr{G},i) = \text{PenroseBanzhaf}^*(\mathscr{G}',i)$ and $\text{ShapleyShubik}^*(\mathscr{G},i) = \text{ShapleyShubik}^*(\mathscr{G}',i)$ for each player i. This reduction is obviously parsimonious; therefore, #P-parsimonious-hardness of PenroseBanzhaf* for threshold hypergraph network flow games is inherited from that for weighted voting games, and #P-many-one-hardness of ShapleyShubik* for threshold hypergraph network flow games is inherited from that for weighted voting games.

Proposition 3.22. *In threshold hypergraph network flow games computing the raw Penrose–Banzhaf power index is* #P*-parsimonious-complete, while computing the raw Shapley–Shubik power index is* #P*-many-one-complete.*

Similarly, an analogon to the power compare problem as studied by Faliszewski and Hemaspaandra [FH09] for weighted voting games, can be defined for and reduced to threshold hypergraph network flow games. The question of whether, given two threshold hypergraph network flow games $\mathscr{G}$ and $\mathscr{G}'$ and a player i occurring in both games, the power of i in $\mathscr{G}$ is greater than in $\mathscr{G}'$, is PP-complete due to analogous argumentation.

The use of hyperedges makes it possible that, in threshold hypergraph network flow games, a coalition of agents can be merged into a single new agent who controls the hyperedge that corresponds to the union of vertices belonging to the hyperedges of the coalition's original agents.

Similarly, it is possible for an agent in such a setting to split into several subagents by partitioning this agent's hyperedge into subsets that each are controlled by one of the new subagents. We define the merging function and the splitting function for threshold hypergraph network flow games as follows:

- The merging function μ_{thnfg} on threshold hypergraph network flow games maps a given threshold hypergraph network flow game $\mathscr{G} \leftrightsquigarrow (H,s,t,w,k)$, with hypergraph $H = (V,E)$, and a given coalition S of agents to the new game $\mu_{\text{thnfg}}(\mathscr{G},S) \leftrightsquigarrow (H_{\&S},s,t,w_{\&S},k)$, where the new hypergraph is $H_{\&S} = (V,E_{\&S})$ with the new set of hyperedges $E_{\&S} = (E \setminus \{e_i \mid i \in S\}) \cup \{e_{\&S}\}$, and the new agent $i_{\&S}$ controls hyperedge $e_{\&S} = \bigcup_{i \in S} e_i$. The new weight function $w_{\&S}$ is given by $w_{\&S}(e_i) = w_i$ for $i \notin S$, and $w_{\&S}(e_{\&S}) = \sum_{i \in S} w_i$.

- The splitting function σ_{thnfg} on threshold hypergraph network flow games maps a given game $\mathcal{G} \leftsquigarrow (H, s, t, w, k)$, with hypergraph $H = (V, E)$, a given agent i, and a given integer $m \geq 2$ to the new game $\sigma_{\mathrm{thnfg}}(\mathcal{G}, i, m) \leftsquigarrow (H_{i \div m}, s, t, w_{i \div m}, k)$, where the new hypergraph is $H_{i \div m} = (V, E_{i \div m})$ with $E_{i \div m} = (E \setminus \{e_i\}) \cup \{e_{n+1}, \ldots, e_{n+m}\}$. Agent i is split into m agents $n+1, \ldots, n+m$ such that $\bigcup_{j=1}^{m} e_{n+j} = e_i$ and $e_{n+j} \cap e_\ell = \emptyset$ for $\ell \in \{1, \ldots, n\} \setminus \{i\}$. The new weight function $w_{i \div m}$ is given by $w_{i \div m}(e_\ell) = w_\ell$ for $\ell \neq i$, and the new agents' weights $w_{n+j} = w_{i \div m}(e_{n+j})$, $1 \leq j \leq m$, satisfy $\sum_{j=1}^{m} w_{n+j} = w_i$.

In contrast to weighted voting games, consistency is not satisfied in general for threshold hypergraph network flow games, neither by μ_{thnfg} nor by σ_{thnfg}. On the one hand, merging two agents via μ_{thnfg} can create new connections between vertices and thus allows new data flows to emerge. On the other hand, existing connections can get lost by splitting an agent via σ_{thnfg} (i.e., splitting the corresponding hyperedge). Therefore, merging and splitting by μ_{thnfg} and σ_{thnfg} might be advantageous for the probabilistic Penrose–Banzhaf index, even for size-two coalitions or a split into two players.

However, just as μ_{wvg} and σ_{wvg} for weighted voting games, both μ_{thnfg} and σ_{thnfg} satisfy independence for threshold hypergraph network flow games: The value of a coalition only depends on the hyperedges of the agents within the coalition, not on other hyperedges that might have been merged or split.

Unanimity Games Aziz et al. [ABEP11] study merging and splitting in unanimity weighted voting games with respect to the Shapley-Shubik index and the normalized Penrose–Banzhaf index. They show that for the normalized Penrose–Banzhaf index, merging is always disadvantageous, whereas splitting is always advantageous. Here, we extend the result for the Shapley-Shubik index to the class of general unanimity games and add a result for the probabilistic Penrose–Banzhaf index. In strong contrast to the normalized index, we show that in unanimity games with respect to the probabilistic Penrose–Banzhaf index, splitting is always disadvantageous or neutral, whereas merging is neutral for size-two coalitions, yet advantageous for coalitions with at least three players. This additionally underlines the differences between the two Penrose–Banzhaf indices.

A simple game $\mathcal{G} = (N, v)$ is called a *unanimity game* if only the grand coalition wins, i.e., $v(C) = 1$ if $C = N$, and $v(C) = 0$ if $C \subsetneq N$. Considering the example of a weighted game, a game represented by $= (w_1, \ldots, w_n; q)$ is a *unanimity weighted voting game* if and only if $\sum_{i=1}^{n} w_i - \min_{i \in N} w_i < q \leq \sum_{i=1}^{n} w_i$.

There is only one possible merging function for unanimity games. Let $\mathcal{G}$ be a unanimity game and let $S \subseteq N$ be a coalition. Define $\mu_{\mathrm{ug}}(\mathcal{G}, S) = (N', v')$ by $N' = \{i_{\&S}\} \cup (N \setminus S)$ and $v'(C) = 1$ if $C = N'$, and $v'(C) = 0$ if $C \subsetneq N'$. Obviously, μ_{ug} satisfies consistency and independence.

Theorem 3.23. *Let $\mathscr{G}$ be a unanimity game with player set N.*

1. $(\mathscr{G}, S) \notin \mu_{ug}$-PenroseBanzhaf-BENEFICIALMERGE *for each $S \subseteq N$ with $\|S\| = 2$,*
2. $(\mathscr{G}, S) \in \mu_{ug}$-PenroseBanzhaf-BENEFICIALMERGE *for each $S \subseteq N$ with $\|S\| \geq 3$.*
3. $(\mathscr{G}, i, m) \notin \sigma_{ug}$-PenroseBanzhaf-BENEFICIALSPLIT *for each $i \in N$ and $m \geq 2$.*

Proof. The first statement follows immediately from Theorem 3.20.

In order to prove the second statement, note that in a unanimity game, any player i can be pivotal only for the coalition $S = N \setminus \{i\}$, and i is always pivotal for this coalition. Thus the raw Penrose–Banzhaf index of each i is always equal to one. It follows that PenroseBanzhaf$(\mathscr{G}, i) = 1/2^{n-1}$ for each player $i \in N$. If an arbitrary coalition S merges, the Penrose–Banzhaf index of a player i in the new game $\mu_{ug}(\mathscr{G}, S)$ is PenroseBanzhaf$(\mu_{ug}(\mathscr{G}, S), i) = 1/2^{n-\|S\|}$. Since $\|S\| \geq 3$, we obtain

$$\text{PenroseBanzhaf}(\mu_{ug}(\mathscr{G}, S), i_{\&S}) - \sum_{i \in S} \text{PenroseBanzhaf}(\mathscr{G}, i) = \frac{2^{\|S\|-1} - \|S\|}{2^{n-1}} > 0.$$

The third statement can be shown by similar arguments. In particular, for any possible split into players with integer weights, we have for each $m \geq 2$,

$$-\text{PenroseBanzhaf}(\mathscr{G}, i) + \sum_{j=1}^{m} \text{PenroseBanzhaf}(\sigma_{ug}(\mathscr{G}, i, m), n + j) = \frac{m - 2^{m-1}}{2^{n+m-2}} \leq 0. \quad \square$$

3.3 Structural Control

In this section we define several control problems for weighted voting games. Doing so, we differentiate between adding and deleting players, two types of control. For each type, we define several possible goals, increasing or decreasing the power of a distinguished player, in relation to the player's power in the original game. In order to define these problems properly, we first have to define, how adding and deleting a player effects the coalitional function for weighted voting games.

In the case of adding players, we are given a set of unregistered possible new players and obtain a new game by adding a subset of the possible new players. Given a weighted voting game $\mathscr{G} = (N, v) \leftrightsquigarrow (w_1, \ldots, w_n; q)$ and a set of players M we want to add, $M \cap N = \emptyset$, with weights $w_{n+1}, \ldots, w_{n+m}$ for $M = \{n+1, \ldots, n+m\}$, $\|M\| = m$, we denote the new game by $\mathscr{G}_{\cup M} = (N \cup M, v_{\cup M})$, represented by $(w_1, \ldots, w_{n+m}; q)$. Inversely, deleting a subset of players $M \subseteq N$ from a weighted voting game $\mathscr{G} = (N, v) \leftrightsquigarrow (w_1, \ldots, w_n; q)$, yields a weighted voting game $\mathscr{G}_{\setminus M} = (N \setminus M, v_{\setminus M})$ represented by $(w_{j_1}, \ldots w_{j_{n-m}}; q)$ with $\{j_1, \ldots, j_{n-m}\} = N \setminus M$, $\|M\| = m$. For different approaches that might also be reasonable here, see Section 3.4.

We first define the problems of adding and deleting players with goals in relation to the old game. By way of example, we present the following decision problem for a power index PI.

STRUCTURAL CONTROL BY ADDING PLAYERS TO INCREASE PI

Given:	A weighted voting game $\mathcal{G} = (N, v)$, a set M of unregistered possible new players, $M \cap N = \emptyset$, their weights $(w_{n+1}, \dots, w_{n+m})$, a distinguished player $a \in N$, and a positive integer k.
Question:	Can at most k players $M' \subseteq M$ be added to $\mathcal{G}$ such that in the new game $\mathcal{G}_{\cup M'}$ it holds that $\mathrm{PI}(\mathcal{G}_{\cup M'}, a) > \mathrm{PI}(\mathcal{G}, a)$?

Analogously, we can ask whether the game can be controlled in order to gain the opposite effect, and non-increase a certain player's index, or to decrease, or non-decrease it. Here hardness in terms of complexity can be seen as a shield to prevent a game from being controlled to improve a player's significance or to worsen a player's significance. In contrast, we could consider the following control question: Is it possible to add a player to a game without changing the distribution of power among the original players? We can ask analogous questions with the same aims for removing players from the game. For instance, we define the following decision problem for a power index PI.

STRUCTURAL CONTROL BY DELETING PLAYERS TO INCREASE PI

Given:	A weighted voting game $\mathcal{G} = (N, v)$, a distinguished player $a \in N$, and a positive integer $k < \|N\|$.
Question:	Can at most k players $M' \subseteq N \setminus \{a\}$ be deleted from $\mathcal{G}$ such that in the new game $\mathcal{G}_{\setminus M'}$ it holds that $\mathrm{PI}(\mathcal{G}_{\setminus M'}, a) > \mathrm{PI}(\mathcal{G}, a)$?

Example 3.24. *Again, consider the weighted voting game $\mathcal{G}$ in Example 3.1. Let $k = 1$, that is a chair is able to remove one player from the game. Consider the Penrose–Banzhaf index. Player 1, 4, and 5 cannot improve from any other player being deleted. Players 2 and 3 can benefit from the other one being removed. Player 6 gains power if 1 is deleted.*

For the Shapley–Shubik index, due to normalization over the permutations of participating players, an increase of power is expected when deleting a player. However, players can also have a disadvantage, if a player leaves the game. For instance, player 1 loses power if 5 is deleted, 2 and 3 lose power if 4 is deleted, 4 loses power if 2 or 3 are deleted, and 5 loses power if 1 is deleted. This suggests a symmetric dependence of the players. In the same way, the power of players 2 and 3 maintains if 6 is removed and the other way around.

From the opposite view, consider the weighted voting game represented by $(2, 3, 4, 5; 10)$, two unregistered players with weights 1 and 2, and $k = 2$. Note that adding them both, ends up in $\mathcal{G}$. Here, the four players have probabilistic Penrose–Banzhaf indices of $1/4$, $1/4$, $1/4$, and $1/2$. The first player (with weight 2) cannot improve by adding any of the two players. The player with weight 3 can take advantage from adding both players or only the one with weight 2. The player with weight 4 improves in every situation adding one or two players. Finally, the player with weight 5 can only benefit if the player with weight 2 is added.

The first and fourth player cannot benefit from adding with respect to the Shapley–Shubik index. The other two can take advantage in the same cases as for the probabilistic Penrose–Banzhaf index.

Next to goals in relation to the old game, we can also compare an index either in relation to the other players' power, or in relation to a constant number. See Section 3.4 for initial results for this idea.

3.3.1 Complexity Results for Adding Players

We distinguish the cases where an upper bound of new players is given as defined above, and where the number of new players is fixed. Also note that the problem of whether it is possible to maintain an index, would be trivial if adding no player at all were allowed.

Theorem 3.25. *Control by adding a given number of players in order to increase (decrease, non-increase, non-decrease, maintain) a distinguished player's probabilistic Penrose–Banzhaf index or Shapley-Shubik index in a weighted voting game is* PP*-hard.*

Proof. We show PP-hardness via similar methods to those in the proof of Theorem 3.7. Reducing from COMPARE-#SUBSETSUM-RR, we map an instance $(a_1,\dots,a_n)$ with $\alpha = \sum_{i=1}^{n} a_i$ to a weighted voting game $\mathcal{G} \leadsto (1,a_1,\dots,a_n;\alpha/2)$, an unregistered player with weight $w_{n+2}=1$, $k=1$, and $a=1$. There is one possible new game obtained by adding the unregistered player to the game $\mathcal{G}_{\cup\{n+2\}}$. We show that

$$\begin{aligned}&\text{PenroseBanzhaf}(\mathcal{G}_{\cup\{n+2\}},1) - \text{PenroseBanzhaf}(\mathcal{G},1) > 0\\ &\quad\iff \#\text{SUBSET SUM}((a_1,\dots,a_n;\alpha/2-2)) > \#\text{SUBSET SUM}((a_1,\dots,a_n;\alpha/2-1)).\end{aligned}$$

It holds that

$$\begin{aligned}&\text{PenroseBanzhaf}(\mathcal{G}_{\cup\{n+1\}},1) - \text{PenroseBanzhaf}(\mathcal{G},1) &(3.5)\\ &= 1/2^n(\|\{C\subseteq\{2,\dots,n+1\} \mid v(C\cup\{n+2,1\})=1, v(C\cup\{n+2\})=0, v(C\cup\{1\})=0\}\|\\ &\quad - \|\{C\subseteq\{2,\dots,n+1\} \mid v(C\cup\{n+2\})=1, v(C\cup\{1\})=1, v(C)=0\}\|)\\ &= 1/2^n(\|\{C\subseteq\{2,\dots,n+1\} \mid 2+\textstyle\sum_{i\in C} a_{i-1} \geq \alpha/2,\ 1+\sum_{i\in C} a_{i-1} < \alpha/2\}\| &(3.6)\\ &\quad - \|\{C\subseteq\{2,\dots,n+1\} \mid 1+\textstyle\sum_{i\in C} a_{i-1} \geq \alpha/2,\ \sum_{i\in C} a_{i-1} < \alpha/2\}\|). &(3.7)\end{aligned}$$

If for some $C \subseteq \{2,\dots,n+1\}$, it holds that $\sum_{i\in C} a_{i-1} < \alpha/2$, but $1+\sum_{i\in C} a_{i-1} \geq \alpha/2$ (in set in (3.7)), it holds that $\alpha/2 - 1 = \sum_{i\in C} a_{i-1}$, since the weights and the quota are integers. If for some $C \subseteq \{2,\dots,n+1\}$ the conditions of the set in (3.6) are satisfied, it holds that $\alpha/2 - 2 = \sum_{i\in C} a_{i-1}$. Therefore, the term in (3.5) is positive if and only if the number of solutions that sum up to $\alpha/2-2$ is greater than $\alpha/2-1$. Thus, verifying whether control by adding players in order to increase the Penrose–Banzhaf index of a player, is PP-hard.

Since PP is closed under complement, the same result holds for the goal of non-increasing an index.

Analogously, we can reduce from COMPARE-#SUBSETSUM-ЯЯ as in the proof of Theorem 3.10 for the goal of decreasing an index, and, by the complement for non-decreasing an index.

Likewise with the methods of the proofs of Theorems 3.14 and 3.15, these results can be adapted to the Shapley–Shubik index. ◻

Remark 3.26. *An upper bound of* $\mathrm{NP}^{\mathrm{PP}}$ *can be established for structural control by adding players. We can guess the subset of new players to be added non-deterministically. Verifying whether the different goals are satisfied is encoded in the* PP*-oracle.*

Theorem 3.27. *Control by adding a fixed number of players in order to increase (decrease, non-increase, non-decrease) a distinguished player's probabilistic Penrose–Banzhaf index or Shapley-Shubik index in a weighted voting game is* PP*-complete.*

Proof. Since the number of players to be added is fixed, there are polynomially many combinations to be added. Therefore we have polynomially many comparisons of power indices. No matter which goal we consider, the comparison can be done in PP by Lemma 2.4 and by the fact that #P is closed under addition and PP under complement. The problem belongs to PP, since PP is closed under union.

Hardness is implied by the case of $k = 1$ player to be added in the proof of Theorem 3.25. By Lemma 3.3, this also holds for any other fixed number of players to be added. ∎

Theorem 3.28. *Control by adding a fixed number of players in order to maintain a distinguished player's probabilistic Penrose–Banzhaf index or Shapley-Shubik index in a weighted voting game is* coNP*-hard and in* PP.

Proof. The upper bound holds by the same argument as in Theorem 3.27. We show coNP-hardness with help of a reduction from $\overline{\text{PARTITION}}$.

Let $(a_1, \dots, a_n)$ be a partition instance with $\alpha = \sum_{i=1}^{n} a_i > 2$.[4] Consider the weighted voting game $\mathscr{G} \leftrightsquigarrow (1, 2a_1, \dots, 2a_n; \alpha + 2)$, an additional unregistered player with weight $w_{n+2} = 1$, and $k = 1$. The agent that is supposed to be promoted is 1. Observe that since all other weights in $\mathscr{G}$ are even, 1 is a null player with

$$\text{ShapleyShubik}(\mathscr{G}, 1) = \text{PenroseBanzhaf}(\mathscr{G}, 1) = 0.$$

We obtain in the only possible new game $\mathscr{G}_{\cup\{n+2\}} \leftrightsquigarrow (1, 2a_1, \dots, 2a_n, 1; \alpha + 2)$ that

$$\text{ShapleyShubik}(\mathscr{G}_{\cup\{n+2\}}, 1) > 0 \iff \text{PenroseBanzhaf}(\mathscr{G}_{\cup\{n+2\}}, 1) > 0$$

if and only if there exists a coalition $C \subseteq \{2, \dots, n+2\}$ with $\sum_{i \in C} w_i = \alpha + 1$ which is the case if and only if $n + 2 \in C$ because this is the only player with an odd weight, and

$$\sum_{i \in C \setminus \{n+2\}} w_i = \alpha \iff \sum_{i \in C \setminus \{n+2\}} 2a_{i-1} = \alpha \iff \sum_{i \in C \setminus \{n+2\}} a_{i-1} = \frac{\alpha}{2}.$$

The latter is the case if and only if $(a_1, \dots, a_n)$ is in PARTITION. To sum up, for both considered indices, the first player's index remains the same if and only if $(a_1, \dots, a_n)$ is not in PARTITION. Therefore, deciding whether adding one unregistered player can be added in order to maintain a player's index is coNP-hard. By Lemma 3.3, this also holds for any other fixed number of players to be added. ∎

[4] If $\alpha > 2$ is not satisfied we can easily decide if the instance is in PARTITION or not, and we can construct a trivial instance.

3.3.2 Complexity Results for Deleting Players

Note that the problem of deleting one player in order to increase an index is not the complement of the problem of adding one player in order to non-increase the same index. This is due to the fact that in the case of adding a player there are some players next to the distinguished player that are guaranteed to be part of the game before and after the structural change. However, if players can be deleted, each player except the distinguished one can be removed from the game.

Initially, we obtain the following two results.

Theorem 3.29. *Control by deleting one player in order to increase a distinguished player's Shapley-Shubik index in a weighted voting game is* NP*-hard (even if only one player can be deleted).*

Proof. We show NP-hardness by means of a reduction from SUBSET SUM. In the proof of Theorem 3.14 we have seen that we can assume that the satisfying solutions all have the same size ℓ with $m > \ell$. Let $(a_1, \ldots, a_n; q)$ be a SUBSET SUM instance and consider the weighted voting game $\mathcal{G} \leftrightsquigarrow (1, a_1, \ldots, a_m, q+1; q+1)$ and player 1 as our distinguished player. Let $k = 1$ and let $\xi = \#\text{SUBSET SUM}((a_1, \ldots, a_n; q))$ denote the number of solutions for the SUBSET SUM instance. Then, for the raw Shapley–Shubik index it holds that $\xi \geq 1$ if and only if deleting some player but 1 can lead to an increase of 1's index.

If: Assume that $\xi = 0$. Then, $\text{ShapleyShubik}^*(\mathcal{G}, 1) = 0$ and remains 0 whichever other player is deleted.

Only if: Assume that $\xi \geq 1$. Then,

$$\text{ShapleyShubik}^*(\mathcal{G}, 1) = \frac{\xi}{2} \cdot \ell!(n+1-\ell)! + \frac{\xi}{2} \cdot (n-\ell)!(\ell+1)!$$

If player $n+2$ is deleted, player 1's new raw index is

$$\text{ShapleyShubik}^*(\mathcal{G}_{\smallsetminus\{n+2\}}, 1) = \xi \cdot \ell!(n-\ell)!$$

This leads to

$$\begin{aligned}
&\text{ShapleyShubik}(\mathcal{G}_{\smallsetminus\{n+2\}}, 1) - \text{ShapleyShubik}(\mathcal{G}, 1) \\
&= \frac{1}{(n+1)!} \cdot \xi \cdot \ell!(n-\ell)! \cdot \frac{2}{2} - \frac{1}{(n+2)!} \cdot \frac{\xi}{2} \cdot \ell!(n-\ell)!(n+1-\ell+\ell+1) \\
&= \frac{1}{(n+1)!} \cdot \frac{\xi}{2}(2-1)\ell!(n-\ell)!
\end{aligned}$$

which is greater than 0, because $\ell!$ and $(n-\ell)!$ are positive. ∎

Theorem 3.30. *Control by deleting one player in order to non-increase or maintain a distinguished player's probabilistic Penrose–Banzhaf index in a weighted voting game is* coNP*-hard (even if only one player can be deleted).*

Proof. Again, we show coNP-hardness by means of a reduction from $\overline{\text{PARTITION}}$. Letting $(a_1,\dots,a_n)$ be a PARTITION instance with $\alpha = \sum_{i=1}^{n} a_i$, we construct the game $\mathscr{G} \leftrightsquigarrow (1,a_1,\dots,a_m,\alpha/2,\alpha/2;\alpha/2+1)$ and consider player 1 as our distinguished player. Let $k=1$ and let $\xi = \#\text{PARTITION}((a_1,\dots,a_n))$ denote the number of solutions for the PARTITION instance. Then, for the raw Penrose–Banzhaf it holds that $\xi \geq 1$ if and only if deleting any player but 1 does not maintain the index of player 1.

If: Assume that $\xi = 0$. Then, PenroseBanzhaf$^*(\mathscr{G},1) = 2$. If player $n+2$ with weight $\alpha/2$ is deleted, the raw index of player 1 is PenroseBanzhaf$^*(\mathscr{G}_{\smallsetminus\{n+2\}},1) = 1$, which ends up in the same probabilistic index. The factor of 2 is due to the fact that the raw index is twice as significant in the new game with one player less than before as in the old game.

Only if: Assume that $\xi \geq 1$. Then, PenroseBanzhaf$^*(\mathscr{G},1) = \xi+2$. If player $n+2$ or $n+3$ is deleted, player 1's new raw index is $\xi+1$. This leads to a higher index since $\xi+2 < 2(\xi+1)$. Deleting player j, $2 \leq j \leq n+1$ leads to a raw index of $\xi/2+2$ which means that in comparison to the old game, the player 1's index is increased: $\xi+2 < 2(\xi/2+2) = \xi+4$.

Especially, if $\xi \geq 1$, deleting any player, cannot lead to a non-increase. Therefore, it also holds that $\xi \geq 1$ if and only if deleting any player but 1 does not non-increase the index of player 1.

Hence, the problems of whether it is possible to maintain or to non-increase a player's probabilistic Penrose–Banzhaf index are coNP-hard. ∎

Observe that from these two constructions (and some values therein that are not calculated here) we cannot draw further conclusion about the complexity of structural control by deleting players. Finding similar or even different techniques for further results are an interesting task for future work. Other challenges follow in the next section.

3.4 Challenges and Future Work

We have analysed the problems of manipulation by a coalition of players that merge together to a single player, by a player annexing other players, or by a player that splits into several players in order to increase power in weighted voting games, in terms of their complexity. In a related setting, we have defined and analysed structural control scenarios. By considering the probabilistic Penrose–Banzhaf power index, and closing a gap between a lower and an upper bound, our results complement previous work. Most remarkably, we were able to pinpoint the exact complexity of the beneficial merging problem by showing its PP-completeness. Differences between complexity can be observed due to distinctions between given weights and several possible weights, and between input numbers given in unary and binary. It can be seen that beneficial annexation is easier to detect than beneficial merging. We have, furthermore, proposed a general framework for merging and splitting that can be applied to various classes of cooperative games with transferable utility.

Thoughts on a PP-oracle For the beneficial splitting problem, so far, we were able to raise the lower bound to PP-hardness, the exact complexity is yet to be determined. For a unary given number of false identities, but unknown weights, we have identified an upper bound of $\mathrm{NP}^{\mathrm{PP}}$. Still, it remains open whether it can be shown to be complete for PP, $\mathrm{NP}^{\mathrm{PP}}$ or something in between.

The class $\mathrm{NP}^{\mathrm{PP}}$ is a huge complexity that contains P^{PP} and therefore, by Toda's theorem [Tod91] the entire polynomial hierarchy. $\mathrm{NP}^{\mathrm{PP}}$ is an interesting class, but somewhat sparse in natural complete problems. The only (natural) $\mathrm{NP}^{\mathrm{PP}}$-completeness results we are aware of are due to Littman et al. [LGM98] who analyse questions related to probabilistic planning, and due to Mundhenk et al. [MGLA00] who study problems related to finite-horizon Markov decision processes. Littman et al. also define ∃-MAJSAT (see Section 2.1). Now, we are interested in a further understanding of natural problems complete for this class.

On our way towards the exact complexity of beneficial merging and control by adding players, we have so far managed to show PP-completeness for the following existential threshold variant of SUBSET SUM.

EXISTENTIAL SUBSET OF SUM (∃-SUBSETSUM)	
Given:	A set $A = \{1,\dots,n\}$, a value function $a : A \to \mathbb{N} \setminus \{0\}$, $i \to a_i$, a positive integer q, and two further positive integers k,t.
Question:	Does there exist a selection A' of the first k elements in $\{1,\dots,k\}$ such that more than t of the remaining combinations of values $a_{k+1},\dots,a_n$ sum up to $q - \sum_{i\in A'} a_i$?

This problem obviously belongs to $\mathrm{NP}^{\mathrm{PP}}$, as the set A' can be chosen nondeterministically, and can verify for the remaining values whether

$$\#\text{SUBSET SUM}((a_{k+1},\dots,a_n; q - \sum_{i\in A'} a_i)) > t$$

with help of an oracle that simulated decisions in probabilistic polynomial time.

The lower bound requires a number of steps, since we have to guarantee the numbers t to remain in the same relation in the reduction. In order to do so, we have travelled through several restrictions on SAT(see, e.g., [Sch78]), following the path of parsimonious reductions by Hunt et al. [HMRS98].

To begin with, we reduce from ∃-MAJSAT with the parameter $t = 2^{n'-k-1}$: $2^{n'-k}$ for the remaining assignments, and a divisor of 2 for the half of them to be satisfied. In the next steps, we follow the parameter through the reduction transforming the general Boolean formula to a 3-SAT instance [Kar72], over existential versions of 1-EX3SAT, 1-EX3MONOSAT [HMRS98], and finally over XC_3 to SUBSET SUM. We obtain a parameter of $t = 2^{\ell}$ with, $\ell < n-k$. All in all, we have shown the following.

Theorem 3.31. *∃-SubsetSum is* $\mathrm{NP}^{\mathrm{PP}}$*-complete.*

Similarly, we next intend to find an existential variant of the COMPARE-#SUBSET SUM problem in order to tackle the open manipulation and control problems stated above.

We conjecture that the following statements hold.

Conjecture 3.32. *For weighted voting games, and for both, the Shapley–Shubik and the Penrose–Banzhaf index, we suppose that*

1. PenroseBanzhaf-BENEFICIALSPLIT *with the number of false identities given in unary and*

2. STRUCTURAL CONTROL BY ADDING PLAYERS TO INCREASE PI

are $\mathrm{NP}^{\mathrm{PP}}$*-complete.*

Open Questions and Generalizations Let alone the questions initiated here, the following problems remain open. A straightforward completion of our results would be to find out whether they can be transferred to other power indices like the normalized Penrose–Banzhaf index. Due to the structure of this index, we claim that other techniques will be needed to solve these problems. The exact placement of most structural control problems in terms of their complexity is also left open.

An interesting task for future research is to study useful properties of merging and splitting functions, such as consistency and independence, in general and when applied to particular classes of games. Natural such functions may, for example, exist in other important classes of cooperative games with transferable utility where each player posesses a certain amount of a divisible resource, such as fractional matching games, bankruptcy games, or market games (see, e.g., [SL09]), or path-disruption games as studied in the next chapter. where merging any two unconnected vertices might influence the value of a coalition of other players. On the one hand, these properties are desirable for the design of a merging or splitting function; on the other hand, they are an approach for an axiomatic evaluation of such functions. The analysis of further properties are an interesting task for future work.

A related question that has arisen in a review was of how to naturally extend the idea of merging to classes of games where players control *several* resources. Which properties do we want to hold in that case? Can a merging function that satisfies independence and consistence be unique for a certain class of games? For unanimity games we have observed that there is only one possible merging function that guarantees unanimity. For weighted voting games, however, such a uniqueness result does not hold, since there are different ways to distribute the player's weights that lead to the same coalitional function. For instance, the games $(1,3,4;8)$ and $(2,3,4;8)$ are semantically the same, even if players merge. Restricting to other classes or requiring other properties might imply uniqueness. Although consistency seems to be an essential property for a merging or splitting function, we have seen a natural merging function on threshold hypergraph network flow games that does not satisfy this property, and we made similar observations for other classes of games.

Regarding structural control, so far we have only yielded results for goals in *relation to the former game*. Alternatively, one might think of a situation where a chair wants to increase a player's significance in comparison to the other players, which can also be achieved if players are added or deleted, the certain player's index remains the same, but all remaining players' indices are distributed in a way that they are below this value.

Besides this, we can also model a scenario, where a player is demanded to exceed a certain *constant index*, and we ask whether it is possible to control a game by adding or deleting a player in order to reach this index. So far, we can tell that if the number of given players to be added or deleted is $k = 0$, our value is $1/2$, and the considered index is the Penrose–Banzhaf power index, the problem is PP-complete. This might change if $k > 0$ is required. We might also study the exact variant of obtaining an exact value.

There seems to be a close connection to the notion of *synergies* in cooperative games, see, e.g. [RMW14]. We may want to have a closer look at related results here.

Next to weighted voting games, of course, other classes of cooperative games with transferable utility might be effected by control scenarios. Then, adding and deleting a player has to be well-defined. Let us consider general (weighted) majority games. Let $\mathscr{G}$ be a majority game with representation $(w_1, w_2, \dots, w_n; \alpha(n))$, that is, $v(C) = 1$ if $\sum_{i \in C} w_i \geq \lfloor \alpha(n) \rfloor + 1$, and $v(C) = 0$ otherwise, for each $C \subseteq N$. Now, if a player is deleted, the number of players n is decremented, such that the threshold $\alpha(n)$ is changed. The new coalitional function is computed as above. Adding a player requires a set of unregistered players, given by their weights and n is increased. As we have seen, for (weighted) threshold games, the new coalitional function is determined similarly, with the only difference that the threshold does not change. One could alternatively think of weights as percentage, and change weights of remaining players proportionally. Thus, the new game $\mathscr{G}_{\cup M}$ is defined differently, by normalizing the sum of weights to the original value. Similar to majority games, here players do not have an absolute but a relative contribution to the game. From a different point of view, adding and deleting players can be viewed in sense of a change over time, as is analysed so far for changing the quota over time [EPZ13]. Similarly, studying a change of players dynamically over time is an interesting task for future work.

Other games that can be interesting to study include games in which the Shapley–Shubik index is easy to compute such as weighted graph games [DP94]. In such games, two indices in two games can be compared in polynomial time, and therefore, if, on the one hand, the coalition that is added to or removed from a game is known, the possibility of control is easy to detect. If, on the other hand, there are possible coalitions to be added, this problem might become interesting again. Eventually, if players correspond to an edge in a game, deleting an edge may be interesting in context of Braess's paradox for non-cooperative congestion games (see, e.g., [NRTV07, pp. 464–465]) where, informally, an extra fast lane might lead to congestion, whereas without this lane traffic may split up to equally slower paths. Can we find a connection to control by deleting a player in a cooperative game with transferable utility?

4 Uncertain Targets in Path-Disruption Games: Bribery and Stability

The contents of this chapter are published in the conference contributions [MRR14] and [RR12]. For completeness, we also briefly mention related results as published in the conference paper [RR11]. The assembled and extended article [RRM16] is to appear. We model and study questions of bribery and stability in probabilistic path-disruption games.

Example 4.1. *Consider the game illustrated in Figure 4.1.*

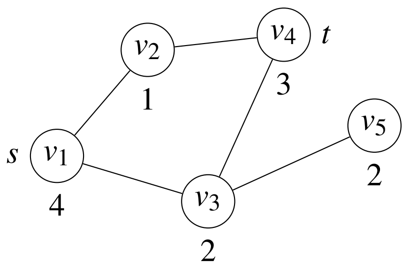


Figure 4.1: Example of a path-disruption game. Costs are written below the vertices. Let the reward be $R = 4$.

Observe that coalition $\{2,3\}$ *is successful and has a positive value of* 1*, and so does each coalition containing* 2 *and* 3*, too. The same holds for the coalition consisting only of player* 4 *as well as the grand coalition.*

Suppose, an adversary in a path-disruption game as defined in Section 2.3 is located on a certain source vertex; however, her target vertex is unknown. In this chapter we expand the original model [BP10] by allowing uncertainty about the targets: In probabilistic path-disruption games, each vertex is a potential target, assigned with the probability that an adversary wants to reach it. We formally define these games and study the computational complexity of problems related to solution concepts and other properties. Settings like these also have a background in non-cooperative game theory such as zero-sum security games [JKV$^+$11], and other network security games [SL08, WW95]. The matter of uncertainty has been studied in strategic games with regard to noise on both the intruder and the defender side [YJTO11].

Moreover, inspired by bribery in the context of voting (see, e.g., [FHH09, FHHR09, FR16]). We have introduced the notion of bribery for path-disruption games with costs [RR11] where adversaries break into the setting and try to change the outcome to their advantage by bribing some of the players. Now that the agents collaborate while, at the same time, they want to win against their adversaries who can actively interfere with the situation in order to achieve their individual goals in opposition to the agents, the game combines aspects of both cooperative and non-cooperative game theory. Relatedly, Bachrach et al. [BMFT11, BS13] study the reliability of players in cooperative games. We analyse the question as to how hard it is to decide whether the adversaries in a (probabilistic) path-disruption game with costs can bribe some of the agents not exceeding their budget, such that no blocking coalition will form that prevents the adversaries from reaching their targets.

Table 4.1 summarizes the complexity results for various stability concepts and problems in probabilistic path-disruption games. Note that the probabilistic model is not used in the papers [BP10] or [RR11]. However, we refer to their work in Table 4.1 whenever the statements can be implied by their proof immediately.

Probabilistic path-disruption games without costs	
single player	multiple players
• monotonic (Prop. 4.4) ¶ • not simple, constant-sum, convex, or superadditive (Prop. 4.5) ¶ • value computation in pol. time (Cor. 4.8) • dummy player verif. coNP-c. (Prop. 4.14) † • core verif. and existence in P (Prop. 4.15) ¶ • core computation in pol. time (Prop. 4.15) ¶	• monotonic (Prop. 4.4) ¶ • not simple, constant-sum, convex, or superadditive (Prop. 4.5) ¶ • value computation in pol. time (Cor. 4.8) • dummy player verif. coNP-c. (Prop. 4.14) † • ε-core verification coNP-c. (Prop. 4.17) † ‖
Probabilistic path-disruption games with costs	
single player	multiple players
• not monotonic, simple, constant-sum, convex, or superadditive (Prop. 4.4,4.5) ¶ • value computation in pol. time (Prop. 4.7) ¶ • dummy player verification coNP-complete • core verification in coNP (Prop. 4.16) ¶	• not monotonic, simple, constant-sum, convex, or superadditive (Prop. 4.4,4.5) ¶ • value verification NP-hard (Cor. 4.8) † • dummy player verification coNP-hard • core verification in coNP (Prop. 4.16) ¶ • ε-core verification coNP-c. (Prop. 4.17) ‖
• BRIBERY NP-compl. (Thm. 4.10, Cor. 4.11) ‡	• BRIBERY Σ_2^p-compl. (Thm. 4.12, Cor. 4.13) § ¶

† [BP10] ‖ [RRM16, RR12]
‡ [RRM16, RR11] ¶ this thesis ([RRM16, MRR14, RR12])
§ [RRM16, MRR14]

Table 4.1: Overview of complexity results of stability and bribery problems in probabilistic path-disruption games. Key: c. stands for complete, verif. for verification.

For path-disruption games in the original model, Bachrach and Porat [BP10] study various problems related to game-theoretic notions. They show that without costs the games are monotonic, that it is possible to compute the core in polynomial time and that verifying whether an imputation is in an ε-core is related to finding a minimal vertex cut. Testing whether an agent is a null player is coNP-complete and consequently, computing the probabilistic Banzhaf index is #P-many-one-complete. In the case of costs, they show that computing the value of a coalition is also related to finding a minimal vertex cut and therefore intractable for several adversaries in general. Moreover, they study the special case of path-disruption games on trees; see also Section 4.2 for an excursion on restrictions to graph classes. For further related results, see also [AS11].

The analogous problems for probabilistic path-disruption games are more general, so any lower bound for the more special variant of a problem immediately is inherited by its generalized variant. On the other hand, upper bounds known for problems on non-probabilistic path-disruption games may be invalid for their more general analogues, or if they are valid, they might be harder to prove. Moreover, we analyse probabilistic path-disruption games on undirected graphs, as this is the more demanding case regarding the computational hardness results. Given an undirected graph, we can simply reduce the problem to the more general case of a directed graph by substituting each undirected edge $\{x,y\}$ by the two directed edges (x,y) and (y,x).

4.1 Probabilistic Path-Disruption Games and Bribery

In path-disruption games as defined in Section 2.3.1, it is assumed that an adversary has one certain target. More generally, we consider the case where the actual target is unknown, though we have probabilities that indicate where the intruder might go. Let us define the notion of *probabilistic path-disruption games* in its most general variant, *with costs and multiple adversaries*.

Definition 4.2. *Let $G = (V,E)$ be an undirected graph with n vertices and m adversaries, each sitting on a given vertex $s_j \in V$, $1 \leq j \leq m$. Moreover, consider every vertex v_i in $V = \{v_1,\dots,v_n\}$ as a potential target and let $p_{j,i}$ be the probability that adversary j (situated on s_j) wants to reach v_i, where $\sum_{i=1}^{n} p_{j,i} = 1$ for each j, $1 \leq j \leq m$. Furthermore, we are given a cost function $c : V \to \mathbb{R}_{\geq 0}$, and a reward R. Using this domain, we define a* probabilistic path-disruption game with costs and multiple adversaries *as follows. Let $N = \{1,\dots,n\}$ be the set of agents, where v_i represents player i, and define the coalitional function v via*

$$v(C) = \tilde{v}(C) \cdot (R - \mu(C))$$

with the minimal costs $m(C)$ defined as

$$\mu(C) = \begin{cases} \min\{c(B) \mid B \subseteq C \textit{ and } \tilde{v}(B) = \tilde{v}(C)\} & \textit{if } \tilde{v}(C) > 0, \\ -1 & \textit{otherwise}, \end{cases}$$

where

$$\tilde{v}(C) = \prod_{j=1}^{m} \sum_{i=1}^{n} p_{j,i} \cdot w(C, j, i)$$

and

$$w(C, j, i) = \begin{cases} 1 & \textit{if } C \textit{ blocks each path from } s_j \textit{ to } v_i, \\ 0 & \textit{otherwise}. \end{cases}$$

Note that if $\tilde{v}(C) = 0$ then the minimal costs do not influence $v(C)$, so they can be any number. If for each j, $1 \leq j \leq m$, there exists exactly one i, $1 \leq i \leq n$, such that $p_{j,i} = 1$ (and we thus have $p_{j,k} = 0$ for all $k \neq i$), we obtain path-disruption games by Bachrach and Porat [BP10] as defined in Definition 2.8. The probabilistic analogues of their other variants of path-disruption games are defined as follows. A *probabilistic path-disruption game with multiple adversaries and without costs* is defined as above, except that neither a cost function nor a reward is given and the coalitional function itself is defined by

$$v(C) = \prod_{j=1}^{m} \sum_{i=1}^{n} p_{j,i} \cdot w(C, j, i).$$

The models *with single adversaries with or without costs* are obtained from the above two variants by setting $m = 1$.

Example 4.3. *Consider the game illustrated in Figure 4.2 as a variant to Example 4.1. Now the adversary's target is uncertain. We only know that remaining at the source or moving to the destination v_2 is impossible (i.e., has probability 0), moving to v_4 is most likely (with a probability of 0.5), and the other two vertices are equally likely targets (with a probability of 0.25 each).*

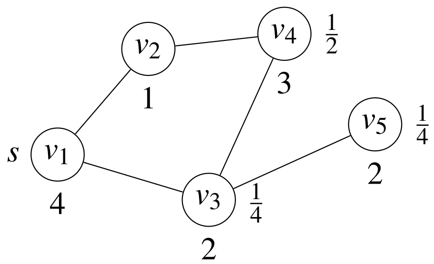


Figure 4.2: Example of a probabilistic path-disruption game with target probabilities written on the right of the vertices. Costs and reward are the same as in Example 4.1.

Now, player 5, e.g., has a more central role with $v(\{5\}) = 1/4 \cdot (4 - 2) = 1/2$. The values of $\{1\}$ and $\{2,3\}$ remain zero and one. Coalition $\{3,4\}$ even has a negative value.

Next, we analyse this new type of game's basic game-theoretic properties. Probabilistic path-disruption games (even without costs) are not simple, as soon as one of the given probabilities $p_{j,i}$ is strictly between 0 and 1. In every other aspect studied below they behave like their restricted predecessor.

Proposition 4.4. *Probabilistic path-disruption games without costs are monotonic, whereas in general they are not.*

Proof. In a given probabilistic path-disruption game without costs, for all agents i, $1 \leq i \leq n$, for all adversaries j, $1 \leq j \leq m$, and for all coalitions A and B, $A \subseteq B \subseteq N$, it holds that $w(A, j, i) \leq w(B, j, i)$, since a coalition can never block fewer paths than a subcoalition. Thus,

$$v(A) = \prod_{j=1}^{m} \sum_{i=1}^{n} p_{j,i} \cdot w(A, j, i) \leq \prod_{j=1}^{m} \sum_{i=1}^{n} p_{j,i} \cdot w(B, j, i) = v(B).$$

Non-monotonicity of the cost-case can be shown by the following single adversary example. Let $G = (\{v_1, \dots, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}\})$, $s = v_1$, $p_{1,4} = 1$, $c(v_i) = 1$ for all i, $1 \leq i \leq 4$, and $R = 1$. Although $\{2\} \subseteq \{2,3\}$, it holds that $v(\{2\}) = 0 > -1 = v(\{2,3\})$. ❑

Proposition 4.5. *Probabilistic path-disruption games are not simple, constant-sum, convex, or superadditive.*

Proof. Obviously, even for the monotonic model without costs, coalitions can have values other than 0 and 1.

By the following counterexample, even without costs, a probabilistic path-disruption game is not a constant-sum game. Consider the graph given in Figure 4.3a, and the game on that graph with six players, without costs, with a reward of 1 and with an adversary travelling from s to t with a probability of 1. Observe that $v(C) = 1$, but also $v(N) = 1$, and $v(N \smallsetminus C) = 1$.

Finally, by the same counterexample, it holds that $C \cap D = \emptyset$ and

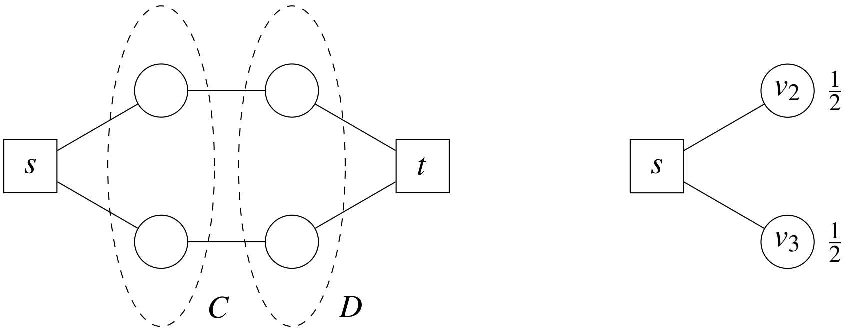
$$v(C \cup D) = 1 < 1 + 1 = v(C) + v(D).$$

Hence, probabilistic path-disruption games are not superadditive, and thus, not convex. ❑

The following proposition underlines that despite these unfortunate properties, probabilistic path-disruption games are a reasonable model, as they do extend to the original path-disruption games.

Proposition 4.6. *There exists a probabilistic path-disruption game without costs and a single adversary that is not strategically equivalent to any non-probabilistic path-disruption game (with the same number of players and without costs and a single adversary).*

Proof. Consider the game $\mathscr{G} = (N, v)$ with three players played on the graph in Figure 4.3b without costs and a reward of 1. The coalitional function of this game is



(a) Not a constant sum game (b) No strategic equivalent

Figure 4.3: Counterexamples of probabilistic path-disruption games

C	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(C)$	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	1

with 2 and 3 being symmetric players. Assume there exists a non-probabilistic path-disruption game $\mathscr{G}' = (N, v')$ without costs and a single adversary such that $\mathscr{G}'$ and $\mathscr{G}$ are strategically equivalent. Then, there exist $\alpha > 0$ and $\beta : N \to \mathbb{R}$ such that $v(C) = \alpha v(C) + \sum_{i \in C} \beta(i)$ holds for each $C \subseteq N$. Consider two cases.

Case 1: Let $s = v_1$ be the same starting point of the adversary. The coalitional function then is

C	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v'(C)$	0	1	a	1	b	1	c	1

with $a,b,c \in \{0,1\}$ and $c \geq a,b$ by monotonicity. The equations $1 = \alpha \cdot 1 + \beta(1)$, $1 = \alpha \cdot 1 + \beta(1) + \beta(2)$, and $1 = \alpha \cdot 1 + \beta(1) + \beta(3)$ imply that $\beta(2) = \beta(3) = 0$. Therefore, we obtain $a = b = c = 1$ and $\alpha = 0$. This, however, contradicts $1 = \alpha \cdot 1 + \beta(2) + \beta(3)$.

Case 2: Let the adversary start at a different vertex, without loss of generality, $s = v_2$. In this case the coalitional function is

C	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v'(C)$	0	a	1	1	b	c	1	1

with $a,b,c \in \{0,1\}$ and $c \geq a,b$ by monotonicity. By $1/2 = \alpha \cdot 1 + \beta(2)$ and $1 = \alpha \cdot 1 + \beta(1) + \beta(2)$, we obtain $\beta(1) = 1/2$. Therefore, we have a contraction in $1 = \alpha \cdot 1 + \beta(2) + \beta(3)$ and $1 = \alpha \cdot 1 + 1/2 + \beta(2) + \beta(3)$. ∎

Another important property of a game representation is that a coalition's value should be tractable. Here, the probabilistic model behaves like the original one. Although the model is more general, we can reduce this problem to that in the original (non-probabilistic) setting in polynomial time. The most challenging case here concerns costs and a single adversary where rather unexpectedly polynomial-time computability still holds.

Proposition 4.7. *Given a probabilistic path-disruption game with costs and a single adversary, and a coalition, its value can be computed in polynomial time.*

Proof. Given a probabilistic path-disruption game $\mathscr{G}$ with costs and a single adversary, consisting of $G = (V,E)$, $s \in V$, $c : V \to \mathbb{Q}_{\geq 0}$, $R \in \mathbb{Q}_{\geq 0}$, and $p_{1,i}$, $1 \leq i \leq n$, and given a coalition $C \subseteq N$, computing $\tilde{v}(C)$ involves at most n computations of $w(C,1,i)$ which, in turn, can be determined in polynomial time using a graph accessibility algorithm (remember that GAP can even be decided in nondeterministic logarithmic space). Either $\tilde{v}(C) = 0$, then we can return 0 as the value of C; or, $\tilde{v}(C) > 0$. Then we consider the graph $G' = (V',E')$ with $V' = V \cup \{v_{n+1}\}$ and

$$E' = E \cup \{\{v_i, v_{n+1}\} \mid 1 \leq i \leq n \text{ with } p_{1,i} > 0 \text{ and } w(C,1,i) = 1\}.$$

Let $t = v_{n+1}$. Define a new cost function $c' : V \to \mathbb{Q}_{\geq 0}$, by setting $c'(v_i) = c(v_i)$ if $i \in C$, and $c'(v_i) = 1 + \sum_{i \in C} c(v_i)$ otherwise. Now, we determine the minimal costs κ (regarding the cost function c') needed to disrupt all paths from s to t in G'. This can be done in polynomial time using the algorithm for MCVC.

In order to calculate $v(C) = \tilde{v}(C) \cdot (R - \kappa)$, we now show that $\kappa = \mu(C)$. By construction of G', C blocks all paths from s to t. Since all other vertices have greater costs, the vertices with minimal costs κ correspond to the players in C. It holds that

$$\mu(C) = \min\{c(B) \mid B \subseteq C \text{ and } \tilde{v}(B) = \tilde{v}(C)\},$$

which is equal to the minimum costs of a coalition $B \subseteq C$ that blocks the same possible targets (that is, vertices with a positive probability of being a target) as C. Since t is only connected to the possible targets blocked by C, this is equal to

$$\min\{c(B) \mid B \subseteq C \text{ and } B \text{ blocks all paths from } s \text{ to } t \text{ in } G'\},$$

which, in turn, is equal to κ by definition. ◻

Using Proposition 4.7, the fact that $\tilde{v}(C)$ can be computed in polynomial time for a coalition C even for multiple adversaries, and a corresponding result for path-disruption games with costs and multiple players by Bachrach and Porat, we obtain the following.

Corollary 4.8. *In a probabilistic path-disruption game without costs, a coalition's value can be determined in polynomial time, but it is* NP*-hard to decide whether the value of a coalition is greater than a given value, for multiple adversaries and costs.*

In the next section, we study the complexity of various problems related to stability concepts in probabilistic path-disruption games. Note, that for this purpose, some concepts have to be redefined that have only been defined for simple games previously, see Section 2.3. Mainly, for a stability concept we ask the questions of verification, whether a given player or payoff vector satisfies the stability concept in a given game, and existence, whether a given game allows stability with respect to a certain concept.

Moreover, we consider another type of influence, namely bribery. The question we ask here is, given a path-disruption game with costs, can the adversaries bribe a coalition $B \subseteq N$ of agents such that no coalition $C \subseteq N$ will be formed that blocks each path from s to t.

For the no-cost case, this problem is well-studied. Considering the simplest form of path-disruption game, single adversary, without costs, and with constant prices for each agent and an infinite budget for the adversary, the answer is yes if and only if $(G,s,t) \in$ GAP, the graph accessibility problem, see Section 2.2: Given a graph G and two distinct vertices, a source vertex s and a target vertex t, can t be reached via a path from s? This problem can be solved in nondeterministic logarithmic space (and thus in deterministic polynomial time). The equivalence holds, since bribery of all agents on a path from s to t will guarantee the adversary a safe travel. If, on the other hand, the number of agents the adversary can bribe is limited by a number k, bribery is possible if and only if there is a path from s to t with length at most k, which is decidable in polynomial time. This problem is also related to generalized connectivity problems, see Section 2.2.

In the following we consider bribery on path-disruption games with costs, at first in the original model, then for the probabilistic model. Here, in contrast to the no-cost case, even if a limited budget may not allow bribing the players on each vertex in any path from the source to the target, successful bribery might still be possible, since for all remaining blocking coalitions the involved costs are too high.

Let PDG be a type of path-disruption game; in particular, distinguish the original non-probabilistic model and the probabilistic model as well as single and multiple adversaries. We focus on the cost-case for the following definition.

PDG-BRIBERY	
Given:	A path-disruption game (of a certain type) inducing $\mathcal{G} = (N,v)$, a price function $\pi : V \to \mathbb{Q}_{\geq 0}$, and a budget $K \in \mathbb{Q}_{\geq 0}$.
Question:	Is there a coalition $B \subseteq N$ such that $\sum_{i \in B} \pi(v_i) \leq K$, and no coalition $C \subseteq N \smallsetminus B$ has a value $v(C) > 0$?

Example 4.9. *Consider the same graph, adversary, and costs as in Example 4.1. Moreover, let $\pi(u_1) = 3$ and $\pi(u_2) = \pi(u_3) = \pi(u_4) = \pi(u_5) = 1$ be the vertices' prices and $K = 1$ the briber's budget. Vertices corresponding to bribable agents are diamond-shaped in Figure 4.4.*

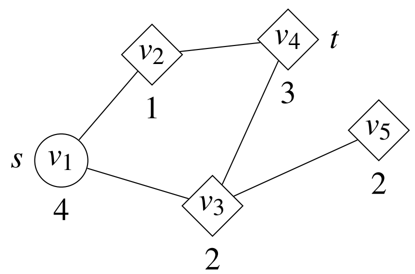


Figure 4.4: Example of bribery in a path-disruption game. Costs and reward $R = 4$ are the same as in Example 4.1.

In this example, bribery is not possible, since every coalition of size at least 2 *or containing players* 1 *cannot be bribed, as its price exceeds the budget. Bribery of* $B = \{2\}$*,* $B = \{3\}$*, or* $B = \{5\}$ *is not successful either, since* $C = \{4\}$ *has a positive value. Bribery of* $B = \{4\}$*, in turn, is not possible since* $v(\{2,3\}) = 1$*.*

If $K = 2$ *and the prices are the same, however, bribery is possible, e.g., for the coalition* $\{2,4\}$*. The remaining coalitions are subsets of* $\{1,3,5\}$*. Without player* 1 *a coalition is not successful. With player* 1*, however, the costs are as high as the reward, therefore, the coalition will not form to block all paths from* v_1 *to* v_4*.*

4.1.1 Complexity of Bribery

The bribery problem for path-disruption games with costs in the original model of path-disruption games with costs is NP-complete for a single adversary [RR11] and Σ_2^p-complete for multiple adversaries [MRR14]. We sketch the proof of the former in Theorem 4.10 and present the latter in Theorem 4.12 for completion. In this thesis, we focus on the probabilistic model, for which the same complexity classification holds, see Corollaries 4.11 and 4.13.

Theorem 4.10 ([RR11])**.** *For path-disruption games with costs and a single adversary,* PDG-BRIBERY *is* NP*-complete.*

Proof Sketch. Firstly, the problem is in NP for the following reasons. Given a path-disruption game with costs consisting of a graph $G = (V,E)$, a cost function $c : V \to \mathbb{Q}_{\geq 0}$, a reward $R \in \mathbb{Q}_{\geq 0}$, a source and a target vertex, $s,t \in V$, inducing $\mathcal{G} = (N,v)$, and given a price function $\pi : V \to \mathbb{Q}_{\geq 0}$, and a bound $K \in \mathbb{Q}_{\geq 0}$, we can nondeterministically guess a coalition $B \subseteq N$. Obviously, it can be tested in polynomial time whether $\sum_{i \in B} \pi(v_i) \leq K$. If this inequality fails to hold, bribery of B is not possible. Otherwise, we can verify whether all coalitions $C \subseteq N \setminus B$ satisfy $v(C) \leq 0$ (which is the case if and only if $\tilde{v}(C) = 0$ or $R \leq \mu(C) < \infty$) in polynomial time by the following algorithm: Let $c' : V \to \mathbb{Q}_{\geq 0}$ be a new cost function with $c'(v_i) = c(v_i)$ if $i \notin B$ and $c'(v_i) = R$ of $i \in B$. Determine the minimal cost κ needed to separate s from t regarding c' with help of the algorithm solving MCVC for $m = 1$. It can be shown that if $\kappa \geq R$, we have that for all $C \subseteq N \setminus B$, the coalitional function is at most 0 and bribery is possible. If, on the other hand, $\kappa < R$, it can be seen that there exists a minimal winning coalition $C \subseteq N \setminus B$ with $\mu(C) = \kappa$ and $v(C) = R - \kappa > 0$, thus, bribery is not possible.

Secondly, the problem is NP-hard. This can be shown by means of a polynomial-time many-one reduction from PARTITION based on the reduction PARTITION $\leq_m^p$ MAXCUT by Karp [Kar72]: Given an instance $(a_1,\ldots,a_n)$, we obtain the MAXCUT instance consisting of the complete graph $G' = (V',E')$ and the edge weight function $w : E' \to \mathbb{N} \setminus \{0\}$ with $w(\{v_i,v_j\}) = a_i \cdot a_j$, and $K = S^2/4$ with $S = \sum_{i=1}^n a_i$. Obviously, the MAXCUT property is satisfied if and only if A belongs to PARTITION. Next, given A and G', we create the following instance X of PDG-BRIBERY in polynomial time. The path-disruption game consists of graph $G = (V,E)$, where $V = V' \cup \{v_{n+1}, v_{n+2}\} \cup \{v_{n+2+i}, v_{2n+2+i} \mid 1 \leq i \leq n\} \cup$

$\{v_{3n+2+j} \mid e_j \in E', 1 \leq j \leq {n(n-1)}/{2}\}$, $E = \{\{u, v_{3n+2+j}\}, \{v_{3n+2+j}, v\} \mid \{u,v\} = e_j \in E'\} \cup \{\{v_{n+1}, v_{n+2+i}\}, \{v_{n+2+i}, v_i\} \mid 1 \leq i \leq n\} \cup \{\{v_i, v_{2n+2+i}\}, \{v_{2n+2+i}, v_{n+2}\} \mid 1 \leq i \leq n\}$ and furthermore of source vertex $s = v_{n+1}$, target vertex $t = v_{n+2}$, reward $R = \left({S^2}/{2}\right) + S$, and cost function $c : V \to \mathbb{Q}_{\geq 0}$, defined by

$$c(v_i) = \begin{cases} R & \text{if } 1 \leq i \leq n+2 \\ a_j & \text{if } n+3 \leq i \leq 2n+2,\, i = n+2+j \\ a_j \cdot \left(\frac{S}{2} + 1\right) & \text{if } 2n+3 \leq i \leq 3n+2,\, i = 2n+2+j \\ w(e_j) & \text{if } 3n+3 \leq i \leq n',\, i = 3n+2+j \end{cases}$$

with $n' = 3n+2+{n(n-1)}/{2}$. Moreover, let $K = {S}/{2}$ and let the price function $\pi : V \to \mathbb{Q}_{\geq 0}$ be defined by

$$\pi(v_i) = \begin{cases} K+1 & \text{if } 1 \leq i \leq n+2 \\ a_j & \text{if } n+3 \leq i \leq 2n+2,\, i = n+2+j \\ K+1 & \text{if } 2n+3 \leq i \leq n'. \end{cases}$$

We briefly describe how the equivalence $(a_1, \ldots, a_n)$ is in PARTITION if and only if bribery is possible in X is shown.

Only if: Suppose there is a subset $A' \subseteq \{1, \ldots, n\}$ with $\sum_{i \in A'} a_i = {S}/{2}$. Then bribery is possible for coalition $B = \{m+2+i \mid i \in A'\} \subseteq N = \{1, \ldots, n'\}$: Note that $\sum_{m+2+i \in B} \pi(v_{m+2+i}) = K$. The fact that $v(C) \leq 0$ holds for each coalition $C \subseteq N \setminus B$ can be shown by a careful case-distinction. If $\tilde{v}(C) = 0$, then $v(C) = 0$ by definition. Otherwise, C contains a minimal winning subcoalition $C' \subseteq C$ with $\tilde{v}(C') = 1$ and $\mu(C) = \sum_{i \in C'} c(v_i)$. It turns out that in every case $\mu(C) \geq R$, thus, bribery is possible.

If: Suppose that there exists a coalition $B \subseteq N$ with $\sum_{i \in B} \pi(v_i) \leq K$ and for all coalitions $C \subseteq N \setminus B$, either $\tilde{v}(C) = 0$ or $\mu(C) \geq R$ holds. Ruling out the elements of B logically, we end up with two main cases for $B \subsetneq \{n+3, \ldots, 2n+2\}$: If $\sum_{i \in B} \pi(v_i) < K$, then we obtain a contradiction; if $\sum_{i \in B} \pi(v_i) = K$, a partition into $A' = \{i \mid n+2+i \in B\}$ and $\{1, \ldots, n\} \setminus A'$ exists. ◻

For probabilistic path-disruption games with costs and a single adversary the NP-hardness lower bound is implied by the special case in Theorem 4.10. Since the verification can be done in polynomial time for a single adversary for each target with a positive probability, NP-membership holds for the single-adversary case.

Corollary 4.11. *For probabilistic path-disruption games with costs and a single adversary,* PDG-BRIBERY *is* NP*-complete.*

Theorem 4.12. *For path-disruption games with costs and multiple adversaries,* PDG-BRIBERY *is* Σ_2^p*-complete.*

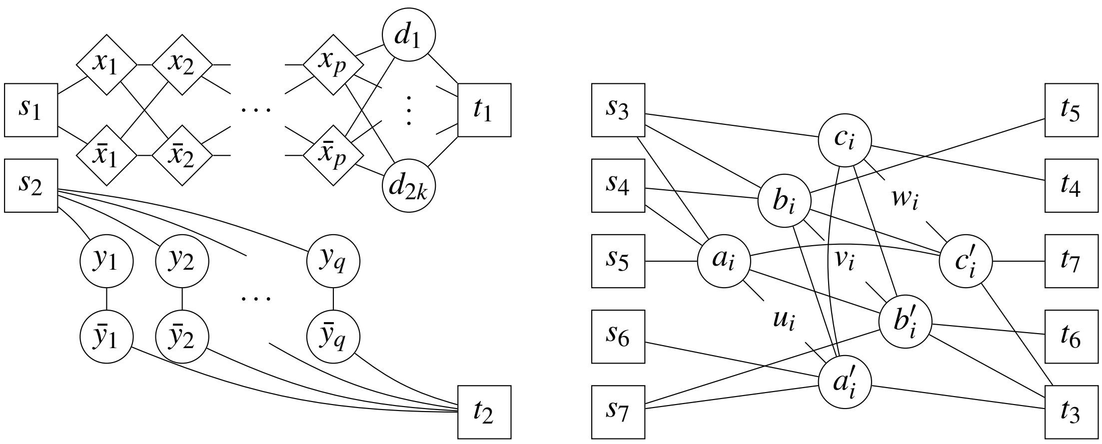
Proof. The problem belongs to Σ_2^p, since it can be written in the corresponding quantifier characterization, see Lemma 2.2: An instance consisting of a path-disruption game $\mathscr{G} =$

(N,v) with multiple adversaries and cost function c, and a reward R, as well as π, and K, belongs to PDG-BRIBERY if and only if

$$(\exists B \subseteq N)(\forall D \subseteq N \setminus B)\left[\sum_{i\in B}\pi(u_i) \leq K \text{ and } \left(\tilde{v}(D) = 0 \text{ or } \sum_{i\in D} c(u_i) \geq R\right)\right].$$

The property in brackets can obviously be tested in polynomial time.

In order to show Σ_2^p-hardness, we reduce from QBF$_2$. Corresponding to the input we are given a formula $F = (\exists X)\,(\forall Y)\,f(X,Y)$, $f(X,Y) = \bigvee_{i=1}^{k}(u_i \wedge v_i \wedge w_i)$, where each implicant i, $1 \leq i \leq k$, has exactly three literals u_i, v_i, and w_i over $X \cup Y$. The graph G for the path-disruption game with multiple adversaries and costs that is part of the PDG-BRIBERY instance to be constructed from F, is built from the three graphs, G_1, G_2, and G_3, shown in Figure 4.5.



(a) Graphs G_1 and G_2 for X and Y variables (b) Graph G_3 for implicant $(u_i \wedge v_i \wedge w_i)$ in f

Figure 4.5: Three graphs for reduction proving Theorem 4.12

In particular, G is constructed from G_1, G_2, and G_3 by identifying, for each occurrence of a literal u_i, v_i, or w_i in f, the vertex in G_3 representing this literal with the vertex representing the corresponding variable ($x \in X$ or $y \in Y$) or its negation ($\bar{x}$ or $\bar{y}$) in G_1 or G_2. Intuitively, the purpose of graph G_1 in Figure 4.5a is to enforce consistency on the part of the briber, the purpose of G_2 in Figure 4.5a is consistency of the coalition, and the purpose of G_3 in Figure 4.5b is to enforce the implicants. The players on vertices in G_1 labelled with X variables or their negations (diamond shape) are bribable for a price of 1 but have 0 cost, sources s_j and targets t_j (rectangle shape) have a cost of $6k+q+1$ and a price of $p+1$, and all other vertices (circle shape) have cost 1 and a price of $p+1$. Let $K = p$ be the briber's budget and $R = 6k+q+1$ be the reward. We show that bribery is possible if and only if the original quantified Boolean formula F is valid.

If: Assume that F is valid, that is, there exists an assignment of the variables in X such that for each assignment of Y there exists an implicant such that each literal is satisfied.

Then, bribery is successful if those players corresponding to x if x is true, and $\bar{x}$ if x is false, are payed: The price limit is met, since p players are bribed. A coalition C can only have a positive value if it contains a successful subcoalition with at most $6k+q$ players other than those on vertices labelled $x \in X$ or $\bar{x}$ that are not bribed, and with no player on a source or target vertex, as their costs are too high to allow $\mu(C) < R$. Since for each $x \in X$, either the player corresponding to x or to $\bar{x}$ is bribed. The other one is free to participate in C. The coalition must include all players on vertices d_i, $1 \leq i \leq 2k$; otherwise, there would be a path from s_1 to t_1. Note that, by construction, for all i, $1 \leq i \leq k$, at least two players on vertices in $\{a_i, b_i, c_i\}$ must be part of the blocking coalition, since otherwise there is either a path from s_4 to t_4 or from s_5 to t_5. Likewise, at least two players on vertices in $\{a'_i, b'_i, c'_i\}$ must participate in C due to paths from s_6 to t_6 or s_7 to t_7. Also notice that if the player on a_i is not in the blocking coalition, the ones on b'_i and c'_i must be because of paths from s_3 to t_3 (and again symmetric statements can be made for a'_i, b_i, etc.). Furthermore, for each $y \in Y$, either the player on y or $\bar{y}$ must be part of the blocking coalition; otherwise, there is a path from s_2 to t_2. Altogether, C includes $6k+q$ vertices with cost 1 each (circle-shaped), leaving for each i, $1 \leq k$, two players on one of $\{a_i, a'_i\}$, $\{b_i, b'_i\}$, or $\{c_i, c'_i\}$ out of C, and for each $y \in Y$, a player on one of y or $\bar{y}$ out as well. Therefore, C represents a consistent assignment to the variables in Y, namely the y or $\bar{y}$ labels of vertices not blocked by C. For each implicant, one path from s_3 to t_3 over either u_i, v_i, or w_i is left to be blocked, which is identified with one $x \in X$ or $\bar{x}$. However, since there is one implicant such that all three literals are satisfied, C cannot succeed. Therefore, bribery is possible.

Only if: Assume that bribery is possible. The briber can bribe up to p players having price 1 (note that all other players not on diamond-shaped vertices are to expensive to bribe). Consider the case where the briber does not play consistently, i.e., either plays (a) neither x nor $\bar{x}$ or (b) both x and $\bar{x}$, for an $x \in X$. In case (b), since the number of bribable players is limited by p, there is some other $x' \in X$ such that case (a) holds. In case (a), a coalition C consisting of the players corresponding to x and $\bar{x}$, to any consistent assignment to the variables in Y, and to vertices a_i, a'_i, b_i, b'_i, c_i, and c'_i for all i, $1 \leq i \leq k$, can form to block all paths from s_j to t_j for $1 \leq j \leq 7$. Since this sums up to $\mu(C) = 6k+q < R$ (i.e., $v(C) > 0$), inconsistent bribery must fail.

Now assume that the briber plays consistently. In this case, suppose that F was not valid. Then for each assignment to the variables in X (corresponding to the possible bribed players), there exists an assignment to the variables in Y such that for each implicant there exists a literal that is not satisfied. Let, without loss of generality, u_i be that literal. Then, the coalition consisting of those players corresponding to variables or their negations in X that are not bribed, of those players corresponding to variables or their negations in Y that are false, and of those players corresponding to b_i, b'_i, c_i, and c'_i for each i, $1 \leq i \leq k$, blocks each path from s_j to t_j for $1 \leq j \leq 7$. This would be a contraction to successful bribery, therefore, F is valid. ◻

In the probabilistic case, again, Theorem 4.12 implies the lower bound immediately. The Σ_2^p upper bound holds obviously by the characterization that a given instance belongs to

PPDGC-MULTIPLE-BRIBERY if and only if

$$(\exists B \subseteq N)(\forall D \subseteq N \setminus B)\left[\sum_{i \in B} \pi(u_i) \leq K \text{ and } (\tilde{v}(D) = 0 \text{ or } c(D) \geq R)\right]$$

by similar arguments as in the previous proof.

Corollary 4.13. *For probabilistic path-disruption games with costs and multiple adversaries,* PDG-BRIBERY *is Σ_2^p-complete.*

4.1.2 Complexity of Stability

Player Properties In a non-simple setting the veto property translates to: a player is a veto player if no coalition has a positive value without it. Nevertheless in the following we will see that in the context of veto players and core stability a probabilistic path-disruption game behaves just like a simple game. We are interested in the questions of verification and existence of veto players as well as the counting problem of how many veto players there are, and the corresponding search problem where the task is to find the veto players. Note that in a probabilistic path-disruption game without costs, a player i is a veto player if and only if it is placed on a vertex with $p_{j,i} = 1$. If at least two vertices have a positive probability of being a target, a player on any of these vertices can be part of a coalition that has a positive value without the other players necessarily being contained. Thus, we can decide in polynomial time whether a given player in a given PPDG without costs is a veto player; testing this property for each of the n players solves the decision of existence in polynomial time, and hence, all veto players can be found and counted in polynomial time. The role of the players placed on the adversaries' source vertices is similar to that of a veto player in a simple game: Every coalition $C \subseteq N$ that contains all players sitting on source vertices has value $v(C) = 1$. For each j, $1 \leq j \leq m$, it holds that $v(N \setminus \{s_j\}) = 1 - p_{j,j}$. The general model does not yield a higher complexity than the original model. In the cost case, these problems are most likely less efficient to solve, since monotonicity cannot be utilized here. Deciding whether a given player is a veto player is in coNP in this case.

For both notions of a null and a dummy player, the verification problem is coNP-complete for probabilistic path-disruption games without costs. The lower bound is inherited by the same result for the non-probabilistic case. The corresponding upper bound holds straightforwardly.

Proposition 4.14. *For both, a null and a dummy player, the problem of whether a given player in a given probabilistic path-disruption game without costs is such a player, is* coNP-*complete.*

Regarding the first notion, null-players, it holds that the Shapley value is 0 if and only if a player is a null player. Hence, the decision whether a given player in a given probabilistic path-disruption game without costs has a positive Shapley value is NP-complete. The hardness proof, of course, also implies coNP-hardness for this problem in the cost case. The best known upper bound for the cost case is Π_2^p. The technique that will be useful for the core in order to gain the first instead of the second level of the polynomial hierarchy cannot be adapted straightforwardly to apply here.

Group Deviation The complexity of problems connected to the core is closely related to those connected to veto players, see Section 2.3.1.

Proposition 4.15. *For a probabilistic path-disruption game with a single adversary and without costs, core verification, core existence, as well as core computation can be solved in polynomial time.*

Proof. Observe that the core of a probabilistic path-disruption game with a single adversary and without costs is non-empty if and only if an agent placed on a vertex with target probability 1. Moreover, in this case, the core consists of only one element. If there is a small probability for at least two targets, the core is empty. Hence, the core can be computed in polynomial time, and it thus can be decided in polynomial time whether the core is non-empty, and also whether a given payoff vector belongs to it. ◻

In respect thereof, the probabilistic model of path-disruption games behaves like a simple game, even though in general it is not. In the multiple-adversary and no-costs case, for a fixed number m of adversaries, deciding whether a payoff vector is in the core of a given game can also be done in polynomial time. On the other hand, if m is not fixed, this cannot be shown straightforwardly. In contrast to the original (non-probabilistic) model of path-disruption games, we suspect this problem (in the no-cost and multiple adversary case) to be coNP-complete. Even with costs the upper bound holds, instead of the second level of the polynomial hierarchy.

Proposition 4.16. *The problem of deciding whether a given payoff vector is in the core of a given probabilistic path-disruption game with costs is in* coNP.

Proof. Let $\vec{q}$ be a given payoff vector, and let a probabilistic path-disruption game with costs be given by a graph $G = (V,E)$, adversaries $s_1,\dots,s_m$ with target probabilities $p_{j,i}$, $1 \leq j \leq m$, $1 \leq i \leq n$, a cost function $c : V \to \mathbb{Q}_{\geq 0}$, and a reward $R \in \mathbb{Q}_{\geq 0}$. Note that for each coalition $C \subseteq N$, there exists a coalition $C' \subseteq C$ with $\mu(C) = \mu(C') = c(C') \leq c(C)$. Therefore,

$$\begin{aligned} R - c(C) \ &\leq \ R - \mu(C) = R - c(C') = v(C') \quad \text{and} \\ \vec{q}(C') \ &\leq \ \vec{q}(C). \end{aligned}$$

Consequently, $\vec{q}$ is in the core of the game implies that

$$R - c(C) \leq R - \mu(C) = v(C) \leq \vec{q}(C),$$

for each $C \subseteq N$. If $\vec{q}$ is not in the core of the game, there exists a coalition $C \subseteq N$ with $v(C) > \vec{q}(C)$. For the corresponding $C' \subseteq N$ it holds that

$$R - c(C') = R - \mu(C) = v(C) > \vec{q}(C) \geq \vec{q}(C').$$

Thus, we only need to test whether $R - c(C) \leq \vec{q}(C)$ for all coalitions $C \subseteq N$, which can be done in coNP. ◻

For the ε-core we study the same question; Given a game $\mathcal{G}$, a payoff vector $\vec{q}$ and a rational bound ε, is the maximal deficit at most ε, or, equivalently, is $\vec{q}$ in the ε-core of $\mathcal{G}$? If only imputations are allowed in the ε-core (as, e.g., Bachrach and Porat require in their definition of the least core), then the least core of a probabilistic path-disruption game with a single adversary and without costs is equal to its core, and thus computable in polynomial time. In general, this does not hold. The following proof extends the analogous proof for the non-probabilistic case [BP10].

Proposition 4.17. *For multiple adversaries and with or without costs, it is* coNP*-complete to decide whether a given payoff vector is in the ε-core of a given probabilistic path-disruption game for a given ε.*

Proof Sketch. Testing whether $\max_{C \subseteq N}(v(C) - \vec{q}(C)) \leq \varepsilon$ is equivalent to testing whether for every coalition $C \subseteq N$ it holds that $\vec{q}(C) \geq v(C) - \varepsilon$. Thus, in order to solve the complement of our problem in NP, we can guess a coalition $C \subseteq N$ nondeterministically and test in polynomial time (see Proposition 4.7 and Corollary 4.8) whether $\vec{q}(C) < v(C) - \varepsilon$. We prove coNP-hardness by means of a reduction from the complement of MCVC. Given an MCVC instance X consisting of a graph $G = (V, E)$ and m vertex pairs (s_j, t_j), $s_j, t_j \in V$, weight function $w : V \to \mathbb{N} \setminus \{0\}$, and a bound $K \in \mathbb{N} \setminus \{0\}$, we construct an instance with the same graph G, adversaries sitting on s_j, $1 \leq j \leq m$, and probabilities $p_{j,i} = 1$ if $v_i = t_j$ for $1 \leq j \leq m$ and $v_i \in V$, and $p_{j,i} = 0$ otherwise. Moreover, we have

$$\vec{q} = \left(\frac{w(v_1)}{\sum_{i=1}^{n} w(v_i)}, \ldots, \frac{w(v_n)}{\sum_{i=1}^{n} w(v_i)} \right) \quad \text{and} \quad \varepsilon = 1 - \frac{2K+1}{2\sum_{i=1}^{n} w(v_i)}.$$

Obviously, this construction can be done in polynomial time. Note that $\vec{q}$ is a *pre-imputation*. We now verify that the given instance X is not in MCVC if and only if $\vec{q}$ belongs to the ε-core of the constructed game.

Only if: Suppose that the given instance does not belong to MCVC, that is, for all subsets $V' \subseteq V$ blocking all paths from s_j to t_j, $1 \leq j \leq m$, it holds that $\sum_{v_\ell \in V'} w(v_\ell) > K$. By construction and the condition that all weights and K are natural numbers, it follows that $\sum_{v_\ell \in V'} w(v_\ell) > K + 1/2$. Thus, for all coalitions $C \subseteq N$ with a positive value (that is, for each adversary j, $1 \leq j \leq m$, C blocks each path from s_j to the only possible target), it holds that

$$\sum_{\ell \in C} q_\ell \;=\; \frac{1}{\sum_{i=1}^{n} w(v_i)} \sum_{\ell \in C} w(v_\ell) \;>\; \frac{2K+1}{2\sum_{i=1}^{n} w(v_i)} \;=\; 1 - \varepsilon.$$

This means that
$$v(C) - \vec{q}(C) \;=\; 1 - \sum_{\ell \in C} q_\ell \;<\; 1 - (1-\varepsilon) = \varepsilon.$$
Since the grand coalition N with $e(N) = 0$ has a positive value and each coalition that cannot disrupt all adversaries' paths has a deficit at most 0, the maximal deficit is less than ε, thus in the constructed instance $\vec{q}$ belongs to the ε-core, as desired.

If: Let the maximal deficit of a coalition in the game be at most ε, that is, for all coalitions $C \subseteq N$ it holds that $v(C) - \vec{q}(C) \leq \varepsilon$. In particular, for each coalition with a positive value, we have $v(C) - \vec{q}(C) = 1 - \vec{q}(C) \leq \varepsilon$. Thus, for all subsets of vertices $V' \subseteq V$ blocking each path from s_j to t_j, $1 \leq j \leq m$, we have $\sum_{v_\ell \in V'} q_\ell \geq 1 - \varepsilon$, which implies
$$\sum_{v_\ell \in V'} \frac{w(v_\ell)}{\sum_{i=1}^{n} w(v_i)} \;\geq\; 1 - \left(1 - \frac{2K+1}{2\sum_{i=1}^{n} w(v_i)}\right) \;=\; \frac{2K+1}{2\sum_{i=1}^{n} w(v_i)},$$
which, in turn, implies $\sum_{v_\ell \in V'} w(v_\ell) \geq K + 1/2 > K$. Thus, there is no subset of vertices satisfying the conditions of MCVC.

For the case with costs, note that the coNP lower bound is trivially inherited from the case without costs. On the other hand, the coNP upper bound can be shown similarly as in the proof of Proposition 4.16. ◻

4.2 Challenges and Future Work

We have expanded the notion of path-disruption games by allowing uncertainty about the adversaries' targets and have discussed the complexity of problems related to various solution concepts and other properties of these more general games. As we have seen, although more general (and perhaps, in some situations, somewhat more realistic), these games behave like their restricted variants in terms of complexity of stability problems. Certain problems can still be be solved efficiently, while others are as hard as (yet no harder than) for the original model of path-disruption games.

In addition, we have studied the complexity of a model of bribery in path-disruption games. While bribery without costs is easy, and can be traced back to questions of connectivity, we have shown that bribery with costs in path-disruption games is NP-complete in the single-adversary case and is Σ_2^p-complete in the multiple-adversary case.

Special Graph Classes In contrast to problems concerning negative influences, for stability verification or existence, a lower complexity is rather desirable. In natural settings, input graphs often have certain properties. From an algorithmic point of view it might be interesting to analyse whether there are special instances that are tractable. Therefore, future work might want to investigate path-disruption games on special classes of graphs. For example, Bachrach and Porat [BP10] already analyse path-disruption games on trees with the result that very often problems that are hard in general become solvable in polynomial time for trees. For the probabilistic variant we obtain similar results.

Proposition 4.18.
- *For probabilistic path-disruption games without costs on trees and on complete graphs, dummy player verification becomes solvable in polynomial time. Similarly null player verification and the problem of determining the Shapley value and a non-simple version of the Penrose–Banzhaf index are easy for this restriction.*
- *The problem of ε-core verification in probabilistic path-disruption games becomes solvable in polynomial time if there are no costs and the game's domain is restricted to be a complete graph.*

Additionally to trees and complete graphs, *planar graphs* might be worth studying or graph properties that can often be found in real life networks, like *small worlds* [WS98]. We suspect that PDGC-MULTIPLE-BRIBERY is NP-complete when restricted to planar graphs, in contrast to the general problem for which we showed Σ_2^p-completeness in Theorem 4.12. Still, this would mean the problem is computationally intractable.

Open Questions For future work, it might be interesting to determine the complexity of problems not settled yet as defined in this section and to consider other problems related to solution concepts for path-disruption games and probabilistic path-disruption games.

Moreover, one might want to vary the model of bribery and to study the resulting problems in terms of their complexity. Even without costs, if the cardinality of a blocking coalition is limited, a bribery problem might become harder again. In the context of voting, variations of bribery in elections are, e.g., the following. In *microbribery* [FHH09, FHHR09], a briber can bribe single positions in a vote selectively instead of the whole vote. Here a coalition might have different prices within the coalition for different reactions to certain probabilities of a vertex to be unblocked. In *swap bribery* [EFS09] and similarly *shift bribery* [BFNT15, BCF$^+$14b], selected positions in a vote can be switched or selected candidates shifted to another position. Here, a restriction might be that a coalition is only bribable for a certain combination of vertices to be unblocked at once. Moreover, there are various versions of *campaign management* [SFE11, FRRS15, FR16]. In these scenarios bribery does not have a bad taste only; resources can be used reasonably to subsidize a generally accepted outcome. Here, the adversary might, for instance not be a malicious intruder, but rather an antidote that is carried through a network in which it costs different effort to oppose a virus. In the context of path-disruption games, another variation might be to consider multiple, independently concurring bribes; another one to define the costs of blocking a vertex in a graph and the prices for bribing the corresponding agents in relation to each other. This might be analysed in connection with the stability of the game and might lead to a new perspective on the topic.

5 Hedonic Games: Axiomatic Properties and Stability

In this chapter, we study different representations of hedonic games. Section 5.1 deals with wonderful stability in enemy-oriented hedonic games, joint work with Rothe, Schadrack, and Schend [RRSS16, RRSS14]. Section 5.2 covers the study of possible and necessary verification and existence in hedonic games with ordinal preferences and thresholds as introduced jointly with Lang, Rothe, Schadrack, and Schend [LRR$^+$15]. In Section 5.3 altruistic hedonic games as recently introduced jointly with Nguyen, Rey, Rothe, and Schend [NRR$^+$16] are presented. Concluding, Section 5.4 contains challenges of all three topics as developed with the same co-authors, respectively. As an illustration, consider the following hedonic game as defined in Section 2.3.2.

Example 5.1. *Let there be five players in $N = \{1,\dots,5\}$ and consider the following profile of preferences as presented here partially. For each $i \in N$, the dots may be filled in by an arbitrary weak order of the remaining coalitions in $\mathscr{N}_i$.*

$$\begin{aligned}
&\succeq_1\colon \{1,2,3\} \succ_1 \dots,\\
&\succeq_2\colon \{1,2,3,5\} \succ_2 \{1,2,3\} \sim_2 \{1,2,5\} \sim_2 \{2,3,5\} \succ_2 \dots,\\
&\succeq_3\colon \{1,2,3,4\} \succ_3 \{1,2,3\} \sim_3 \{1,3,4\} \sim_3 \{2,3,4\} \succ_3 \dots,\\
&\succeq_4\colon \{2,4,5\} \succ_4 \{3,4\} \sim_4 \{4,5\} \succ_4 \dots,\\
&\succeq_5\colon \{2,4,5\} \succ_5 \{2,5\} \sim_5 \{4,5\} \succ_5 \dots.
\end{aligned}$$

In the following we investigate different types of encoding such a preference profile succinctly. The common structure of decision problems is the following.

CORE STABILITY VERIFICATION (CSV) and STRICT CORE STABILITY VERIFICATION (SCSV) are the problems of whether a given coalition structure Γ in a given hedonic game is core-stable (strictly core-stable, respectively), that is whether no coalition is (weakly) blocking Γ. In the literature it is also common to ask the complement question as to whether there exists a (weakly) blocking coalition under the same problem name. We carefully distinguish both notations and use the positive problem naming.

CORE STABILITY EXISTENCE (CSE) and STRICT CORE STABILITY EXISTENCE (SCSE) contain each instance of a hedonic game that allows a (strictly) core-stable coalition structure.

Example 5.2. *For the game in Example 5.1, there exists a core-stable coalition structure, namely* $\{\{1,2,3\},\{4,5\}\}$.

It may be the case that such a stable coalition structure always exists for a certain representation of hedonic games. If not, we are interested in the computational complexity of this decision problem. By definition core-stable verification is a problem in coNP, since the complement problem can be decided in nondeterministic polynomial time by choosing a coalition and verifying whether this coalition blocks the given coalition structure in polynomial time. The corresponding existence problem can be written with an existential quantifier (does there exist a coalition structure) followed by an universal quantifier (such that all coalitions) and a question decidable in polynomial time (do not block the coalition structure), which satisfies the quantifier characterization of Σ_2^p (Lemma 2.2). Therefore, by its nature, CORE STABILITY EXISTENCE belongs to Σ_2^p. Table 5.1 provides an overview of the history of complexity results of six stability problems in three types of hedonic games.

	FRIEND-ORIENTED	ENEMY-ORIENTED	ADDITIVELY SEPARABLE
CSV	• in coNP	• coNP-complete † §	• coNP-complete †
CSE	• in P (always exists) *	• in P (always exists) *	• NP-hard †† • NP-h. (symmetric) ‡ • Σ_2^p-complete §§
SCSV	• in coNP	• coNP-complete † §	• coNP-complete †
SCSE	• in P (always exists) * §	• in Σ_2^p, NP-hard § • DP-hard (Thm. 5.12) ¶ • coDP-h. $\Rightarrow$ Θ_2^p-h. (Cor. 5.63) ¶	• NP-hard †† • in Σ_2^p § • Σ_2^p-complete ‖
WSV		• coNP-complete (Thm. 5.7) ¶	
WSE		• in Θ_2^p § • DP-hard (Thm. 5.10) ¶ • coDP-h. $\Rightarrow$ Θ_2^p-c. (Thm. 5.62) ¶	

* [DBHS06] § [Woe13a]
† [SD07] §§ [Woe13b]
†† [SD10] ‖ [Pet15]
‡ [ABS13] ¶ this thesis ([RRSS14])

Table 5.1: Overview of the history of complexity results of core, strict core, and wonderful stability in friend-based, enemy-oriented, and additively separable hedonic games. Key: hardness is abbreviated by h. and completeness by c., a grey entry signalizes that only trivial lower and upper bounds are known.

For additively separable hedonic games these problems have been studied intensely: Sung and Dimitrov show that CSV and SCSV are coNP-complete [SD07] and CSE and SCSE are NP-hard [SD10]. Aziz et al. [ABS13] confirm this result for the restriction to symmetric values and finally Woeginger [Woe13b] pinpoints the complexity of CSE by showing Σ_2^p-

completeness. The strict variant SCSE has been settled very recently by Peters [Pet15] who shows Σ_2^p-completeness for this problem as well. While in friend-oriented hedonic games a strictly core-stable and in enemy-oriented hedonic games a core-stable coalition structure always exist [DBHS06, Woe13a], SCSE under enemy-oriented preferences is intractable. Woeginger [Woe13a] shows that this problem is NP-hard and contained in Σ_2^p. He, moreover, introduces the notion of wonderful stability (Definition 2.7) as a desirable concept for enemy-oriented hedonic games and presents complexity of its existence problem as a challenging open question. So far, there has been a gap between NP-hardness and Θ_2^p-membership [Woe13a]. We do not close this gap completely, but raise the lower bound to DP-hardness and show that coDP-hardness would be sufficient to prove Θ_2^p-completeness. In Section 5.1 we will elaborate on these complexity results of wonderful stability verification and existence, summarized in the third part of the table, and sketch the related hardness proofs for strict core stability existence.

Additionally to questions of group deviations, we consider the verification and existence problems of other stability concepts such as stability due to no deviations by single players and coalitions structures that stand out in direct comparison to others, as defined in Section 2.3.2.

Example 5.3. *For the game in Example 5.1, the coalition structure* $\{\{1,2,3\},\{4,5\}\}$ *is also Nash-stable and strictly popular. Indeed, it satisfies every concept with one exception, it is not perfect. There is no perfect coalition structure in this game.*

Next to networks of friends, we are also interested in a refinement where players are ranked. For the singleton encoding, Cechlárová and Romero-Medina [CR01] introduce $\mathscr{B}$- and $\mathscr{W}$-preferences. For $\mathscr{B}$-preferences, they show that if rankings are strict, a strict core-stable coalition structure always exists. In contrast to that, if weak rankings are allowed, the existence problem is NP-complete [CH03]. Cechlárová and Hajduková also show that the corresponding verification problems are in P. For $\mathscr{W}$-preferences, Cechlárová and Romero-Medina show that there are similarities to stable roommate problems [Cec02]; in the strict case, the existence problems for the core and the strict core are tractable, however core existence is NP-complete in general [CH04]. The strict-core case is open. Aziz et al. [AHP12] study single player deviations for both extensions amongst others. They show that the existence and verification problems for Nash stability are NP-complete for both strict and weak rankings in the case of $\mathscr{B}$-preferences. For individual stability this only holds in general, but not in the strict case, which is left open. For $\mathscr{W}$-preferences Nash stability existence can be decided in polynomial time for the strict case and is open in general. In all other cases, individually and contractually individually stable coalition structures always exist. Pareto-optimality has been studied by Aziz et al. [ABH13] who show that existence and verification are tractable for $\mathscr{W}$-preferences and intractable for $\mathscr{B}$-preferences.

Combining these two encodings, we propose representing players' preferences ordinally and with a double threshold dividing co-players into friends, enemies and neutral players. In order to derive preferences over coalitions, we suggest generalizing Bossong-Schweigert extensions [BS06] to rankings of friends and enemies. Leaving some players but those at the top and the bottom unranked is also inspired by truncated ballots in a voting context [BFLR12]. A related model can be found in the context of matching theory: Responsive preferences are studied in bipartite many-to-one matching markets and consider the composition of one participant to another, although not in distinction of friends or enemies (see, e.g., [Rot85, RS92, Sot12]). In this context of many-to-one matching markets an agent on the one side has *responsive preferences* over assignments of the agents on the other side, if for any two assignments that differ in only one agent, the assignment containing the most preferred agent is preferred.

Since the generalized Bossong-Schweigert extension allows uncertainties between coalitions from a player's point of view, we introduce the notions of possibility and necessity to stability concepts. Computational complexity results for possible and necessary stability verification and existence are summarized in Table 5.2. We define the new representation of games with ordinal preferences and thresholds in Section 5.2 and elaborate on the complexity results in Sections 5.2.1 to 5.2.4.

	Verification		Existence		
γ	Possible	Necessary	Possible	Certain	Necessary
perfection	in P (Prop. 5.32)	in P (Prop. 5.32)	in P (Prop. 5.32)	in P (Prop. 5.32)	in P (Prop. 5.32)
ind. rationality	in P (Prop. 5.33)	in P (Prop. 5.33)	in P (Obs. 5.27)	in P (Obs. 5.27)	in P (Obs. 5.27)
contr. ind. stability	in NP	in P (Thm. 5.34)	in NP	in Π_2^p	in NP
ind. stability	in NP	in P (Thm. 5.34)	in NP	in Π_2^p	in NP
Nash stability	in NP	in P (Thm. 5.34)	NP-complete (Thm. 5.35)	NP-hard, in Π_2^p (Cor. 5.37)	NP-complete (Thm. 5.36)
core stability	coNP-hard, in Σ_2^p (Thm. 5.38)	in coNP	in Σ_2^p	in Π_3^p	in Σ_2^p
str. core stability	coNP-hard, in Σ_2^p (Thm. 5.38)	in coNP	in Σ_2^p	in Π_3^p	in Σ_2^p
Pareto optimality	coNP-hard, in Σ_2^p (Thm. 5.41)	coNP-complete (Thm. 5.41)	in P (Obs. 5.27)	in P (Obs. 5.27)	in Σ_2^p
popularity	coNP-hard, in Σ_2^p (Thm. 5.39)	coNP-complete (Thm. 5.39)	in Σ_2^p	in Π_3^p	in Σ_2^p
str. popularity	coNP-hard, in Σ_2^p (Thm. 5.39)	coNP-complete (Thm. 5.39)	coNP-hard, in Σ_2^p (Thm. 5.40)	coNP-hard, in Π_3^p (Thm. 5.40)	coNP-hard, in Σ_2^p (Thm. 5.40)

Table 5.2: Overview of complexity results of possible and necessary stability verification and existence problems in hedonic games with ordinal preferences and thresholds. A grey entry signalizes that only trivial lower and upper bounds are known. All results are part of this thesis and published in [LRR+15].

In particular, for Nash-stability and despite exponentially many possible extensions in the number of players, verification and existence are not harder than the more restricted models. We also obtain (rather) low completeness results of the verification problems for necessary stability in coalition comparison. We present various lower bounds and put up a number of new questions open to be specified.

All models of representing hedonic games studied in the literature so far are based upon selfish players only. Inspired by the notion of altruism in non-cooperative games (see, e.g., [HS09, CKKS11, AS12, RS13a]), measures of influence in social networks (see, e.g., [RBSG11]), as well as preference extensions and influence in decision making processes (see, e.g., [SB14, GLP15, GVQ15]), we introduce hedonic games with altruistic influences. The underlying encoding is a network of friends as defined in Section 2.3.2 with symmetric friendship relations for stability reasons.

Example 5.4. *Consider the example of four players,* 1*,* 2*,* 3*, and* 4*, and let* 1 *be friends with* 2*, but neither with* 3 *nor with* 4*, while* 2 *and* 3 *are friends with each other, but not with* 4*. The corresponding network is displayed in Figure 5.1. Now, in the friend-oriented*

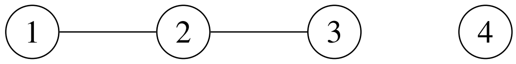


Figure 5.1: Example of a network of friends

extension model player 2 *prefers teaming up with* 1 *and* 3 *to forming a coalition with* 1 *and* 4*. Player* 1*, on the other hand, is indifferent between coalitions* $\{1,2,3\}$ *and* $\{1,2,4\}$*, because they both contain the same number of* 1*'s friends and the same number of* 1*'s enemies. Intuitively, however,* 1 *would have an advantage from being in a coalition with* 2 *and* 3*, since* 2 *and* 3*—being friends—can be expected to cooperate better than* 2 *and* 4*. Also,* 1 *can be expected to care about her friend* 2*'s interests and thus might prefer a coalition in which* 2 *is satisfied (*$\{1,2,3\}$*) to one in which* 2 *is less satisfied (*$\{1,2,4\}$*).*

Since such an advantage as expressed in this example cannot be expressed by friend-oriented preferences, we propose to refine friend-oriented hedonic games, in which this idea of players caring about their friends' preferences is taken into account. We define degrees of altruism, from being selfish at first, over aggregating opinions of a player and her friends equally, to altruistically letting one's friends decide first. The latter is the most altruistic case, as we assume that from a player's perspective only friends can be consulted, while agents further away cannot be communicated with or cannot be trusted. In a social network, for example, the whole set of players other than friends might not even be known, rather than being enemies.

Taking friends' opinions into account does not contradict the idea of hedonic games; on the contrary: In hedonic games player i's happiness depends only on the coalition that contains i. Here, only the notion of happiness is redefined; as i is also interested in her friends' satisfaction (with varying degrees), this idea is now part of i's utility. Still, this utility only depends on the coalition containing i by only considering friends' opinions within that same coalition.

The proposed games satisfy a number of desirable properties. They are compactly representable but not fully expressive. However, they can express different hedonic games than those representable by popular compact models in the literature (see Section 2.3.2).

Once more, we study the complexity of verification and existence problems for several stability concepts. Related work for friend-oriented hedonic games has been described above in the context of hedonic games with ordinal preferences and thresholds. Our complexity results for the selfish-first, equally treated, and altruistic preferences can be found in Table 5.3.

	VERIFICATION			EXISTENCE		
γ	SF	EQ	AL	SF	EQ	AL
perfection ¶	in P			in P		
ind. rationality ¶	in P	in P	in P	in P	in P	in P
contr. ind. stability ¶				in P (always exist, Cor. 5.55)		
ind. stability ¶	in P (Prop. 5.53)					
Nash stability ¶				in P (always exist, Thm. 5.54)		
core stability	in coNP			in P (5.56)	in Σ_2^p	
str. core stability						
Pareto optimality				in P	in P	in P
popularity						
str. popularity (Thm. 5.58) ¶	coNP-complete	in coNP		coNP-hard, in Σ_2^p	in Σ_2^p	

¶ this thesis ([NRR+16])

Table 5.3: Overview of complexity results of stability verification and existence problems in hedonic games with altruistic influences. A grey entry signalizes that only trivial lower and upper bounds are known.

5.1 Complexity of Wonderful Stability in Enemy-Oriented Hedonic Games

The following analysis has been worked on in collaboration with Rothe, Schadrack, and Schend [RRSS16, RRSS14]. In this thesis, we will address the problems related to wonderful stability as well as sketch associated results for strict core stability.

The problems of (strict) core stability verification and existence translate to the analogous questions for a wonderfully stable coalition structure in an enemy-oriented hedonic game represented by an undirected graph. Undirected edges signify that friendship relations are symmetric. Note that we can make use of this restriction without loss of generality, since, in the context of stability, a single directed friendship is equivalent to no friendship relation at all between two players (see, e.g., [Woe13a]).

WONDERFUL STABILITY VERIFICATION (WSV)	
Given:	An undirected graph (V,E) and a partition into cliques Π of V.
Question:	Is Π wonderfully stable?

WONDERFUL STABILITY EXISTENCE (WSE)	
Given:	An undirected graph (V,E).
Question:	Does (V,E) allow a wonderfully stable partition?

Example 5.5. *Consider the players from Example 5.1 who are part of the following network of games[1]. Extending preferences in an enemy-oriented way, we obtain a corresponding profile.*

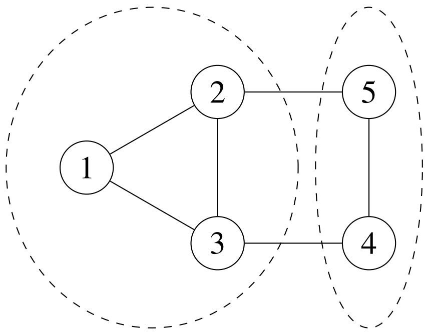


Figure 5.2: Example of a wonderfully stable partition in a graph corresponding to an enemy-oriented hedonic game

It can be observed that the strictly core-stable coalition structure $\{\{1,2,3\},\{4,5\}\}$ *(Example 5.2) even corresponds to a wonderfully stable partition in this graph.*

The following useful property holds for graphs consisting of several independent components by definition.

Property 5.6. *Let G be an undirected graph consisting of k independent components G_i $1 \leq i \leq k$. There exists a wonderfully stable partition Π for G if and only if there exist wonderfully stable partitions Π_i for all components G_i, $1 \leq i \leq k$. In particular, in that case it holds that $\Pi = \bigcup_{i=1}^{k} \Pi_i$.*

In order to establish the complexity of WONDERFUL STABILITY VERIFICATION, we make use of the same proof technique that Sung and Dimitrov [SD07] used for the core stability problem in hedonic games with enemy-oriented preferences.

Theorem 5.7. WONDERFUL STABILITY VERIFICATION *is* coNP-*complete.*

Proof. Just as for core stability, the verification problem for wonderfully stable partitions belongs to coNP due to the characterization that a given partition into cliques Π of the vertices in a given graph is not wonderfully stable if and only if there exists a clique in the graph that blocks Π. In contrast to core stability, here, blocking only depends on a single

[1] Note that in figures of networks we use player names instead of vertex names for a better readability.

vertex. Hence, for a clique P in the graph it can be verified in polynomial time whether its cardinality is greater than the size of the current clique in Π of any vertex in P.

Hardness for coNP is shown via a reduction from CLIQUE as in [SD07]. Given an instance of CLIQUE (which, for an undirected graph $G = (V,E)$ and a positive integer k, asks whether G has a clique of size at least k), we construct the following graph $G' = (V',E')$. The vertex set V' is obtained from V by adding, for each $v \in V$, $k-2$ vertices. We connect each of the $k-2$ new vertices and v to form a clique of size $k-1$, for each $v \in V$. The edge set E' consists of these new edges and all edges in E. Let Π be the partition into $\|V\|$ cliques such that each $(k-1)$-clique as constructed above forms one part. This can obviously be achieved in polynomial time. We claim that there is a clique of size k in G if and only if there exists a clique $P \subseteq V'$ that blocks Π in G', namely, Π is not wonderfully stable.

Only if: If there is a clique P of size k in G, the same clique can be found in G'. The vertices $v \in P$ thus have a clique number $\omega_{G'}(v)$ of at least k. Since the size of all cliques in Π is $k-1$, there exists a vertex v in the clique P with $\omega_{G'}(v) \geq k > \|\Pi(v)\|$; therefore, P blocks Π in G'.

If: If there is no clique of size k in G, there is no clique of size k in G', either, and $\omega_{G'}(v) = k-1$ holds for each $v \in V'$. Furthermore, $\|\Pi(v)\| = k-1$, for each $v \in V'$. Thus, there is no blocking clique for Π in G'. ◻

Woeginger [Woe13a] studies WONDERFUL STABILITY EXISTENCE and describes the gap between the lower bound of NP-hardness and an upper bound of Θ_2^p as a challenging question. He conjectures that WSE is Θ_2^p-complete.

We improve the lower bound by showing DP-hardness in three steps. Firstly, we show coNP-hardness.

Theorem 5.8. WONDERFUL STABILITY EXISTENCE *is* coNP-*hard.*

Proof. Again, we reduce from CLIQUE to the complement of WSE. Given an instance (G,k) of CLIQUE, we construct the same graph G' as in the proof of Theorem 5.7 as an instance for the complement of WSE. We may assume that $k \geq 3$; otherwise, we would test in polynomial time whether E is empty or not and reduce to an appropriate trivial instance. We now show that there is a clique of size k in G if and only if there is no wonderfully stable partition for G'.

Only if: If there is a clique P of size k in G, the same clique can be found in G'. As in the proof of Theorem 5.7, P blocks the partition that consists of the $\|V\|$ cliques of size $k-1$ constructed in the reduction or any other partition. On the other hand, if a partition contains P, then each of the $(k-1)$-cliques mentioned above blocks this partition, since the new vertices are now in a clique of size at most $k-2$, but their clique number is $k-1$.

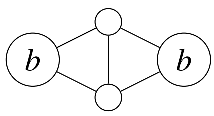
If: If there is no clique of size k in G, the partition as in the proof of Theorem 5.7 is wonderfully stable, since there is no blocking clique. ◻

Secondly, we show that the problem is also NP-hard, a fact already mentioned without proof by Woeginger [Woe13a]. Combining the two results, it can be implied, that WSE is unlikely to be in either NP or coNP, unless the polynomial hierarchy collapses.

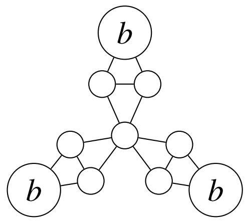
Theorem 5.9 ([Woe13a]). WONDERFUL STABILITY EXISTENCE *is* NP-*hard.*

Proof. We show NP-hardness via a reduction from XC_3 in the restricted variant where each element in B occurs at most three times, for any instance $(B, \mathscr{S})$. Furthermore, we can assume that each element occurs at least once; otherwise, we would reduce to a trivial no-instance of WSE.

Given such an instance, we construct the following graph $G = (V, E)$. For each $S_i \in \mathscr{S}$, add three vertices to V that are connected to each other as a 3-clique. Label the three vertices with the three elements of S_i. For each element $b \in B$, consider the following three cases. Firstly, if b occurs only once in a set of $\mathscr{S}$, no changes are made. Secondly, if b occurs twice, the subgraph in Figure 5.3a is inserted between the two vertices labelled with b. Third, if b occurs three times, the subgraph in Figure 5.3b is inserted between the three vertices labelled with b. Since it is easy to determine how often an element of B occurs in a set of $\mathscr{S}$ and the number of new vertices is limited by $7\|B\|$, G can be constructed in polynomial time.



(a) Two vertices



(b) Three vertices

Figure 5.3: Construction between vertices labelled $b \in B$ for the proof of Theorem 5.9

We now show that there is an exact cover of B by sets in $\mathscr{S}$ if and only if there is a wonderfully stable partition for G.

Only if: If there exists an exact cover of B by $k = \|B\|/3$ sets in $\mathscr{S}$, we include the 3-cliques corresponding to these sets into the partition Π that shall be wonderfully stable. The remaining vertices (those from the inserted connecting subgraphs, and those corresponding to the S_i that are not part of the exact cover) are distributed as follows. Again, consider the three cases of occurrence: If an element b occurs only once, the only vertex labelled with b is already in a clique in Π. If an element b occurs twice, one vertex labelled b remains. This vertex forms a 3-clique with the two connecting vertices as in Figure 5.3a. In this case, we put this 3-clique into Π. If an element b occurs three times, two vertices with the same label remain. From the structure of the connecting subgraph as in Figure 5.3b, the two vertices connected to the vertex that is already in a part of the partition, form a 3-clique with the vertex in the middle. The other two pairs of vertices again form 3-cliques with the remaining vertices labelled b. If these 3-cliques are added to Π, the partition is complete. It remains to show that Π is wonderfully stable. Since each part of Π is a clique of size 3 and each vertex in G has a clique number of 3, the conditions for a wonderfully stable partition are satisfied.

If: If there exists a wonderfully stable partition Π in G, all cliques in Π have size 3, since by construction each vertex $v \in V$ has a clique number $\omega_G(v) = 3$. Since the connecting subgraphs from Figures 5.3a and 5.3b are constructed such that exactly one labelled vertex is not part of a 3-clique, we have that, for each element $b \in B$, the one corresponding vertex has to be part of another 3-clique that does not contain an unlabelled vertex. Thus, there exist exactly $\|B\|/3$ cliques that consist of three labelled vertices, corresponding to sets in $\mathscr{S}$ in which each element of B occurs exactly once. That is, there exists an exact cover of B in $\mathscr{S}$. ◻

Thirdly, we combine the two latter results to prove DP-hardness of WSE with help of Wagner's Lemma 2.1.

Theorem 5.10. WONDERFUL STABILITY EXISTENCE *is* DP-*hard.*

Proof. Again, consider the NP-hard problem XC_3. Given two instances of XC_3, $(B_1, \mathscr{S}_1)$ and $(B_2, \mathscr{S}_2)$, where $(B_2, \mathscr{S}_2) \in XC_3$ implies $(B_1, \mathscr{S}_1) \in XC_3$, we construct the following graph $G = (V, E)$. G consists of two disconnected subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, that is, $G = (V_1 \cup V_2, E_1 \cup E_2)$. G_1 is obtained from $(B_1, \mathscr{S}_1)$ by the construction given in the proof of Theorem 5.9. G_2 is built in two steps. In the first step, the XC_3 instance $(B_2, \mathscr{S}_2)$ is transformed into an instance of CLIQUE: For each set $S_i \in \mathscr{S}$, create a vertex v_i. If two sets S_i and S_j are disjoint, connect the corresponding vertices by an edge $\{v_i, v_j\}$. Let $k = \|B_2\|/3$. In the second step, add, for each $v \in V$, $k-2$ vertices and edges as in the proof of Theorem 5.8. This construction can obviously be done in polynomial time. Note that, again, the proof only works for $k \geq 3$. If $k \leq 2$, reduce to an appropriate trivial WSE instance. We claim that $(B_1, \mathscr{S}_1) \in XC_3$ and $(B_2, \mathscr{S}_2) \notin XC_3$ if and only if there exists a wonderfully stable partition for G. Note that by the conditions stated above and by Lemma 2.1 this equivalence implies DP-hardness of WSE.

Only if: Suppose it holds that $(B_1, \mathscr{S}_1) \in XC_3$ and $(B_2, \mathscr{S}_2) \notin XC_3$. Since $(B_1, \mathscr{S}_1)$ is in XC_3, G_1 has a wonderfully stable partition by the proof of Theorem 5.9. Since additionally $(B_2, \mathscr{S}_2)$ is not in XC_3, there are no $k = \|B\|/3$ pairwise disjoint sets in $\mathscr{S}$, thus there is no clique of size k in G. By the proof of Theorem 5.8, G_2 then also has a wonderfully stable partition. Since G_1 and G_2 are not connected, that is, the clique number of each vertex remains unchanged ($\omega_G(v) = \omega_{G_1}(v)$ if $v \in V_1$, and $\omega_G(v) = \omega_{G_2}(v)$ if $v \in V_2$), and since there are no additional vertices in G, G has a wonderfully stable partition as well (see also Property 5.6).

If: Suppose it holds that $(B_1, \mathscr{S}_1) \notin XC_3$ or $(B_2, \mathscr{S}_2) \in XC_3$. If $(B_1, \mathscr{S}_1) \notin XC_3$, then by the proof of Theorem 5.9, there is no wonderfully stable partition for G_1. Thus, by Property 5.6, there is no wonderfully stable partition for G. On the other hand, if $(B_2, \mathscr{S}_2) \in XC_3$, there exists an exact cover of B in $\mathscr{S}$, that is, there are $k = \|B\|/3$ pairwise disjoint sets in $\mathscr{S}$. By construction, these sets are represented by k vertices in G_2, each connected to one another, thus forming a k-clique. By the proof of Theorem 5.8, it follows that there is no wonderfully stable partition for G_2. Again, by Property 5.6, there is no wonderfully stable partition for G either. ◻

The following examples illustrates the construction from the proof of Theorem 5.10.

Example 5.11. $B_1 = \{1,2,3,4,5,6\}$, $\mathscr{S}_1 = \{\{1,2,3\},\{1,2,4\},\{4,5,6\},\{2,4,6\}\}$,
$B_2 = \{1,2,3,4,5,6,7,8,9\}$, $\mathscr{S}_2 = \{\{1,2,3\},\{4,5,6\},\{6,7,8\},\{3,7,9\}\}$.
Figure 5.4 shows the graph G that is constructed from these two instances. The thick edges show a wonderfully stable partition. In G_1 there are no blocking 3-cliques, since each vertex takes part in a 3-clique. In G_2 there is no 3-clique, thus the outer 2-cliques are wonderfully stable.

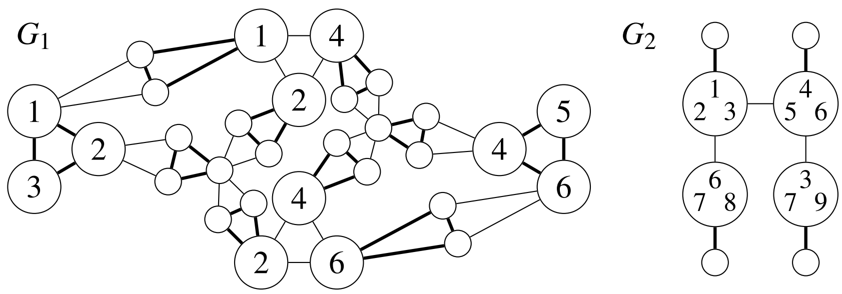


Figure 5.4: Example of DP-hardness construction in proof of Theorem 5.10

Similarly, it can be shown that STRICT CORE STABILITY EXISTENCE for hedonic games with enemy-oriented preferences is DP-hard. This demagnifies the gap between the lower bound and the known Σ_2^p upper bound, but still leaves it open.

Theorem 5.12 ([RRSS14]). *Under enemy-oriented preferences,* STRICT CORE STABILITY EXISTENCE *is* DP*-hard.*

Proof Sketch. As for WSE, coNP-hardness is shown by a reduction from CLIQUE to the complement of SCSE.

Let (G,k) be a CLIQUE instance with a graph $G = (V,E)$ and an integer $k \geq 4$. We construct an SCSE instance represented by the graph $G' = (V',E')$. Let $V' = V \cup V_1 \cup V_2$, where V_1 contains $k-2$ new vertices for each of the vertices $v \in V$ and V_2 contains $k-3$ new vertices for each $v \in V$, such that $\|V'\| = \|V\| + \|V\|(2k-5)$. Every vertex $v \in V$ is connected to its $k-2$ associated vertices from V_1, any two of which are also connected by an edge, thus forming a $(k-1)$-clique with the corresponding vertex v. Moreover, the $k-3$ vertices from V_2 associated with any $v \in V$ are connected to one of the vertices from V_1 in the $(k-1)$-clique containing v, and they are also connected among each other, thus forming a $(k-2)$-clique with the single vertex v' from V_1 they are connected to. E' contains all edges from E and the additional edges described above. See Figure 5.5 for an illustration.

It can be seen that G has a clique of size at least k if and only if there is no strictly core-stable coalition structure in the game $\mathscr{H}'$ represented by G'.

Recalling Lemma 2.10, we know that in graphs where all vertices have the same fixed clique number, every wonderfully stable partition Π of G corresponds also to a strictly core-stable coalition structure in the game represented by G, and vice versa. Hence, NP-hardness

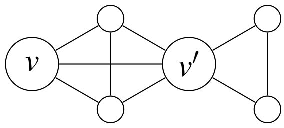


Figure 5.5: Construction for proof of Theorem 5.12: Connecting vertices from V_1 and V_2 to $v \in V$ for $k = 5$

for SCSE follows straightforwardly from the NP-hardness proof for WSE, see the proof of Theorem 5.9.

DP-hardness of SCSE then follows, analogously to the proof of Theorem 5.10, with the help of Wagner's Lemma 2.1. ☐

Consider the class of graphs where all vertices have the same fixed clique number k. We can show NP-membership of WSE restricted to instances of this graph class (denoted by k-WSE). Together with a lower bound that follows from the construction for proving Theorem 5.9, this implies NP-completeness.

Theorem 5.13. *For $k \geq 3$, k-*WSE *is* NP-*complete.*

Proof. By assumption, all vertices in the given graph G have the same clique number k. The graph has to have $\ell \cdot k$ vertices for some $\ell \in \mathbb{N}$; otherwise, a wonderfully stable partition could never be found. Thus, the problem of deciding whether G has a wonderfully stable partition is equivalent to the problem of deciding whether there is a clique cover of size ℓ for G, which is an NP-complete problem, see Section 2.2. That means, k-WSE belongs to NP, since a partition of the vertices into ℓ sets can be guessed non-deterministically and tested for whether each set is a clique.

For the lower bound, it follows from Theorem 5.9 that WSE on graphs with a fixed clique number of $k = 3$ is NP-hard. We can extend this NP-hardness to any fixed clique number $k \geq 3$ by reducing k-WSE to $(k+1)$-WSE. We may assume that an instance for k-WSE has $\ell \cdot k$ vertices (otherwise, we reduce to a trivial no-instance). Given such a graph, we construct an instance of $(k+1)$-WSE by adding ℓ vertices to the original graph. We connect each new vertex to each original vertex and leave the new vertices unconnected among each other. It is easy to see that there is a wonderfully stable partition into ℓ k-cliques in the original graph if and only if there is a wonderfully stable partition into ℓ cliques of size $(k+1)$ each in the constructed graph. ☐

Since by Lemma 2.10 the problems WSE and SCSE are equivalent for graphs in this class, NP-completeness also holds for the analogously restricted variant of SCSE to games on graphs with a fixed clique number of at least 3.

5.2 Representing Hedonic Games with Ordinal Preferences and Thresholds

The following model is introduced in a conference contribution jointly with Lang, Rothe, Schadrack, and Schend [LRR+15]. We suggest the following representation combining ideas from the friends-and-enemies encoding and the singleton encoding (see Section 2.3.2). Each player partitions the other players into three sets: friends, enemies, and other players she considers as neutral, does not know, or care about. For further refinements, additionally, a weak ranking over the set of friends and the set of enemies is expressed.

Definition 5.14. *Let $N = \{1,\dots,n\}$ be a set of agents. For each $i \in N$, a* weak ranking with double threshold for agent i*, denoted by $\trianglerighteq_i^{+0-}$, consists of a partition of $N \smallsetminus \{i\}$ into three sets:*

- *N_i^+ (i's* friends*), together with a weak order $\trianglerighteq_i^+$ over N_i^+,*
- *N_i^- (i's* enemies*), together with a weak order $\trianglerighteq_i^-$ over N_i^-, and*
- *N_i^0 (the* neutral agents *for i).*

We also write $\trianglerighteq_i^{+0-}$ in the form $(\trianglerighteq_i^+ \mid N_i^0 \mid \trianglerighteq_i^-)$.

If player i is indifferent between a subset of friends or a subset of enemies $X = \{a_1,a_2,\dots,a_x\}$, let $X_\sim$ denote $a_1 \sim_i a_2 \sim_i \cdots \sim_i a_x$ in i's weak ranking with double threshold. For each player i, the neutral agents are not ordered which signifies indifference among them; all friends are preferred to neutral agents, and those to all enemies. This induces a *weak order induced by $\trianglerighteq_i^{+0-}$* defined via $f \triangleright_i j$, for each $f \in N_i^+$ and $j \in N_i^0$, $j \sim_i k$, for each $j,k \in N_i^0$. and $j \triangleright_i e$, for each $j \in N_i^0$ and $e \in N_i^-$.

Example 5.15. *Consider the set of players $N = \{1,\dots,10\}$. An example of a weak ranking with double threshold for agent* 1 *is*

$$\trianglerighteq_1^{+0-} \;=\; (2 \triangleright_1 3 \sim_1 6 \mid \{4,7,10\} \mid 5 \triangleright_1 8 \sim_1 9).$$

In other words, player 1 *likes* 2*,* 3*, and* 6 *of which she prefers* 2 *and is indifferent between the other two;* 1 *does not like* 5*,* 8 *and* 9 *of which the latter two are equally worse than* 5*; and* 1 *does not know or care about the other three players. The induced weak order is* $2 \triangleright_1 3 \sim_1 6 \triangleright_1 7 \sim_1 4 \sim_1 10 \triangleright_1 5 \triangleright_1 8 \sim_1 9$.

Generalizing the Bossong–Schweigert extension principle [BS06], we can extend the given preferences of the players to a preference over the relevant coalitions. Note that this preference over coalitions might be incomplete; there may be coalitions that remain incomparable.

Definition 5.16. *Let $\trianglerighteq_i^{+0-}$ be a weak ranking with double threshold for agent i. The* extended order $\succeq_i^{+0-}$ *is defined as follows: For every $C, D \subseteq N$, $C \succeq_i^{+0-} D$ holds if and only if the following two conditions hold:*

1. *There is an injective function $\sigma : D \cap N_i^+ \to C \cap N_i^+$ such that each $y \in D \cap N_i^+$ satisfies $\sigma(y) \trianglerighteq_i y$.*
2. *There is an injective function $\theta : C \cap N_i^- \to D \cap N_i^-$ such that each $x \in C \cap N_i^-$ satisfies $x \trianglerighteq_i \theta(x)$.*

Finally, $C \succ_i^{+0-} D$ holds if and only if $C \succeq_i^{+0-} D$ and not $(D \succeq_i^{+0-} C)$ hold.

Intuitively, for a player i, the best coalitions are the ones containing all friends of i's and no enemies, the worst containing all enemies and no friends, and in between more and better friends as well as fewer and less unpopular enemies implicate an improvement.

Example 5.17. *Consider $N = \{1,2,3,4,5,6\}$ and the first players' weak ranking with double threshold $\trianglerighteq_1^{+0-} = (2 \triangleright_1 3 \sim_1 6 \mid \{4\} \mid 5)$. Figure 5.6 illustrates the just presented extension principle.*

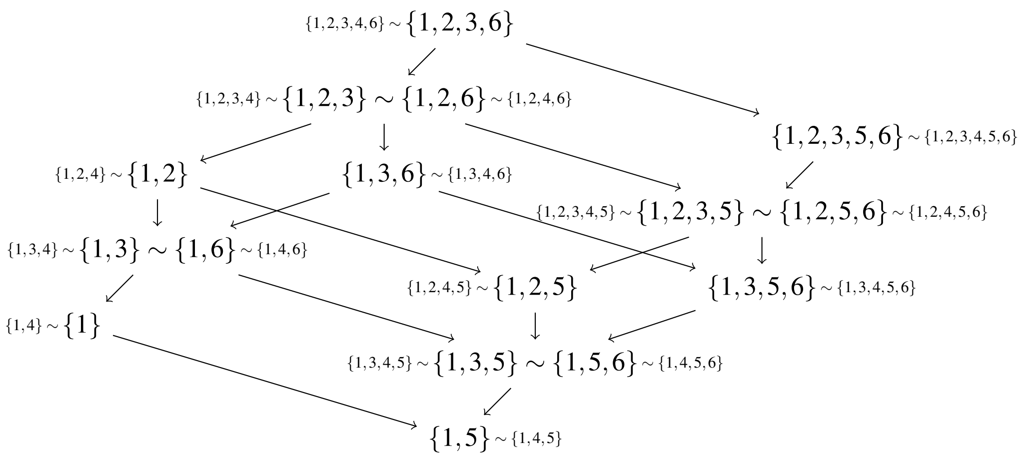


Figure 5.6: Example of the generalized Bossong-Schweigert extension of preference $\trianglerighteq_1^{+0-} = (2 \triangleright_1 3 \sim_1 6 \mid \{4\} \mid 5)$. A downward arc means player 1 prefers the upper coalition to the lower one according to $\succeq_1^{+0-}$. Transitivity is implied.

It can be seen, that in a symmetric way, coalitions with better friends are preferred to less good friends, more friends are preferred to fewer friends, but there is no comparison in between, e.g., $\{1,2\}$ and $\{1,3,6\}$. Also, a coalition is preferred without the enemy, but we cannot compare a coalition to one with more friends and more enemies at the same time, e.g., $\{1\}$ and $\{1,3,5\}$. Adding the neutral player does not make any difference.

The exact relation between two coalitions C and D ($C \succ_i D$, $D \succ_i C$, $C \sim_i D$, or undecided) from player i's point of view can be determined in polynomial time in the number of players by the following characterizations. These propositions are inspired by Bouveret et al. [BEL10] who characterize the original Bossong–Schweigert order in the context of fair division.

Proposition 5.18. *1. Let $\trianglerighteq_i^{+0-}$ be a weak ranking with double threshold for agent i, and let C and D be two coalitions containing i. Consider the orders $f_1 \trianglerighteq_i f_2 \trianglerighteq_i \cdots \trianglerighteq_i f_\mu$ with $\{f_1, f_2, \ldots, f_\mu\} = C \cap N_i^+$ and $f_1' \trianglerighteq_i f_2' \trianglerighteq_i \cdots \trianglerighteq_i f_{\mu'}'$ with $\{f_1', f_2', \ldots, f_{\mu'}'\} = D \cap N_i^+$, as well as $e_1 \trianglerighteq_i e_2 \trianglerighteq_i \cdots \trianglerighteq_i e_\nu$ with $\{e_1, e_2, \ldots, e_\nu\} = C \cap N_i^-$ and $e_1' \trianglerighteq_i e_2' \trianglerighteq_i \cdots \trianglerighteq_i e_{\nu'}'$ with $\{e_1', e_2', \ldots, e_{\nu'}'\} = D \cap N_i^-$. Then, $C \succeq_i^{+0-} D$ if and only if*

a) $\mu \geq \mu'$ and $\nu \leq \nu'$,

b) for each k, $1 \leq k \leq \mu'$, it holds that $f_k \trianglerighteq_i f_k'$, and

c) for each ℓ, $1 \leq \ell \leq \nu$, it holds that $e_{\nu-\ell+1} \trianglerighteq_i e_{\nu'-\ell+1}'$.

2. Let $w_i : N \to \mathbb{R}$ be compatible *with $\trianglerighteq_i^{+0-}$ if and only if*

- *for each $j \in N_i^+$, we have $w_i(j) > 0$;*
- *for each $j \in N_i^-$, we have $w_i(j) < 0$;*
- *for each $j \in N_i^0$, we have $w_i(j) = 0$; and*
- *for all $j, k \in N_i^+ \cup N_i^-$, we have $j \triangleright_i k$ if and only if $w_i(j) > w_i(k)$.*

Then, $C \succeq_i^{+0-} D$ holds if and only if any w_i compatible with $\trianglerighteq_i^{+0-}$ satisfies $\sum_{j \in C} w_i(j) \geq \sum_{j' \in D} w_i(j')$.

Proof. Each of these two characterizations show polynomial computability of $\succeq_i^{+0-}$:

1. Obviously, if (a) to (c) hold, the two injective functions $\sigma : D \cap N_i^+ \to C \cap N_i^+$, and $\theta : C \cap N_i^- \to D \cap N_i^-$ mapping $f_k' \mapsto f_k$ for each k, $1 \leq k \leq \mu'$, and $e_{\nu-\ell+1} \mapsto e_{\nu'-\ell+1}'$ for each ℓ, $1 \leq \ell \leq \nu$, satisfy $\sigma(f_k') \trianglerighteq_i f_k'$ and $e_{\nu-\ell+1} \trianglerighteq_i \theta(e_{\nu-\ell+1})$, for the same range of k and ℓ. On the other hand, if there are two injective functions with the desired requirements, (a) holds. If there was a k with $f_k' \triangleright_i f_k$ (or an ℓ with $e_{\nu'-\ell+1}' \triangleright_i e_{\nu-\ell+1}$), this would imply $\sigma(f_k') = f_j$ for a $j < k$ (or $\theta(e_{\nu-\ell+1}) = e_{\nu-j+1}'$ with $j > \ell$, respectively). This, however, implies that either a requirement is violated for f_1' (or e_ν), or that σ (or θ) is not injective, a contradiction.
2. Assume that $C \succeq_i^{+0-} D$. For the set of friends N_i^+ we obtain an injective function $\sigma : D \cap N_i^+ \to C \cap N_i^+$ such that for each $y \in D \cap N_i^+$, we have $\sigma(y) \trianglerighteq_i y$. Hence, for each compatible w_i, $w_i(\sigma(y)) \geq w_i(y)$ holds. Thus, since σ is injective,

$$\sum_{j \in C \cap N_i^+} w_i(j) \geq \sum_{j \in \sigma(D \cap N_i^+) \subseteq C \cap N_i^+} w_i(j) = \sum_{j' \in D \cap N_i^+} w_i(\sigma(j')) \geq \sum_{j' \in D \cap N_i^+} w_i(j').$$

Similarly, for N_i^-, and $\theta : C \cap N_i^- \to D \cap N_i^-$ injective, it holds that

$$0 \geq \sum_{j \in C \cap N_i^-} w_i(j) \geq \sum_{j \in C \cap N_i^-} w_i(\theta(j)) = \sum_{j' \in \theta(C \cap N_i^-) \subseteq D \cap N_i^-} w_i(j') \geq \sum_{j' \in D \cap N_i^-} w_i(j').$$

For each player $j \in N_i^0$, we have $w_i(j) = 0$; therefore, in total,

$$\sum_{j \in C} w_i(j) \geq \sum_{j' \in D} w_i(j'). \tag{5.1}$$

Now assume that for each compatible w_i, (5.1) holds. Thus,

$$\sum_{j \in C \cap N_i^+} w_i(j) - \sum_{j' \in D \cap N_i^-} w_i(j') \geq \sum_{j' \in D \cap N_i^+} w_i(j') - \sum_{j \in C \cap N_i^-} w_i(j).$$

Assume there were no injective function mapping from each summand from the right-hand side to one at least as large on the left hand side; then, there exists an assignment to the values of w_i compatible with $\trianglerighteq_i^{+0-}$ that does not satisfy the inequality, a contradiction. ◻

As we have seen above, the generalized Bossong–Schweigert extension can leave uncertainties between two coalitions in a player's preference order. As one possibility to deal with these incomparabilities, in [LRR+15] it is suggested to determine the relation between incomparable coalitions by adapting scoring vectors such as the Borda scoring rule, which is well-known from voting theory (see, e.g., [BR16]). By this rule for each player $i \in N$, a function f^i_{Borda} is defined that assigns values to other players depending on the position in i's weak ranking with double threshold and on the interpretation whether friends and enemies are evaluated optimistically or pessimistically, compatible with $\succeq_i^{+0-}$ in the sense of Proposition 5.18.2. This function f^i_{Borda} induces a relation $\succeq_i \in \text{Ext}(\succeq_i^{+0-})$ as defined next in Definition 5.19. Moreover, observe that a hedonic game induced by a Borda-like comparability function is additively separable. Therefore, with regard to verification and existence problems of stability concepts, known upper bounds for additively separable hedonic games (see, e.g., [ABS13]) are inherited. Lower bounds may remain valid, depending on the input and the scoring function used. In some cases, the setting might be a special case of additively separable hedonic games such that hard problems in those games are tractable here. A careful revision of known hardness results adapted to this setting leads to the results shown in [LRR+15] and summed up in Table 5.4.

In this thesis we leave incomparabilities open and consider every possible extension that does not conflict with transitivity.

Definition 5.19. *Let i be a player in a set N. A complete preference relation $\succeq_i$ over all coalitions containing i* extends *$\succeq_i^{+0-}$ if and only if it contains it; that is, if $C \succeq_i^{+0-} D$ implies $C \succeq_i D$, for all coalitions $C, D \subseteq \mathcal{N}_i$. Let* $\text{Ext}(\succeq_i^{+0-})$ *be the set of all complete preference relations extending $\succeq_i^{+0-}$.*

	VERIFICATION	EXISTENCE
perfection, ind. rationality	in P	in P
single player deviation	in P	NP-complete (sp, so) for ind. stability; (sp, {so, o, sp, p}) for Nash stability
group deviation	coNP-complete	Σ_2^p-complete (sp, {sp, p}) for core stability; coNP-hard, in Σ_2^p for str. core stability
Pareto optimality	in NP	in P

Table 5.4: Complexity results of verification and existence problems in hedonic games induced by Borda-like comparability functions [LRR⁺15]. In brackets the scoring function for friends and enemies is specified, from pessimistic (o) to optimistic (p) over strong (s) variants.

On this background we define games in which each player i has friends, enemies, and neutral co-players, and preferences $\trianglerighteq_i^{+0-}$ over the former two sets such that i's preference relation $\succeq_i^{+0-}$ over coalitions can be derived by the generalized Bossong–Schweigert extension and completed by all possible extensions $\succeq_i \in \operatorname{Ext}(\succeq_i^{+0-})$.

Definition 5.20. *A* hedonic game with ordinal preferences and thresholds[2] *is a tuple $H = \langle N, \trianglerighteq_1^{+0-}, \dots, \trianglerighteq_n^{+0-} \rangle$, where $N = \{1, \dots, n\}$ is a set of players, and $\trianglerighteq_i^{+0-}$ denotes the ordinal preferences with thresholds for player $i \in N$ as defined in Definition 5.14.*

Example 5.21. *Let $A = \{1,2,3\}$, $\trianglerighteq_1^{+0-} = (2 \triangleright_1 3 \mid \emptyset \mid)$, $\trianglerighteq_2^{+0-} = (3 \mid \emptyset \mid 1)$, and $\trianglerighteq_3^{+0-} = (1 \mid \{2\} \mid)$. The generalized Bossong–Schweigert orders are*

$$\{1,2,3\} \succ_1^{+0-} \{1,2\} \succ_1^{+0-} \{1,3\} \succ_1^{+0-} \{1\}$$

for player 1,

$$\begin{array}{ccc} & \{2,3\} & \\ \succ_2^{+0-}\swarrow & & \searrow\succ_2^{+0-} \\ \{2\} & & \{1,2,3\} \\ \succ_2^{+0-}\searrow & & \swarrow\succ_2^{+0-} \\ & \{1,2\} & \end{array}$$

for player 2, and for player 3

$$\{1,3\} \sim_3^{+0-} \{1,2,3\} \succ_3^{+0-} \{3\} \sim_3^{+0-} \{2,3\}.$$

So, two preferences are already complete, and there are three complete preferences extending $\succeq_2^{+0-}$, one setting $\{2\} \succ_2 \{1,2,3\}$, another setting $\{2\} \sim_2 \{1,2,3\}$, and the third setting $\{1,2,3\} \succ_2 \{2\}$, leaving all other relations the same.

[2] Note that in [LRR⁺15] this game is denoted by *FEN-hedonic game*, for *friends*, *enemies*, and *neutral players*.

The possible extensions in a hedonic game with ordinal preferences and thresholds are anonymous and symmetric. Moreover, we obtain the following properties.

Proposition 5.22. *The extensions in a hedonic game with ordinal preferences and thresholds are necessarily monotonic. For each hedonic game with ordinal preferences and thresholds there exists an extension that is independent and for some one that is not.*

Proof Sketch. If $A \succeq_i B$ holds for each extension of a hedonic game with ordinal preferences and thresholds, then $A \succeq_i^{+0-} B$. By definition there exist two injective functions $\sigma : B \cap N_i^+ \to A \cap N_i^+$ and $\theta : A \cap N_i^- \to B \cap N_i^-$. Now, if a player $j \neq i$ ascends in i's ranking, it holds for each $k \in N \setminus \{i,j\}$ with $j \trianglerighteq_i k$, that $j \trianglerighteq_i^{+0-'} k$ in the new weak ranking with double threshold $\succeq_i^{+0-'}$, and there exists an $\ell \in N \setminus \{i,j\}$ with $\ell \trianglerighteq_i j$ and $j \trianglerighteq_i^{+0-'} k$. Since j is in A but not in B, the following cases can occur: Player j is added to the image set of σ, then the function remains injective. Player j is already in the image of σ, then if $j = \sigma(y) \triangleright_i^{+0-} y$, for some $y \in B \cap N_i^+$, it still holds that $\sigma(y) \triangleright_i^{+0-'} y$ (and $\sim_i^{+0-} \implies \trianglerighteq_i^{+0-'}$). Player j is in the domain of θ, then if $j \triangleright_i^{+0-} \theta(j)$, then it still holds that $j \triangleright_i^{+0-'} \theta(j)$ (and $\sim_i^{+0-} \implies \trianglerighteq_i^{+0-'}$). Player j is removed from the domain of θ, then the function remains injective. Therefore, $A \succ_i^{+0-} B$ implies $A \succ_i^{+0-'} B$, and $A \sim_i^{+0-} B$ implies $A \succeq_i^{+0-'} B$. Which means that for all extensions the relation maintains or even increases.

An extension of a hedonic game with ordinal preferences and thresholds is not necessarily independent. Consider the generalized Bossong–Schweigert extension of player 1's preference in Example 5.17. The relation between coalitions $\{1\}$ and $\{1,3,5\}$ and that between $\{1,2\}$ and $\{1,2,3,5\}$ is open. They can be dissolved differently in an extension violating independence. On the other hand all these indifferences can be dissolved in a way that the extension is independent: Define for each player i a function $w_i : N \to \mathbb{R}$ that is compatible with $\trianglerighteq_i^{+0-}$ as defined in Proposition 5.18.2. Then, extend the preferences additively. Since this is a proper extension by this proposition, and since additively separable hedonic games satisfy independence, we have found the independent extension. □

Remark 5.23. *Consider, as an example in an additively separable hedonic game, a coalition $\{i, f, e\}$ where player i has a positive value for f, and a negative value for e. In comparison to $\{i\}$ this coalition is preferred by player i if f has a greater absolute value than e in the additively separable representation, is considered indifferent if f and e have the same absolute value, and is less preferred otherwise. If we do not provide values but ordinal preferences and thresholds and consider f as a friend and e as an enemy of i's, $\{i, f, e\}$ and $\{i\}$ are incomparable from i's perspective; thus, all three scenarios are possible in an extension of a hedonic game with ordinal preferences and thresholds.*

Proposition 5.24. *There exists an extension of a hedonic game with ordinal preferences and thresholds that is not additive. For each additively separable hedonic game $(N, \succeq)$, there exists a hedonic game with ordinal preferences and thresholds and an extension such that $(N, \succeq)$ is represented.*

Proof. From Proposition 5.22 we know that there exists an extension that is not independent. Since all additively separable hedonic games are independent, the first statement holds. Secondly, consider an additively separable hedonic game with a player's values $u_i(j)$ for each player $j \in N$ with $u_i(i) = 0$. Then define the weak ranking with double threshold for i by $N_i^+ = \{j \mid u_i(j) > 0\}$, $N_i^0 = \emptyset$, $N_i^- = \{j \mid u_i(j) < 0\}$, and $j \trianglerighteq_i^{+0-} k$ if $u_i(j) \geq u_i(k)$ for each $j,k \in N_i^+$ or $j,k \in N_i^-$. Note that now u_i is compatible with $\trianglerighteq_i^{+0-}$ as defined in Proposition 5.18.2. By this the characterization in this proposition the additive extension of the values is an equivalent to a proper extension of the just constructed hedonic game with ordinal preferences and thresholds. ◻

Consequently, we obtain the following corollary that distinguishes hedonic games with ordinal preferences and thresholds from the representations it is based on.

Corollary 5.25. *There exists an extension of a hedonic game with ordinal preferences and thresholds that cannot be represented as a friend-oriented hedonic game or by $\mathscr{W}$-preferences.*

In order to study the existence and verification of stability concepts as defined in Section 2.3.2, we define the notions of *possible* and *necessary* stability.

Definition 5.26. *Let γ be a stability concept for hedonic games, $\langle N, \trianglerighteq_1^{+0-}, \ldots, \trianglerighteq_n^{+0-} \rangle$ be a hedonic game with ordinal preferences and thresholds, and Γ be a coalition structure. Γ satisfies* possible γ *if and only if there exists a profile $\langle \succeq_1, \ldots, \succeq_n \rangle \in \times_{i=1}^n \mathrm{Ext}(\succeq_i^{+0-})$ such that Γ satisfies γ in $\langle N, \succeq_1, \ldots, \succeq_n \rangle$. Γ satisfies* necessary γ *if and only if for each $\langle \succeq_1, \ldots, \succeq_n \rangle \in \times_{i=1}^n \mathrm{Ext}(\succeq_i^{+0-})$, Γ satisfies γ in $\langle N, \succeq_1, \ldots, \succeq_n \rangle$.*

Observation 5.27. *A possibly perfect coalition structure in a hedonic game with ordinal preferences and thresholds is always necessarily perfect. It may, however, be non-unique due to neutral players. Furthermore, note that there always exists a necessarily individual rational coalition structure (namely, the coalition structure where every agent is alone) and there always certainly exists a Pareto-optimal coalition structure (perhaps a different one for different extensions).*

Proposition 5.28. *Consider a hedonic game with ordinal preferences and thresholds $\langle N, \trianglerighteq_1^{+0-}, \ldots, \trianglerighteq_n^{+0-} \rangle$.*

1. *A coalition structure Γ is (necessarily and possibly) perfect if and only if for each player i, $N_i^+ \subseteq \Gamma(i)$ and $N_i^- \cap \Gamma(i) = \emptyset$.*
2. *A coalition structure Γ is possibly individually rational if and only if for each $i \in N$, $\Gamma(i)$ contains at least a friend of i's or only neutral agents.*
3. *A coalition structure Γ is necessarily individually rational if and only if for each $i \in N$, $\Gamma(i)$ does not contain any enemies of i's.*

4. *A coalition structure* Γ *is necessarily individually stable if and only if it is necessarily individually rational and no player i can join a coalition that she would possibly prefer and the members of which do not see her as an enemy.*

5. *A coalition structure* Γ *is necessarily contractually individually stable if and only if it is necessarily individually rational, no player i can join a coalition that she would possibly prefer and the members of which do not see her as an enemy, and at the same time no* j *in* $\Gamma(i)$ *considers i as a friend.*

Proof. 1. A coalition structure is perfect if and only if each player is in one of her favourite coalitions, that is, each player is together with all her friends and no enemies.

2. For each $i \in N$, i necessarily prefers $\{i\}$ to $\Gamma(i)$ if and only if $\Gamma(i)$ contains no friend and at least one enemy of i's.

3. For each $i \in N$, i possibly prefers $\{i\}$ to $\Gamma(i)$ if and only if $\Gamma(i)$ contains an enemy of i's.

4. Note that a player j possibly prefers a coalition C to $C \cup \{i\}$ if and only if j necessarily prefers C to $C \cup \{i\}$ if and only if i is an enemy of j's. Assume that Γ is necessarily individually stable. Then, for each $i \in N$, if i prefers to move to another (possibly empty) coalition C in Γ, there is a player in C that prefers player i not being in the coalition. If C is empty, there is no such player, thus, Γ has to be individually rational. Hence, C is non-empty and there has to be a player in C that sees i as an enemy. Now assume that Γ is not individually stable, that is, there is a player i and a coalition $C \in \Gamma \cup \{\emptyset\}$ such that i prefers $C \cup \{i\}$ to $\Gamma(i)$ and, for each $j \in C$, $C \cup \{i\} \succeq_j C$. If $C = \emptyset$, then Γ is not individually rational. Otherwise, each j does not see i as an enemy.

5. Additionally to the conditions of individual stability, contractually individual stability is violated if every member of $\Gamma(i)$ agrees to i's departure, which is the case if and only if no j in $\Gamma(i)$ considers i as a friend. ◻

Observe the following relations between possible and necessary stability concepts (see Figure 2.2 for a comparison to the general case).

Observation 5.29. *If there exists a necessarily strictly popular coalition structure, it is unique, whereas there can be more than one possibly strictly popular coalition structure. If there exists a necessarily strictly popular coalition structure, it is necessarily Pareto optimal. If there exist possibly strictly popular coalition structures, each of them is possibly Pareto-optimal. A necessarily strictly popular coalition structure does not need to be possibly individually rational. Even if the possible core is non-empty, a necessarily strictly popular coalition structure does not need to be possibly core-stable. The same holds for the concepts of Nash stability, individual stability, contractual individual stability, and strict* *core stability. If there exists a unique perfect partition, it is necessarily the unique necessarily strictly popular coalition structure.*

Example 5.30. *Consider the hedonic game with ordinal preferences and thresholds from Example 5.21. Observe that there does not exist a (possibly) perfect coalition structure. While* $\{\{1,2,3\}\}$ *is possibly Nash-stable, there does not exist a necessarily Nash-stable coalition structure, as in each of five cases, player* 1 *or player* 2*, at least possibly, wants to move to another coalition. Coalition structure* $\{\{1,2,3\}\}$ *is possibly individually rational, but not necessarily due to player* 2*;* $\{\{1,2\},\{3\}\}$ *is not possibly individually rational; the other three coalition structures are necessarily individual rational.*

For $\{\{1,3\},\{2\}\}$ *it holds that player* 2 *possibly wants to move to* $\{1,3\}$ *and* 1 *and* 3 *do not see* 2 *as an enemy, thus necessary individual stability is not satisfied. Also, since in* $\{2\}$ *there is no other player who considers* 2 *a friend, contractually individual stability is not satisfied either. Observe that this coalition structure is, however, possibly individually stable.*

Coalition structure $\{\{1\},\{2,3\}\}$ *is not necessarily individually stable, as player* 3 *wants to move to* $\{1,3\}$ *where* 1 *welcomes him. Player* 2*, however, considers* 3 *a friend, thus, as* 2 *does not want to move, and* 1 *is considered an enemy by* 2 *when moving to* $\{2,3\}$*, this coalition structure is necessarily contractually individually stable.*

We are interested in axiomatic properties and characterizations of stability concepts in hedonic games with ordinal preferences and thresholds. However, for some concepts no general statements can be made as to whether there exists a coalition structure satisfying a stability concept γ (possibly or necessarily). In these cases we ask how hard it is to decide whether for a given hedonic game with ordinal preferences and thresholds, a given coalition structure possibly or necessarily satisfies γ, and to decide whether there exists a coalition structure in a given hedonic game with ordinal preferences and thresholds that possibly or necessarily satisfies γ.

Here, we redefine the verification and existence problems of stable coalition structures to the notions of possible and necessary verification and existence. Again, let γ be one of the previously defined stability concepts for hedonic games.

POSSIBLE γ VERIFICATION	
Given:	A hedonic game with ordinal preferences and thresholds and a coalition structure Γ.
Question:	Does Γ satisfy possible γ?

NECESSARY γ VERIFICATION	
Given:	A hedonic game with ordinal preferences and thresholds and a coalition structure Γ.
Question:	Does Γ satisfy necessary γ?

For example, POSSIBLE CORE STABILITY VERIFICATION thus asks the following: Given a hedonic game with ordinal preferences and thresholds and a coalition structure Γ, is Γ possibly core-stable, or equivalently, is Γ not necessarily blocked by a coalition? In detail that is, does there *exist* a profile of preferences extending the generalized Bossong–Schweigert extension such that *each* coalition does not block Γ? Contrarily, NECESSARY CORE STABILITY VERIFICATION asks for an equal instance, is Γ necessarily core-stable, that is, is Γ not possibly blocked by any coalition? In detail we have, *for each* profile of preferences extending the generalized Bossong–Schweigert extension, does *each* coalition not block Γ? Due to the quantifier characterization (Lemma 2.2) the former problem belongs to Σ_2^p, while the latter belongs to coNP, since two universal quantifiers can be combined to one nondeterministic path.

Note that two interpretations of necessary existence can be distinguished, the first one asking whether there always exists a coalition structure that satisfies γ, while the second one is asking whether a particular coalition structure necessarily satisfies γ. Intuitively this distinction makes sense, since in the first case the setting might provide a central authority with partial knowledge of the agents' preferences and requires the knowledge that whatever the possible preferences are, there is always some coalition structure satisfying γ; in the second case, the choice of coalition structure is independent of the agents' possible preferences.

CERTAIN γ EXISTENCE	
Given:	A game in the representation with ordinal preferences and thresholds.
Question:	Is there necessarily a coalition structure satisfying γ, that is, for all profiles of preferences extending the generalized Bossong–Schweigert extension, does there exist a coalition structure satisfying γ?

NECESSARY γ EXISTENCE	
Given:	A game in the representation with ordinal preferences and thresholds.
Question:	Is there a necessarily γ coalition structure, that is, is there a coalition structure that satisfies γ in all profiles of preferences extending the generalized Bossong–Schweigert extension?

Note that an instance for the first problem is also one for the latter one; however, no generalizations about a dependence in complexity can be made.

Example 5.31. *For example, consider the following game with three players,* $A = \{1,2,3\}$, *with* $\trianglerighteq_1^{+0-} = (2 \mid \{3\} \mid)$, $\trianglerighteq_2^{+0-} = (1 \mid \{3\} \mid)$, *and* $\trianglerighteq_3^{+0-} = (1 \mid \emptyset \mid 2)$. *We obtain the following generalized Bossong–Schweigert orders:* $\{1,2\} \sim_1^{+0-} \{1,2,3\} \succ_1^{+0-} \{1\} \sim_1^{+0-} \{1,3\}$, $\{1,2\} \sim_2^{+0-} \{1,2,3\} \succ_2^{+0-} \{2\} \sim_2^{+0-} \{2,3\}$, *and* $\{1,3\} \succ_3^{+0-} \{3\} \succ_3^{+0-} \{2,3\}$ *and* $\{1,3\} \succ_3^{+0-} \{1,2,3\} \succ_3^{+0-} \{2,3\}$, *while* 3 *is undecided between* $\{3\}$ *and* $\{1,2,3\}$. *Any coalition structure in which players* 1 *and* 2 *are not in the same coalition cannot possibly* *be Nash-stable. On the one hand,* $\{\{1,2\},\{3\}\}$ *is Nash-stable if and only if an extension provides* $\{3\} \succeq_3 \{1,2,3\}$. *On the other hand,* $\{\{1,2,3\}\}$ *is Nash-stable if and only if* $\{1,2,3\} \succeq_3 \{3\}$ *in an extension. Thus, for every extension, there certainly exists a Nash-stable coalition structure. However, there is no necessarily Nash-stable coalition structure.*

Here, we focus on the second interpretation. Possible existence is unambiguous, asking whether there is some coalition structure satisfying γ for some extension.

POSSIBLE γ EXISTENCE	
Given:	A game in the representation with ordinal preferences and thresholds.
Question:	Is there a coalition structure that satisfies possible γ?

For example, CERTAIN CORE STABILITY EXISTENCE asks: Is there necessarily a core-stable coalition structure, that is, *for all* profiles of preferences extending the generalized Bossong–Schweigert extension, does there *exist* a coalition structure that is not blocked by *any* coalition?

NECESSARY CORE STABILITY EXISTENCE asks: Is there a necessarily core-stable coalition structure, that is, does there *exist* a coalition structure Γ such that *for each* profile of preferences extending the generalized Bossong–Schweigert extension, Γ is not blocked by *any* coalition?

POSSIBLE CORE STABILITY EXISTENCE asks: Is there a possibly core-stable coalition structure, that is, does there *exist* a profile of preferences extending the generalized Bossong–Schweigert extension and does there *exist* a coalition structure that is not blocked by *any* coalition? Due the structure of quantifiers, the first problem belongs to Π_3^p, while the latter two problems belong to Σ_2^p (see Lemma 2.2).

5.2.1 Complexity of Possible and Necessary Stability

Computational Complexity results for possible and necessary stability verification and existence are summarized in Table 5.2.

Proposition 5.32. *All variants of verification and existence problems regarding perfection are in* P.

Proof. Verification of whether a coalition structure is possibly and necessarily perfect is easy by Proposition 5.28.

Existence can be decided by, e.g., the following algorithm: Start with player 1 and let $\Gamma(1) := \{1\} \cup N_1^+$. Sequentially, for each $i \in \Gamma(1)$, add N_i^+ to $\Gamma(1)$ until there are no further possible changes. Check whether, for each $i \in \Gamma(1)$, $N_i^- \cap \Gamma(1) = \emptyset$. If not, output "there is no perfect coalition structure"; if so, start over with $N \setminus \Gamma(1)$. It might be the case that a friend cannot be added, because he is already assigned to another coalition. If he is on his own, add him anyway; otherwise, output "there is no perfect coalition structure." Continue until each player is allocated to a coalition. Then, output "there is a perfect coalition structure." Note that this algorithm works in polynomial time. ∎

All problems regarding individual rationality are in P as well by the characterizations in Proposition 5.28 and Observation 5.27.

Proposition 5.33. POSSIBLE INDIVIDUAL RATIONALITY VERIFICATION *and* NECESSARY INDIVIDUAL RATIONALITY VERIFICATION *are in* P.

Proof. Given a hedonic game with ordinal preferences and thresholds and a coalition structure Γ, it can be decided whether Γ is necessarily or only possibly or not at all individually rational in polynomial time: For each $i \in N$, test the following cases:

- If i has an enemy and no friends in $\Gamma(i)$, output "Γ is not possibly individually rational";
- if i has both a friend and an enemy in $\Gamma(i)$, set a boolean value p to true.

Now, if p is true, at least one player i is undecided between $\Gamma(i)$ and $\{i\}$, thus, output "Γ is possibly, but not necessarily individually rational". Otherwise, each player has only friends or neutral players in $\Gamma(i)$, thus, output "Γ is necessarily individually rational". ∎

5.2.2 Complexity of Nash, Individual, and Contractually Individual Stability

Proposition 5.28 does not provide a characterization of Nash stability. Nevertheless, it can be verified in polynomial time whether a given coalition structure in a given hedonic game with ordinal preferences and thresholds is necessarily Nash-stable.

Theorem 5.34. NECESSARY NASH STABILITY VERIFICATION, NECESSARY INDIVIDUAL STABILITY VERIFICATION, *and* NECESSARY CONTRACTUALLY INDIVIDUAL STABILITY VERIFICATION, *are in* P.

Proof. Given a hedonic game with ordinal preferences and thresholds and a coalition structure Γ, verify the following steps for each $i \in N$: For each (of at most n coalitions) $C \in \Gamma \cup \{\emptyset\}$, $C \neq \Gamma(i)$, determine the relation between $\Gamma(i)$ and $C \cup \{i\}$. This can be done in polynomial time by Proposition 5.18. If $C \cup \{i\} \succ_i \Gamma(i)$, output "$\Gamma$ is not Nash-stable." If the relation is undecided, output "Γ is possibly not Nash-stable." Otherwise, if this is not true for any player or coalition in $\Gamma \cup \{\emptyset\}$, output "$\Gamma$ is necessarily Nash-stable."

Similar algorithms work for individual and contractually individual stability. In particular, the characterizations in Proposition 5.28 can be verified in polynomial time. ∎

Note that this cannot easily be transferred to the possible variants since resolving an undecided relation might influence another relation for the same player.

Theorem 5.35. POSSIBLE NASH STABILITY EXISTENCE *is* NP-*complete.*

Proof. The problem belongs to NP, since it is enough to decide whether there exist a coalition structure of N and a profile of preferences extending the generalized Bossong–Schweigert extension such that for each player $i \in N$ and each coalition $C \in \Gamma \cup \{\emptyset\}$, $\Gamma(i) \succeq_i C \cup \{i\}$. The latter can be tested in polynomial time in $n = \|N\|$, since there are at most n coalitions in Γ and the relation between two coalitions from a common player's perspective can be decided in polynomial time by Proposition 5.18.

NP-hardness can be shown via a polynomial-time many-one reduction from XC_3. Given a set B with $3k$ elements and a family $\mathscr{S}$ of subsets $S \subseteq B$ with $\|S\| = 3$, is there an exact cover of B in $\mathscr{S}$, that is, is there a subset $\mathscr{S}' \subseteq \mathscr{S}$ such that $\cup_{S \in \mathscr{S}'} S = B$ and $\|\mathscr{S}'\| = k$? Without loss of generality it can be assumed that $k \geq 2$ (otherwise, we reduce to a trivial instance) and each element in B occurs at most three times in a set in $\mathscr{S}$ (see Chapter 2). Given such an XC_3 instance, we construct the following game. This construction is inspired by the construction of the proof that it is NP-hard to decide whether there exists a Nash-stable coalition structure in an additively separable hedonic game [SD10, Theorem 3]. Here, however, several adjustments had to be made in order to guarantee necessary preferences over coalitions (see Remark 5.23). Let

$$N = \{\alpha_i \mid 1 \leq i \leq 3k-1\} \cup \{\beta_r \mid r \in B\} \cup \{\zeta_{S,\ell} \mid S \in \mathscr{S}, 1 \leq \ell \leq 3k-2\}$$

and let the player's preferences be defined as follows.

- $\trianglerighteq^{+0-}_{\alpha_i} = \big(\alpha_{i+1} \mid \{\alpha_j : i \neq j \neq i+1\} \mid \{\text{other players}\}_\sim\big)$, for each i, $1 \leq i \leq 3k-2$,

  $\trianglerighteq^{+0-}_{\alpha_{3k-1}} = \big(\mid \{\alpha_j : j \neq 3k-1\} \mid \{\text{other players}\}_\sim\big)$,

- $\trianglerighteq^{+0-}_{\beta_r} = \big(\{\alpha_i : 1 \leq i \leq 3k-1\}_\sim \triangleright_{\beta_r} \bigcup_{r \in S} Q_{S\sim} \triangleright_{\beta_r} \{\beta_{r'} : r' \neq r\}_\sim \mid \emptyset \mid \{\text{other players}\}_\sim\big)$, for each $r \in B$,

- $\trianglerighteq^{+0-}_{\zeta_{S,\ell}} = \big(\zeta_{S,\ell+1} \mid \{\zeta_{S,\ell'} : \ell \neq \ell' \neq \ell+1\} \cup \{\beta_r : r \in S\} \mid \{\text{other players}\}_\sim\big)$, for each $S \in \mathscr{S}$, and ℓ, $1 \leq \ell \leq 3k-3$,

  $\trianglerighteq^{+0-}_{\zeta_{S,3k-2}} = \big(\mid \{\zeta_{S,\ell'} : \ell' \neq 3k-2\} \cup \{\beta_r : r \in S\} \mid \{\text{other players}\}_\sim\big)$, for each $S \in \mathscr{S}$,

where $Q_S = \{\zeta_{S,\ell} \mid 1 \leq \ell \leq 3k-2\}$ for each $S \in \mathscr{S}$. Moreover, let $P_S = \{\beta_r \mid r \in S\} \cup Q_S$. This profile can be constructed in polynomial time, since there are $n \leq 3k + 3k + 3k \cdot (3k-2)$ players, and a each player's preference can be written in linear time in n. This profile is visualized in Figure 5.7.[3]

[3] Note that in figures of networks we relax the distinction of player names and vertex names for the sake of readability.

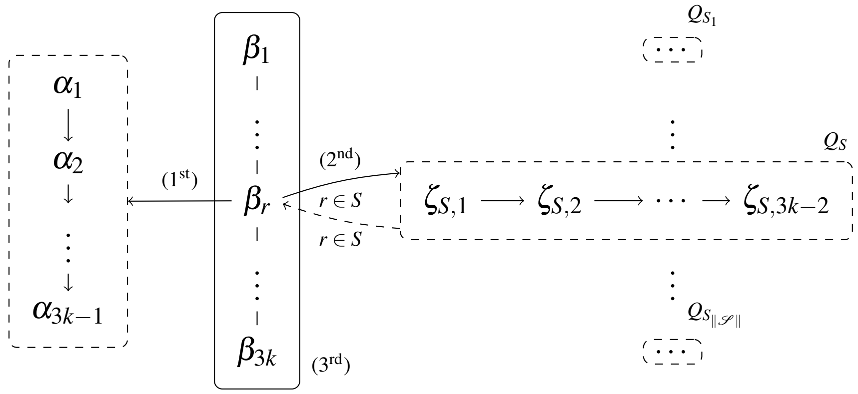


Figure 5.7: Network of friends for construction in proof of Theorem 5.35. A solid line represents a friendship-relation (with priorities if required) a dashed line stands for an at least neutral relation.

We show that $(B, \mathscr{S})$ is a positive instance for XC_3 if and only if there exists a possibly Nash-stable coalition structure in the generalized Bossong–Schweigert extension of the constructed game.

Only if: Assume there exists a solution $\mathscr{S}'$ for $(B, \mathscr{S})$. Consider the coalition structure

$$\Gamma = \{\{\alpha_i \mid 1 \leq i \leq 3k-1\}\} \cup \{P_S \mid S \in \mathscr{S}'\} \cup \{Q_S \mid S \notin \mathscr{S}'\}.$$

No α_i, $1 \leq i \leq 3k-1$, wants to move, since all of them are in one of their favourite coalitions (all their friends, and no enemies). No $\zeta_{S,\ell}$, $S \in \mathscr{S}$, $1 \leq \ell \leq 3k-2$, wants to move, since all of them are also in one of their favourite coalitions and they are indifferent between any β_r, $r \in S$, being in the coalition or not.

Since P_S only contains friends of β_r's, moving to the empty set is out of the question for β_r. Moving to any $Q_{S'}$ with $r \notin S'$, (or $Q_{S'}$ with $r \in S'$, $S' \neq S$, if such a coalition exists) would mean a loss of friends and in the first case an increase of enemies. For any $P_{S'}$, $r \notin S'$, it holds that $P_S \succ_{\beta_r}^{+0-} P_{S'} \cup \{\beta_r\}$, since the second contains three friends ($\beta_{r'}$, $\beta_{r''}$, and $\beta_{r'''}$), two of which are interchangeable, and the third less liked than a friend $\zeta_{S,1}$ in P_S. Additionally there are no enemies in P_S but in $P_{S'} \cup \{\beta_r\}$. Observe that, since $\mathscr{S}'$ is an exact cover of B, there is no $P_{S'} \in \Gamma$ with $r \in S'$, $S' \neq S$.

The remaining coalition a β_r, $r \in B$, with $\Gamma(\beta_r) = P_S$, might want to move to is $C = \{\alpha_i \mid 1 \leq i \leq 3k-1\}$. P_S and $C \cup \{\beta_r\}$ are incomparable for β_r. There are more ($3k > 3k-1$) friends in P_S than in $C \cup \{\beta_r\}$, but each friend in $C \cup \{\beta_r\}$ is preferred to one in P_S. A single incomparability, however, implies that there is a possible extension in which β_r does not prefer $C \cup \{\beta_r\}$ to P_S. All in all, Γ is possibly Nash-stable.

If: Assume there is a possibly Nash-stable coalition structure Γ. Γ cannot contain any coalition that contains a strict subset of $C := \{\alpha_1, \ldots, \alpha_{3k-1}\}$ by the following arguments.

As player α_{3k-2}'s only friend is α_{3k-1}, he will always want to move to the coalition α_{3k-1} is contained in. Hence any coalition structure dividing these two players into different coalitions is not possibly Nash-stable. For the same reasons, α_{3k-3} will always follow α_{3k-2} and so on; thus, any coalition structure Γ' with $\Gamma'(i) \neq \Gamma'(i+1)$, $1 \leq i \leq 3k-2$, is not possibly Nash-stable. Also, as soon as there is another player $x \notin C$ in a coalition C' with $C \cup \{x\} \subseteq C'$, player α_{3k-1} necessarily prefers being alone. Thus, Γ does not contain any strict superset of C. However, no player wants to deviate from C itself, as there are no enemies in this coalition, and everyone's friends. Therefore, it holds that

$$C \in \Gamma. \tag{5.2}$$

Analogously, all $\zeta_{S,\ell}$, $1 \leq \ell \leq 3k-2$, have to be together in one coalition in Γ, separately for each $S \in \mathscr{S}$. More precisely, for $S \in \mathscr{S}$, each $\zeta_{S,\ell}$ follows $\zeta_{S,\ell+1}$ sequentially, for ℓ, $3k-3 \geq \ell \geq 1$, to a superset D_S of Q_S. D_S cannot contain any $\zeta_{S',\ell'}$ with $S' \neq S$, $1 \leq \ell' \leq 3k-2$, nor any β_r with $r \notin S$, since $\zeta_{S,3k-2}$ is indifferent between everything but her enemies and will deviate.

This leaves us the following combinations to consider: For each $S \in \mathscr{S}$, Γ contains

$$D_S = Q_S \cup R_S, \tag{5.3}$$

where $R_S \subseteq \{\beta_r \mid r \in S\}$. If R_S contained one or two elements β_r and $\beta_{r'}$, β_r would necessarily prefer $C \cup \{\beta_r\}$ to D_S, since both coalitions contain $3k-1$ friends (or D_S even less in the first case) and no enemies, and each friend in the first coalition is ranked higher than one in the latter. Hence,

$$D_S = Q_S \quad \text{or} \quad D_S = P_S.$$

If a β_r was alone with other $\beta_{r'}$s not in a P_S, he would be with at most $3k-1$ friends, and would rather move to C with the same number of, but higher ranked, friends. This implies that for each $r \in B$, there exists an $S \in \mathscr{S}$ such that $\Gamma(\beta_r) = P_S$, which means that there is an exact cover of B in $\mathscr{S}$. ◻

Theorem 5.36. NECESSARY NASH STABILITY EXISTENCE *is* NP-*complete.*

Note that the following proof as well as those in Subsection 5.2.4 cannot be found in the conference contribution [LRR+15].

Proof. The problem belongs to NP, since it can be verified in polynomial time in the number of players whether a nondeterministically chosen coalition structure is necessarily Nash-stable by Theorem 5.34.

NP-hardness can be shown similarly to the proof of Theorem 5.35. We add a player α_{3k} and change the order of friends for each β_r and obtain the following game as constructed from a given XC$_3$-instance $(B, \mathscr{S})$ using the same denotations as previously:

$$N = \{\alpha_i \mid 1 \leq i \leq 3k\} \cup \{\beta_r \mid r \in B\} \cup \{\zeta_{S,\ell} \mid S \in \mathscr{S}, 1 \leq \ell \leq 3k-2\}$$

and

- $\trianglerighteq^{+0-}_{\alpha_i} = \big(\alpha_{i+1} \mid \{\alpha_j : i \neq j \neq i+1\} \mid \{\text{other players}\}_{\sim}\big)$, for each i, $1 \leq i \leq 3k-1$,

  $\trianglerighteq^{+0-}_{\alpha_{3k}} = \big(\mid \{\alpha_j : j \neq 3k\} \mid \{\text{other players}\}_{\sim}\big)$,

- $\trianglerighteq^{+0-}_{\beta_r} = \big(\bigcup_{r\in S} Q_{S\sim} \triangleright_{\beta_r} \{\beta_{r'} : r' \neq r\}_{\sim} \triangleright_{\beta_r} \{\alpha_i : 1 \leq i \leq 3k\}_{\sim} \mid \emptyset \mid \{\text{other players}\}_{\sim}\big)$, for each $r \in B$,

- $\trianglerighteq^{+0-}_{\zeta_{S,\ell}} = \big(\zeta_{S,\ell+1} \mid \{\zeta_{S,\ell'} : \ell \neq \ell' \neq \ell+1\} \cup \{\beta_r : r \in S\} \mid \{\text{other players}\}_{\sim}\big)$, for each $S \in \mathscr{S}$, and ℓ, $1 \leq \ell \leq 3k-3$,

  $\trianglerighteq^{+0-}_{\zeta_{S,3k-2}} = \big(\mid \{\zeta_{S,\ell'} : \ell' \neq 3k-2\} \cup \{\beta_r : r \in S\} \mid \{\text{other players}\}_{\sim}\big)$, for each $S \in \mathscr{S}$.

It holds that $(B,\mathscr{S})$ is a positive XC_3 instance if and only if there exists a necessarily Nash-stable coalition structure in the generalized Bossong–Schweigert extension of the constructed game.

Only if: Assume, $\mathscr{S}'$ is a solution for $(B,\mathscr{S})$. Let $C := \{\alpha_1,\dots,\alpha_{3k}\}$ and consider $\Gamma = \{C\} \cup \{P_S \mid S \in \mathscr{S}'\} \cup \{Q_S \mid S \notin \mathscr{S}'\}$.

Analogously to above, it holds that no α_i, $1 \leq i \leq 3k$, and no $\zeta_{S,\ell}$, $S \in \mathscr{S}$, $1 \leq \ell \leq k$, wants to move. Each β_r, $r \in B$, now necessarily prefers P_S to joining any other existing or empty coalition. Thus, Γ is necessarily Nash-stable.

If: Let Γ be a necessarily Nash-stable coalition structure. Analogously to above, $C \in \Gamma$ (see Equation (5.2)), because of players α_i, $1 \leq i \leq 3k$ and $D_S = Q_S \cup R_S \in \Gamma$ (see Equation (5.3)), because of players $\zeta_{S,\ell}$, $S \in \mathscr{S}$, $1 \leq \ell \leq k$. R_S cannot contain one or two elements; otherwise, $\beta_r \in R_S$ would possibly prefer moving to $C \cup \{\beta_r\}$, which would lead to a not necessarily Nash-stable Γ. Consequently, $D_S = Q_S$ or $D_S = P_S$. A β_r outside of P_S in Γ would also imply a possible deviation to $C \cup \{\beta_r\}$. Thus, all β_r, $r \in B$, are covered by disjoint P_S in Γ; hence $\{S \mid P_S \in \Gamma\}$ is a solution for the XC_3 instance $(B,\mathscr{S})$. ◻

Note that the same construction can be used in order to show NP-hardness for certainty of a Nash-stable coalition structure: If, there exists a Γ that is Nash-stable for every extension, it also holds that, for every extension, there exists a an adequate Γ. On the other hand, since in the proof above (now assuming that, for every extension, there exists a Nash-stable coalition structure Γ that might depend on the extension) $C \cup \{\beta_r\}$ is possibly preferred by β_r to D_S (which, in turn, is preferred to a coalition R without any $\zeta_{S,\ell}$), there exists an extension which does not allow D_S (nor R) to be in Γ. Regarding this extension, $\Gamma(\beta_r) = P_S$, for each $r \in B$; thus, there is a solution for $(B,\mathscr{S})$.

Corollary 5.37. CERTAIN NASH STABILITY EXISTENCE *is* NP-*hard.*

5.2.3 Complexity of Core Stability

So far we have focused on single-player deviations. In this subsection we turn to group deviations.

Theorem 5.38. POSSIBLE CORE STABILITY VERIFICATION *and* POSSIBLE STRICT CORE STABILITY VERIFICATION *are* coNP-*hard.*

Proof. Hardness for coNP of both problems can be shown with help of the reduction from CLIQUE to the complement of the core stability verification problem in the enemy-based representation [SD07]. Note that the representation is a special case of the representation with ordinal preferences and thresholds, where there are no neutral agents and only indifferences between all friends and between all enemies in a player's preference. Furthermore, note that the enemy-based-extension [DBHS06] is a possible extension in $\times_{i=1}^{n} \mathrm{Ext}(\succeq_i^{+0-})$. While a "clique" of friends is necessarily preferred by all members to a coalition containing fewer friends or even more enemies, there is not necessarily a blocking coalition in the construction if there is no such clique (for example, there is no blocking coalition in the enemy-based extension). ◫

5.2.4 Complexity of Pareto Optimality and Popularity

With techniques related to those in the proof of Theorem 5.35, we can show that the questions of whether a given coalition structure is possibly strictly popular or popular or Pareto-optimal are coNP-hard, necessarily strictly popular or popular or Pareto-optimal are coNP-complete, and it is coNP-hard to decide whether there exists a strictly popular coalition structure, for both, the possible and the necessary case.

Theorem 5.39. POSSIBLE STRICT POPULARITY VERIFICATION *and* POSSIBLE POPULARITY VERIFICATION *are* coNP-*hard;* NECESSARY STRICT POPULARITY VERIFICATION *and* NECESSARY POPULARITY VERIFICATION *are* coNP-*complete.*

Proof. Influenced by the previously mentioned proof [SD10, Theorem 3] and based on ideas in the proof of Theorem 5.35 we show coNP-hardness of these problems by means of a reduction from XC_3 with slight variance for the four cases.

To begin with, given an XC_3 instance $(B, \mathscr{S})$, we construct the following game:

$$N = \{\alpha_{r,i} \mid r \in B, 1 \le i \le 3k+3\} \cup \{\beta_r \mid r \in B\} \cup \{\zeta_{S,\ell} \mid S \in \mathscr{S}, 1 \le \ell \le 3k+1\}$$

and

- $\trianglerighteq_{\alpha_{r,1}}^{+0-} = \big(\beta_r \sim_{\alpha_{r,1}} \{\alpha_{r,j} : j \neq 1\}_\sim \;\big|\; \emptyset \;\big|\; \{\text{other players}\}_\sim\big)$, for each $r \in B$,

  $\trianglerighteq_{\alpha_{r,i}}^{+0-} = \big(\{\alpha_{r,j} : j \neq i\}_\sim \;\big|\; \{\beta_r\} \;\big|\; \{\text{other players}\}_\sim\big)$, for each $r \in B$, and each i, $2 \le i \le 3k+3$,

- $\trianglerighteq^{+0-}_{\beta_r} = \left(\bigcup_{r\in S} Q_{S\sim} \rhd_{\beta_r} \{\beta_{r'} : r' \neq r\}_\sim \rhd_{\beta_r} C_{r\sim} \;\middle|\; \emptyset \;\middle|\; \{\text{other players}\}_\sim\right)$, for each $r \in B$,

- $\trianglerighteq^{+0-}_{\zeta_{S,\ell}} = \left(\{\zeta_{S,\ell'} : \ell' \neq \ell\}_\sim \;\middle|\; \{\beta_r : r \in S\} \;\middle|\; \{\text{other players}\}_\sim\right)$, for each $S \in \mathscr{S}$, and ℓ, $1 \le \ell \le 3k+1$,

where, again, $Q_S = \{\zeta_{S,\ell} \mid 1 \le \ell \le 3k+1\}$ and $P_S = \{\beta_r \mid r \in S\} \cup Q_S$, for each $S \in \mathscr{S}$, and $C_r = \{\alpha_{r,i} \mid 1 \le i \le 3k+3\}$, for each $r \in B$. Furthermore, let

$$\Gamma = \{C_r \cup \{\beta_r\} \mid r \in B\} \cup \{Q_S \mid S \in \mathscr{S}'\}$$

be the coalition structure of interest. This construction is polynomial in k (as are the variants below). This profile is visualized in Figure 5.8.

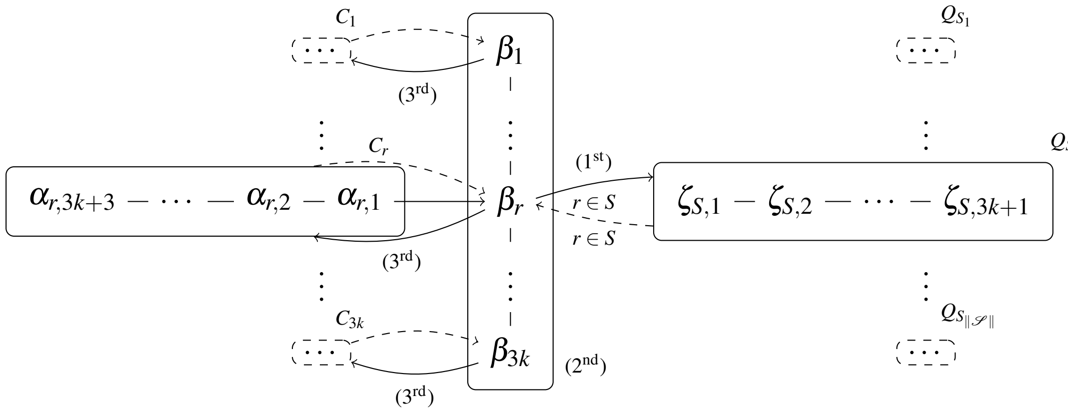


Figure 5.8: Network of friends for construction in proof of Theorem 5.39. A solid line represents a friendship-relation (with priorities if required) a dashed line stands for an at least neutral relation.

We show that Γ is possibly strictly popular if and only if there is no solution for $(B, \mathscr{S})$.

Only if: Assuming $(B, \mathscr{S})$ has a solution $\mathscr{S}'$, we consider the coalition structure $\Gamma' = \{C_r \mid r \in B\} \cup \{P_S \mid S \in \mathscr{S}'\} \cup \{Q_S \mid S \notin \mathscr{S}'\}$. There are $3k$ players, $\alpha_{r,1}$, $r \in B$, that necessarily prefer $\Gamma(\alpha_{r,1})$ to $\Gamma'(\alpha_{r,1})$, and $3k$ players, β_r, $r \in B$, that necessarily prefer $\Gamma'(\beta_r)$ to $\Gamma(\beta_r)$. All other players are indifferent between their coalitions in Γ and Γ'. Thus, Γ is necessarily prevented from being strictly popular.

If: Assume Γ is not possibly strictly popular, that is, for each extension, there exists another coalition structure Γ' that beats Γ in pairwise comparison. All players $\alpha_{r,i}$, $r \in B$, $1 \le i \le 3k+3$, and $\zeta_{S,\ell}$, $S \in \mathscr{S}$, $1 \le S \le 3k+1$, are in one of their favourite coalitions in Γ; hence, cannot improve in Γ'. Therefore, there are at most $3k$ players, namely of the form β_r, $r \in B$, that vote in favour of Γ'.

If, for some $r \in B$, not all players $\alpha_{r,i}$, $1 \leq i \leq 3k+3$, are together, they are all worse off in comparison to Γ. This cannot be counterbalanced by $3k$ players; consequently, they have to be in one coalition in Γ'. For the same reason, for each $S \in \mathscr{S}$, the $3k+1$ players in Q_S cannot be separated in Γ'.

If some β_r, $r \in B$ wants to improve by adding friends to $C_r \cup \{\beta_r\}$, all $3k+3$ players in C_r will disapprove; hence, this can also not be the case in Γ'.

Thus, for each $r \in B$, $\alpha_{r,1}$ is separated from β_r, which sums up in a number of $3k$ players in favour of Γ in comparison to Γ'. That means, in order for Γ' to be successful, each β_r has to prefer $\Gamma'(\beta_r)$ to $\Gamma(\beta_r)$. It necessarily holds that $Q_S \cup \{\beta_r, \beta_{r'}\} \succ_{\beta_r} Q_S \cup \{\beta_r \mid r \in S\} \succ_{\beta_r} \{\beta_{r'} \mid r \in B\}$, for a player $\beta_{r'}$ with $r, r' \in S, r \neq r'$, but $C_r \cup \{\beta_r\}$ and $Q_S \cup \{\beta_r, \beta_{r'}\}$ are indifferent. However, there exists an extension in which this indifference is solved by favour of $\Gamma(\beta_r)$, thus for this extension, $Q_S \cup \{\beta_r, \beta_{r'}\}$ cannot be in Γ, (nor any less preferred coalition by β_r). Also, there cannot be any enemies of players Q_S in the same coalition, since otherwise there would be at least $3k+1$ more players that disapprove. This leaves only one possibility: There is a coalition structure, such that all players β_r are in a coalition P_S with $r \in S$. Consequently, there is an exact cover of B by sets in $\mathscr{S}$.

Thus, POSSIBLE STRICT POPULARITY VERIFICATION is coNP-hard. For POSSIBLE POPULARITY VERIFICATION we consider the following modification:

$$\trianglerighteq^{+0-}_{\alpha_{3k,1}} = \left(\{\alpha_{3k,j} : j \neq 1\}_{\sim} \;\middle|\; \{\beta_{3k}\} \;\middle|\; \{\text{other players}\}_{\sim}\right).$$

Everything else remains the same. Note that now, we need Γ' to strictly defeat Γ, in order to obtain that Γ is not possibly popular. The argumentation is analogous to above, except that now only $3k-1$ players of the form $\alpha_{r,1}$, $r \in B$, dislike being in a different coalition than β_r. Then, for each extension, there is such a Γ' if and only if all players β_r, $r \in B$, can be placed in a P_S, $r \in S$. Thus, it holds that Γ is possibly popular if and only if there is no solution for $(B, \mathscr{S})$.

Next, we consider a modification where, for each $r \in B$, there are only $3k+2$ players $\alpha_{r,i}$, where $\alpha_{3k,1}$'s preferences depend on the strict or not strict case, and $\trianglerighteq^{+0-}_{\beta_r} = \left(C_{r\sim} \triangleright_{\beta_r} \bigcup_{r \in S} Q_{S\sim} \triangleright_{\beta_r} \{\beta_{r'} : r' \neq r\}_{\sim} \;\middle|\; \emptyset \;\middle|\; \{\text{other players}\}_{\sim}\right)$. This has the consequence that possibly

$$P_S \succ_{\beta_r} C_r \cup \{\beta_r\}$$

and necessarily

$$C_r \cup \{\beta_r\} \succ_{\beta_r} Q_S \cup R_S \succ_{\beta_r} \{\beta_{r'} \mid r \in B\},$$

for each $r \in B$.

Similar to previous argumentation, it can be shown that in the game where $\alpha_{3k,1}$ has β_{3k} as a friend, it holds that Γ is necessarily strictly popular if and only if there is no solution for $(B, \mathscr{S})$.

Analogously to the third and second cases, it can be shown that if β_{3k} is neutral for $\alpha_{3k,1}$, it holds that Γ is necessarily popular if and only if there is no solution for $(B, \mathscr{S})$.

Finally, the coNP upper bound for the latter two problems holds, since both, an extension and a coalition structure Γ' can be chosen nondeterministically, and it can be verified in polynomial time whether Γ' violates the conditions for the input coalition structure to be popular or strictly popular, respectively. ◻

Theorem 5.40. *For strict popularity, all three existence problems are* coNP*-hard.*

Proof. For the question of existence, we choose the same constructions as the first and the third one in the proof of Theorem 5.39, without fixing Γ.

We have seen that, for the first construction, if there is no solution for the given XC_3 instance there exists a possibly strictly popular coalition structure (namely, Γ, that was given in the verification problem). Now we show that, if there is a solution, not only Γ is beaten in pairwise comparison, but also there is no other strictly popular coalition structure. Observe that Γ and Γ' (the coalition structure related to the XC_3 solution) tie up in pairwise comparison with the maximal amount of positive votes each ($3k$). Thus, these two cannot be strictly popular. Any other coalition structure can also only possibly gain at most $3k$ positive votes; hence, there is no coalition structure that beats every other one in pairwise comparison.

For the third construction, we have seen that Γ is necessarily strictly popular, thus, there exists a necessarily strictly popular coalition structure, and at the same time there certainly exists a strictly popular coalition structure. By analogous arguments to above, where all other coalition structures than Γ and Γ' are even necessarily worse off, it can be seen that, if there is a solution for the original XC_3 instance, there is no necessarily strictly popular coalition structure at all. ◻

Theorem 5.41. POSSIBLE PARETO OPTIMALITY VERIFICATION *is* coNP*-hard and* NECESSARY PARETO OPTIMALITY VERIFICATION *is* coNP*-complete.*

Proof. Consider a similar, but even simpler reduction from XC_3 than previously. Letting $(B, \mathscr{S})$ with $\|B\| = 3k$ be a given XC_3 instance, we construct the following game: $N = \{\beta_r \mid r \in B\} \cup \{\zeta_{S,\ell} \mid S \in \mathscr{S}', 1 \leq \ell \leq 3k-3\}$ and

- $\trianglerighteq^{+0-}_{\beta_r} = \left(\bigcup_{r \in S} Q_{S\sim} \rhd_{\beta_r} \{\beta_{r'} : r' \neq r\}_\sim \mid \emptyset \mid \{\text{other players}\}_\sim\right)$, for each $r \in B$,

- $\trianglerighteq^{+0-}_{\zeta_{S,\ell}} = \left(\zeta_{S,\ell+1} \mid \{\zeta_{S,\ell'} : \ell \neq \ell' \neq \ell+1\} \cup \{\beta_r : r \in S\} \mid \{\text{other players}\}_\sim\right)$, for each $S \in \mathscr{S}$, and ℓ, $1 \leq \ell \leq 3k-4$,

  $\trianglerighteq^{+0-}_{\zeta_{S,3k-3}} = \left(\mid \{\zeta_{S,\ell'} : \ell' \neq 3k-3\} \cup \{\beta_r : r \in S\} \mid \{\text{other players}\}_\sim\right)$, for each $S \in \mathscr{S}$,

where Q_S and P_S are defined as above. Moreover, let $\Gamma = \{\{\beta_r \mid r \in B\}\} \cup \{Q_S \mid S \in \mathscr{S}'\}$. This profile can be determined in polynomial time and is visualized in Figure 5.9.

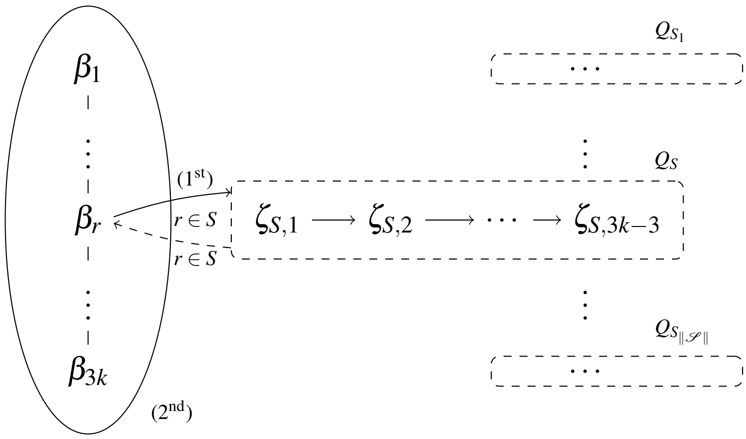


Figure 5.9: Network of friends for construction in proof of Theorem 5.41. A solid line represents a friendship-relation (with priorities if required) a dashed line stands for an at least neutral relation.

It holds that $(B,\mathscr{S})$ belongs to XC_3 if and only if Γ is not possibly Pareto-optimal.

Only if: Consider a solution $\mathscr{S}'$ for $(B,\mathscr{S})$, assuming there is one, coalition structure $\Gamma' = \{P_S \mid S \in \mathscr{S}'\} \cup \{Q_S \mid S \notin \mathscr{S}'\}$ necessarily Pareto dominates Γ: Each player $\zeta_{S,\ell}$, $S \in \mathscr{S}$, $1 \leq \ell \leq 3k-3$, is indifferent between Q_S and P_S, as β is considered as neutral. Furthermore, each β_r, $r \in B$, necessarily strictly prefers P_S to $\Gamma(\beta_r)$, since two friends can be mapped to two indifferent friends, and $3k-3$ players can be mapped to higher ranked players, and β_r has got no enemies in either coalition.

If: Assume there exists a coalition structure Γ' that necessarily Pareto dominates Γ, that is, for each player i, $\Gamma'(i) \succeq_i \Gamma(i)$ and for at least one player j, $\Gamma'(j) \succ_j \Gamma(j)$. From the point of view of players $\zeta_{S,\ell}$, $S \in \mathscr{S}$, $1 \leq \ell \leq 3k-3$, the players in Q_S have to be together in one coalition in Γ' and without any enemies. A player β_r necessarily prefers P_S to $\{\beta_{r'} \mid r' \in B\}$ and the latter possibly to every other coalition containing β_r. Since Γ' necessarily Pareto dominates Γ, there is an extension, for which the only possible Γ' assigns each β_r, $r \in B$, to a P_S which implies that there is an exact cover of B in $\mathscr{S}$.

For the second case, NECESSARY PARETO OPTIMALITY VERIFICATION, we slightly change the construction: Now there are only $3k-2$ players $\zeta_{S,\ell}$ for each $S \in \mathscr{S}$ and each β_r, $r \in B$, prefers each $\beta_{r'}$, $r \neq r'$ to $\zeta_{S,\ell}$, $r \in S$, $1 \leq \ell \leq 3k-2$. Observe that with analogous argumentation, changing the relations of possible and necessary preferences, $(B,\mathscr{S})$ is a positive instance if and only if Γ is possibly not Pareto-optimal.

The fact that we can verify, for a nondeterministically chosen extension and coalition structure, whether it Pareto dominates a given coalition structure, in polynomial time, completes the proof. ◻

Questions left open and approaches of how to tackle them can be found in Section 5.4.

5.3 Altruistic Hedonic Games

The following model is accepted as a conference contribution jointly with Nguyen, Rey, Rothe, and Schend [NRR$^+$16]. We propose the idea of expanding preference extensions in hedonic games from single players' decisions to altruistic influences. Our model is based on friend-oriented hedonic games.

In order to compare two coalitions A and B under consideration of friends' opinions it would be an intuitive idea to collect the preference relations between A and B of all friends in the intersection of A and B, and decide which one is preferred according to majority or a similar evaluation method. A weak point of this idea, however, is the fact that we can indeed compare two coalitions easily, but may have trouble with comparing three or more coalitions. Assume, for example, that coalitions A and B have a common friend who prefers A to B; B and a third coalition C have another common friend who prefers B to C; and C and A have a third common friend who prefers C to A. Such a scenario exists even in rather small (seven vertices) and symmetric networks, as shown in [NRR$^+$16], and leads to an irrational aggregated opinion among friends. If the player whose preference order we study is indifferent between A, B, and C, we obtain an intransitive order whichever positive impact the friends' opinion has on our player.

Alternatively, we could detect all those cycles in the aggregated friends' opinion, and dissolve it into indifference. This, however, might lead to the need of a comparison of an exponential number of coalitions in the number of players in order to compare two coalitions. Hence, we would have a conflict with the compact representation of our model, and it would have been easier to specify an arbitrary preference as an input in the first place.

Furthermore, if we asked other friends than those concerned directly, we would lose the hedonic aspect of the game, as the happiness of a player would depend on other coalitions. Nevertheless, we can consider a friend's opinion on a single coalition A, regardless of whether he is contained in B, and compare two separate assimilable values for the two coalitions. This idea leads us to the following modelling.

From a player i's point of view, the utility for a coalition A in the game is on the one hand determined by i's own evaluation of A and on the other hand by the average value of i's friends. Based on the friend-oriented extension principle (see Section 2.3.2), we obtain a friend j's opinion on a coalition containing both player i and j, by the value $u_j(A) = n\|A \cap N_j^+\| - \|A \cap N_j^-\|$. This can have an influence on i's utility on a coalition and thus on her preference relation in the following ways.

Considering friends to be equally important and focussing on the average valuation, three main cases turn out to be reasonable to be distinguished. These three cases correspond to different *degrees of altruism*: A player may be selfish at first and ask her friends only in case of indifference, treat her friends and herself equally, or be truly altruistic by asking her friends first and deciding herself only in case of indifference. By assigning a weight to player i's own contribution in comparison to her friends' influence on her preference, we obtain these priorities.

Definition 5.42. *Let i be a player in a network of friends. Her extended preferences are*

selfish first *if i initially decides upon her preference over two coalitions friend-orientedly (see Section 2.3.2) and, if and only if she is indifferent between them, she asks her friends for a vote. For $M > n^5$, we define:*

$$A \succeq_i^{+sf} B \iff M(n\|A \cap N_i^+\| - \|A \cap N_i^-\|) + \sum_{a \in A \cap N_i^+} \frac{n\|A \cap N_a^+\| - \|A \cap N_a^-\|}{\|A \cap N_i^+\|}$$
$$\geq M(n\|B \cap N_i^+\| - \|B \cap N_i^-\|) + \sum_{b \in B \cap N_i^+} \frac{n\|B \cap N_b^+\| - \|B \cap N_b^-\|}{\|B \cap N_i^+\|}.$$

equally treated *if i and her friends "vote" friend-orientedly at the same time, equally taking part in the decision. Formally, we define:*

$$A \succeq_i^{+eq} B \iff \sum_{a \in A \cap (N_i^+ \cup \{i\})} \frac{n\|A \cap N_a^+\| - \|A \cap N_a^-\|}{\|A \cap (N_i^+ \cup \{i\})\|}$$
$$\geq \sum_{b \in B \cap (N_i^+ \cup \{i\})} \frac{n\|B \cap N_b^+\| - \|B \cap N_b^-\|}{\|B \cap (N_i^+ \cup \{i\})\|}.$$

altruistic *if i first asks her friends for their opinion on a coalition they are contained in and adopts their average opinion; if and only if the consensus is indifference, the player decides for herself. For $M > n^5$, we define:*

$$A \succeq_i^{+al} B \iff n\|A \cap N_i^+\| - \|A \cap N_i^-\| + M \sum_{a \in A \cap N_i^+} \frac{n\|A \cap N_a^+\| - \|A \cap N_a^-\|}{\|A \cap N_i^+\|}$$
$$\geq n\|B \cap N_i^+\| - \|B \cap N_i^-\| + M \sum_{b \in B \cap N_i^+} \frac{n\|B \cap N_b^+\| - \|B \cap N_b^-\|}{\|B \cap N_i^+\|}.$$

In [NRR$^+$16] it is shown that $M > n^5$ is sufficient for the intuitive priorities and for the definitions to be equivalent.

In all three cases, we normalize by the number of friends whose opinion is considered, to obtain a true comparability between two coalitions from the friends' points of view, and not prefer a coalition merely because it contains more friends (which would only repeat the friend-based nature of the model).

The following network of friends provides an example of the three variants and how they are distinct and gradually represent the three different approaches to altruism in hedonic games.

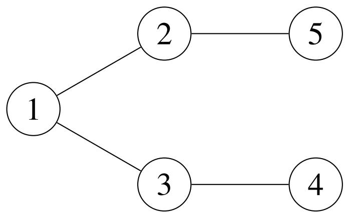


Figure 5.10: A network of friends illustrating the distinct degrees of altruism in Example 5.43

Example 5.43. *Consider the game with five players $N = \{1,2,3,4,5\}$ and friendship relations represented by the network in Figure 5.10.*

Player 1*'s extended preferences depend on the degree of altruism. The following table lists the related positive utilities of coalitions in the friend-oriented order. The utilities for non-acceptable coalitions are not mentioned, since the preferences over those coalitions are the same in all models considered, as they do not contain any friends.*

friend-oriented, C:	$\{1,2,3\}$	$\{1,2,3,4\}$	$\{1,2,3,5\}$	N	$\{1,2\}$	$\{1,3\}$	$\{1,2,4\}$	$\{1,2,5\}$	$\{1,3,4\}$	$\{1,3,5\}$	$\{1,2,4,5\}$	$\{1,3,4,5\}$
$u_1(C)$	10	9	9	8	5	5	4	4	4	4	3	3
$u_2(C)$	4	3	9	8	5	–	4	10	–	–	9	–
$u_3(C)$	4	9	3	8	–	5	–	–	10	4	–	9
selfish-first		6	6		5	5	**4**	**10**	10	4	9	9
equally-treated	6	7	7	**8**	5	5	4	7	7	4	6	6
altruistic	4	6	6	8	5	5	4	**10**	**10**	4	9	9

Table 5.5: A player's utilities in a hedonic game with different altruistic influences

One can see that all four weak preference orders are different: Under the friend-oriented preference extension, player 1*'s weak preference order is the one given in the first line according to the values of u_1. For the selfish-first extension, the main order remains the same; however, indifferences can be dissolved, as is the case here with $\{1,2,5\} \succ_1^{+sf} \{1,2,4\}$. Therefore, values in cells left blank are irrelevant, while a dash indicates that a value does not exist. Under equally-treated preferences, the grand coalition is the most preferred one. Intuitively, this is the case because all friends have a large number of friends at the same time. Finally, under altruistic preferences, player* 1*'s friends consider $\{1,2,5\}$ and $\{1,3,4\}$ the best coalition. Since they agree on that, player* 1 *altruistically adopts this opinion without considering her own opinion.*

The utility of a coalition from player i's point of view can also be deduced from the corresponding network of friends itself.

Proposition 5.44. *Let G be a network of friends. Moreover, let λ denote the number of edges in $\{\{v_i, v_j\} \mid j \in C \cap N_i^+\}$, that is, the number of friends $\|C \cap N_i^+\|$. Let μ denote the number of edges between friends of i's, that is, $\|\{\{v_j, v_k\} \mid j, k \in C \cap N_i^+\}\|$, and let $\nu = \|\{\{v_j, v_k\} \mid j \in C \cap N_i^+, k \in C \cap N_j^+, k \notin C \cap N_i^+\}\|$. Then, i's utility of a coalition $C \in \mathcal{N}_i$ under selfish-first preferences is*

$$M \cdot \lambda(n+1) + M + n + 2 - (M+1)\|C\| + \frac{(n+1)(2\mu+\nu)}{\lambda};$$

under equal-treatment

$$\frac{(2\lambda+2\mu+\nu)(n+1)}{\lambda+1} - \|C\| + 1;$$

and under altruistic-treatment preferences

$$M(n+2) + \lambda(n+1) + 1 - (M+1)\|C\| + \frac{M(n+1)(2\mu+\nu)}{\lambda}.$$

5.3.1 Axiomatic Analysis

We study hedonic games with altruistic influences with regard to their axiomatic properties. We formulate several properties (see also Section 2.3.2) as adapted from decision making literature [Tid06, End13, BBP04, LR16] and analyse hedonic games with altruistic influences with respect to them. Notably, all three relations in Definition 5.42 are reflexive, transitive, and total, therefore indeed preference extensions. They are also anonymous/neutral.

A player's value of a coalition and therefore a player's utility can be computed in polynomial time. Hence, the following property holds.

Proposition 5.45. *For a network of friends and under all three degrees of altruism, the relation of two coalitions containing a player from this player's point of view, can be determined in polynomial time.*

It is our purpose of friend's influence that independence is not satisfied. On the contrary, we want to encourage the idea of two coalitions being evaluated differently depending on a new player added who might be friends with varied players in the two coalitions. Consider the introducing example shown in Figure 5.1. For all three degrees of altruism, player 1 is indifferent between $\{1,3\}$ and $\{1,4\}$, but prefers $\{1,2,3\}$ to $\{1,2,4\}$. Monotonicity is satisfied for selfish-first preferences, but not always for more altruistic extensions. It might be the case that two players i and j become friends; however, there are two coalitions $A, B \in \mathcal{N}_i$ of which i has preferred A before, but now prefers B because j does not have any friends in A besides i. In this case the average value in A can be decreased while it is increased if j is popular in A. What is more, we define the following type of monotonicity. Let $i \in N$ be a player in a network of friends. Here the only possibility for player $j \neq i$ to advance in i's opinion is to turn from being an enemy to being a friend (i.e., adding an edge $\{v_i, v_j\}$ in the corresponding network). In terms of influences of friends on the preference, we say that $\succ$ is *type-I-monotonic*[4] if for two coalitions $A, B \in \mathcal{N}_i$ with $j \in A \cap B$, it holds that (1) if $A \succ_i B$ and $A \succeq_j^+ B$, then $A \succ_i' B$, and (2) if $A \sim_i B$ and $A \succeq_j^+ B$, then $A \succeq_i' B$.

For a network of friends and a preference extension under altruistic influences based on friend-orientation, we define *friend-oriented unanimity*: Let $i \in N$ be a player in the network. Let $A, B \in \mathcal{N}_i$ be coalitions with $A \cap N_i^+ = B \cap N_i^+$. We say that $\succeq_i$ is *friend-orientedly unanimous* if $A \succeq_j^+ B$ for each $j \in (N_i^+ \cup \{i\}) \cap A$ implies that $A \succ_i B$. Note that the definition of friend-oriented unanimity covers all cases where the same subset of friends is consulted and they all have a unanimous opinion in terms of friend-oriented preferences, in particular the case where all friends' opinions are considered: $N_i^+ \subseteq A \cap B$. For the notion of symmetry, the game is not changed if two players are swapped, here corresponds to swapping two vertices is an automorphism.

Having defined and inspected these properties, we obtain the following proposition.

Proposition 5.46 ([NRR+16])**.** *For all three degrees of altruism, friend-oriented unanimity and symmetry are satisfied. For selfish-first preferences, type-I-monotonicity holds.*

Note that the opposite implication of symmetry only holds if the interchangeable players have a distance of at most two. Furthermore, note that an even stronger graph-theoretic property holds: Two coalitions C and D are interchangeable (have the same value) from player i's point of view if they have the same graph structure from player i's point of view. That is, there exists an automorphism π of G with $\pi(i) = i$, $\pi(j) = k$ and $\pi(k) = j$ for $j \in C$ and $k \in D$, and $\pi(\ell) = \ell$ for $\ell \notin C \cup D$.

Proposition 5.47 ([NRR+16])**.** *Hedonic games with altruistic influences express different hedonic games than friend-oriented and additively separable hedonic games.*

Proof Sketch. On the one hand, there exists a hedonic game with altruistic influences that is not additively separable: We have seen that the game in Figure 5.1 is not extended independently for all three degrees of altruism. Therefore the game is not additively separable. This, in turn, implies that it cannot be represented by a friend-oriented hedonic game.

On the other hand, for each friend-oriented hedonic game $\mathscr{H} = (N, \succeq)$, it is the case that from the point of view of one player i, there exists a network that extends to the same preference order. However, in this network, all of i's friends cannot have any other friends but i. This means that an arbitrary friend-oriented preference order of a friend of i's cannot be represented. As a consequence, not all additively separable hedonic games can be represented by a hedonic game with altruistic influences. 🕮

5.3.2 Complexity Results

In this section, we study questions of verification and existence of stable coalition structures in our model. If, for some concept, such a coalition structure does not always exist, we are interested in the computational complexity of deciding whether or not such a coalition structure exists in a given game. These results are encapsulated in Table 5.3.

4 In [NRR+16] we denote the former variant as type-II-monotonic.

Observation 5.48. *A coalition structure Γ is individually rational under all three degrees of altruism (Definition 5.42) if and only if for each $i \in N$, $\Gamma(i) \cap N_i^+ \neq \emptyset$ or $\Gamma(i) = \{i\}$.*

Under selfish-first preferences, it is easy to figure out which coalition is the most preferred one for each player, namely, the unique coalition of i and all her friends. Thus, it is also easy to find out whether there exists a perfect coalition structure, which is the rare case if and only if each connected component in the underlying graph is a clique.

In order to study equal treatment and altruism consider the following lemmas.

Lemma 5.49. *Let $\langle N, \succeq \rangle$ be a hedonic game represented as a network of friends and with preferences under any of the three degrees of altruism.*

1. *For each player i, each of her friends $j \in N_i^+$ assigns a positive value to any coalition $C \in \mathscr{N}_i \cap \mathscr{N}_j$.*
2. *If a player has at least one friend, her favourite coalition contains at least one friend.*

Proof.
1. Due to symmetric friendship relationships, a friend always has at least one friend in a coalition she is asked to evaluate. Therefore, if a valuation of a friend is considered to influence a preference, it is always positive.
2. Suppose a coalition contains player i and none of her friends, than the overall value is at most zero. If there is at least one friend, the value is positive by the first statement of this lemma. □

Lemma 5.50. *Let $\langle N, \succeq \rangle$ be a hedonic game represented as a network of friends and preferences under equal treatment of friends. Let C be player i's most preferred coalition. If a friend j is in $N_i^+ \cap C$, then $N_j^+ \setminus N_i^+ \subseteq C$.*

Proof. Assume there is a player $k \in N_j^+ \setminus N_i^+$ with $k \notin C$. Then, $C \cup \{k\} \succ_i^{+eq} C$, since i asks the same number of friends and the value of $C \cup \{k\}$ increases by n for at least one player and decreases by 1 for at most $n-2$ players:

$$\begin{aligned}
&\frac{\sum_{a \in (N_i^+ \cup \{i\}) \cap (C \cup \{k\})} u_a(C \cup \{k\})}{\|(N_i^+ \cup \{i\}) \cap (C \cup \{k\})\|} - \frac{\sum_{a \in C \cap (N_i^+ \cup \{i\})} u_a(C)}{\|(N_i^+ \cup \{i\}) \cap C\|} \\
&= \frac{\sum_{a \in C \cap (N_i^+ \cup \{i\})} (u_a(C \cup \{k\}) - u_a(C))}{\|N_i^+ \cap C\| + 1} = \frac{\sum_{a \in C \cap (N_i^+ \cup \{i\}) \cap N_k^+} (n) - \sum_{a \in C \cap (N_i^+ \cup \{i\}) \setminus N_k^+} (1)}{\|N_i^+ \cap C\| + 1} \\
&\geq \frac{n-(n-2)}{\|N_i^+ \cap C\| + 1} > 0.
\end{aligned}$$
□

As a necessary condition for a perfect coalition structure under equal treatment, we can state the following proposition.

Proposition 5.51. *Whenever there exists a perfect coalition structure under equal preferences, it is unique and consists of all connected components.*

Proof. Let C be a coalition in a perfect coalition. Due to Lemma 5.49.2, C is connected. Suppose C is a proper subset of a connected component. Then, there exists an edge $\{v_k, v_\ell\}$ with $k \in C$ and $\ell \notin C$. By Lemma 5.49.2, there exists another friend j of k's in C.

Case 1: Assume there exists a player j with $\ell \notin N_j^+$. Then, by Lemma 5.50 this is a contradiction to C being j's favourite coalition, because $C \cup \{\ell\} \succ_j^{+eq} C$.

Case 2: Therefore, for each $j \in N_k^+ \cap C$, it holds that $\ell \in N_j^+$ (and $j \in N_\ell^+$ by symmetry).

(a) Assume there exists another player $x \in C$ with $\ell \notin N_x^+$. By Lemma 5.49.2, there exists a player $j \in N_k^+$ with $x \in N_j^+$ (and $j \in N_x^+$). Again, with $\ell \in N_j^+$ this is a contradiction to C being x's most preferred coalition by Lemma 5.50.

(b) Finally, for each player $x \in C$, $\{v_x, v_\ell\}$ is an edge in the network graph. This implies that $u_\ell(C \cup \{\ell\}) = n \cdot \|C\| - 0$. Thus, comparing coalitions $C \cup \{\ell\}$ and C from k's point of view, and letting f denote $\|N_k^+ \cap C\|$ we obtain:

$$
\begin{aligned}
&\frac{u_k(C \cup \{\ell\}) + \sum_{j \in N_k^+ \cap C} u_j(C \cup \{\ell\}) + u_\ell(C \cup \{\ell\})}{1+f+1} - \frac{u_k(C) + \sum_{j \in N_k^+ \cap C} u_j(C)}{1+f} \\
&= \frac{(1+f)(u_k(C)+n+\sum_{j \in N_k^+ \cap C}(u_j(C)+n) + n \cdot \|C\|) - (2+f)(u_k(C) + \sum_{j \in N_k^+ \cap C} u_j(C))}{(2+f)(1+f)} \\
&= \frac{(1+f)n - u_k(C) + f(1+f)n - \sum_{j \in N_k^+ \cap C} u_j(C) + (1+f)(n \cdot \|C\|)}{(2+f)(1+f)} \\
&= \frac{(1+f)n - n \cdot f + \|N_k^- \cap C\| + f(1+f)n - \sum_{j \in N_k^+ \cap C} u_j(C) + (1+f)(n \cdot \|C\|)}{(2+f)(1+f)} \\
&\geq \frac{n + \|N_k^- \cap C\| + f(1+f)n - f\|C\|n + (1+f)(n \cdot \|C\|)}{(2+f)(1+f)} \\
&= \frac{n + \|N_k^- \cap C\| + f(1+f)n + n \cdot \|C\|}{(2+f)(1+f)} > 0.
\end{aligned}
$$

Therefore, $C \cup \{\ell\} \succ_k^{+eq} C$ holds, which means that C has to be the whole connected component. ∎

Corollary 5.52. *If there exists a perfect coalition structure, all connected components have a diameter of at most two.*

Next, we show the polynomial-time decidability of the verification problems for single-player deviations under selfish-first preference extensions.

Proposition 5.53. *For all three degrees of altruism, it can be tested in polynomial time whether a given coalition structure in a given game is Nash-stable, individually stable, or contractually individually stable.*

Proof. Let Γ be a coalition structure. By definition, we need to check if for each player $i \in N$ and for each existing coalition C in Γ or for the empty coalition, i prefers $\Gamma(i)$ to being added to C. For n players, there are at most $n+1$ such coalitions, and the preference relation can be verified in polynomial time by Proposition 5.45. Analogous arguments hold for individual and contractually individual stability. ◻

Theorem 5.54. *For all three degrees of altruism, a Nash-stable coalition structure always exist.*

Proof. Let $E = \{i \mid N_i^+ = \emptyset\}$ be the set of players without friends. The coalition structure $\{\{i\} \mid i \in E\} \cup \{N \setminus E\}$ is Nash-stable. For each $i \in E$, $u_i(N \setminus E) < 0$, since there are no friends to be evaluated positively nor to be asked for their valuation. Therefore, they would rather stay alone. For each $i \notin E$, $u_i(N \setminus E) > 0$, since there is at least one friend who leads to a positive value and i herself contributes a positive value by Lemma 5.49.1. Thus i would rather like to stay in $N \setminus E$ than to move alone to the empty coalition or to an enemy. ◻

Since Nash stability implies individual stability, which, in turn, implies contractually individual stability, the following corollary holds.

Corollary 5.55. *Individually and contractually individually stable coalition structures always exist.*

Similarly, a (strictly) core-stable coalition structure always exists for selfish-first preferences.

Theorem 5.56 ([NRR+16])**.** *In hedonic games with selfish-first preferences, the coalition structure consisting of the connected components is always strictly core-stable.*

A Pareto-optimal coalition structure, of course, always exists. On the other hand, for all three degrees of altruism, there exists a game such that no coalition structure is strictly popular.

Example 5.57. *Consider the game given in Example 5.43.*

1. *Under selfish-first preferences, coalition structures* $\{\{1,2,5\},\{3,4\}\}$ *and* $\{\{1,3,4\},\{2,5\}\}$ *are more popular than all other coalition structures, but are in a tie.*

2. *For equally treated preferences, even three coalition structures are in a tie:* $\{\{1,2,3,4\},\{5\}\}$, $\{\{1,2,3,5\},\{4\}\}$, *and the one consisting of only the grand coalition,* $\{\{1,2,3,4,5\}\}$.

3. *Under altruistic preferences,* $\{\{1,3,4\},\{2,5\}\}$ *is more popular than* $\{\{1,2,3,4\},\{5\}\}$, *which, in turn, is more popular than* $\{\{1,2,3,4,5\}\}$. *However, the number of players who prefer* $\{\{1,2,3,4,5\}\}$ *to* $\{\{1,3,4\},\{2,5\}\}$ *is equal to the number of players who prefer, vice versa,* $\{\{1,3,4\},\{2,5\}\}$ *to* $\{\{1,2,3,4,5\}\}$. *Furthermore,* $\{\{1,2,5\},\{3,4\}\}$ *is more popular than* $\{\{1,2,3,5\},\{4\}\}$*; the two coalition structures behave analogously due to symmetries. There is no other coalition structure that is not beaten by any of the above-mentioned coalition structures. Hence, no coalition structure is strictly popular.*

Theorem 5.58. *1. The question as to whether a given coalition structure in a given game with selfish-first preferences is strictly popular is* coNP*-complete;*

2. the question of whether there exists a strictly popular coalition structure in a given game under selfish-first preferences is coNP*-hard.*

Proof. 1. The problem belongs to coNP, since the complementary problem can be decided by nondeterministically choosing another coalition structure and verifying whether a larger number of players prefer the latter to the former than the other way around. This verification can be done in polynomial time by Proposition 5.45.

We show coNP-hardness by means of a polynomial-time many-one reduction from XC_3 to the complement of our problem. For a given XC_3 instance consisting of sets $B = \{1, \ldots, 3k\}$ and a family $\mathscr{S}$ of subsets $S \subseteq B$ with $\|S\| = 3$ we, again, may assume that each $b \in B$ occurs at most three times in the sets within $\mathscr{S}$ [GJ79]. The following construction is, once again, inspired by methods used by Sung and Dimitrov [SD10] which are adopted in a non-trivial way, though, and in a different way than in Section 5.2. Given such an XC_3 instance $(B, \mathscr{S})$, we consider the set of players

$$N = \{\beta_b \mid b \in B\} \cup \{\zeta_{S,\ell} \mid S \in \mathscr{S}, 1 \leq \ell \leq 3k\} \cup \{\eta_{S,j} \mid S \in \mathscr{S}, 1 \leq j \leq 3k+3\}.$$

The network of friends[5] as displayed in Figure 5.11 is the following:

- all players in $\{\beta_b \mid b \in B\}$ are friends with each other,
- β_b and $\zeta_{S,\ell}$ are each others' friends if $b \in S$, for each $S \in \mathscr{S}$, $1 \leq \ell \leq 3k$,
- for each $S \in \mathscr{S}$, all players in $Q_S = \{\zeta_{S,\ell}, \eta_{S,j} \mid 1 \leq \ell \leq 3k, 1 \leq j \leq 3k+3\}$ are each others' friends, and
- there are no other friendship relations.

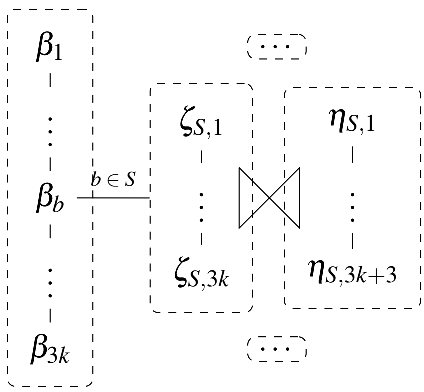


Figure 5.11: Network of friends for construction in proof of Theorem 5.58

[5] Note again that for the sake of readability player names rather than vertex names are specified in figures.

Moreover, define the coalition structure

$$\Gamma = \{\{\beta_b \mid b \in B\}\} \cup \{Q_S \mid S \in \mathscr{S}\}.$$

We show that Γ is strictly popular if and only if there exists no exact cover of B in $\mathscr{S}$.

Only if: Assume there exists an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ such that $\bigcup_{S \in \mathscr{S}'} S = B$ and $\|\mathscr{S}'\| = k$. Then, for the coalition structure

$$\Delta = \{\{\beta_b \mid b \in S\} \cup Q_S \mid S \in \mathscr{S}'\} \cup \{Q_S \mid S \in \mathscr{S} \smallsetminus \mathscr{S}'\},$$

it holds that

$$\|\{i \mid \Delta(i) \succ_i^{+sf} \Gamma(i)\}\| = 3k + k \cdot 3k = k(3k+3) = \|\{j \mid \Gamma(j) \succ_j^{+sf} \Delta(j)\}\|.$$

Hence, Γ cannot be strictly popular.

If: If, on the other hand, Γ is not strictly popular, there exists some coalition structure Δ that is preferred to Γ by at least as many players as the number of those who prefer Γ to Δ.

Consider the following cases.

- If for an $S \in \mathscr{S}$, the players $\eta_{S,j}$, $1 \leq j \leq 3k+3$, do not play together in Q_S or as soon as another player β_b, $b \in B$, is added to their clique, there are $3k+3$ dissatisfied players.
- If the players β_b, $b \in B$, do not join some Q_S, $b \in S$, their best choice is teaming up, which leads to coalition structure Γ.
- Consequently, at least one Q_S is changed in Δ. The $3k+3$ negative votes can only be balanced if at least as many other players prefer their coalition in Δ.
- $3k$ votes can be compensated by players $\zeta_{S,\ell}$, $1 \leq \ell \leq 3k$. As soon as there are more players from this type, another $Q_{S'}$ is altered. The only way to improve the situation for such a player is to invite some extra players β_b, $b \in S$. One extra player β_b, $b \in S$ would be the same as two such extra players and one player of the form $\eta_{S,k}$ less from a selfish point of view. However, then the friend's influences would be employed, in favour of the full clique Q_S.
- The remaining three negative votes have to be settled by players β_b, $b \in B$. They only improve their situation if they join some Q_S, $b \in S$; and if they do so, all at once. Otherwise, they would be dissatisfied in comparison to their coalition in Γ. Indifference is not possible.
- If more than k sets Q_S, say $x > k$, are altered by this, there are $3k + x \cdot 3k$ positive votes and $x \cdot (3k+3)$ negative votes, which means that Γ is more popular by a difference of $3(x-k)$ votes. Therefore, the players β_b, $b \in B$, cannot make up for negative votes if they move alone or in pairs to a total of more than k sets Q_S.

Having eliminated all other possibilities, the one which remains is a coalition structure Δ where for each $b \in B$, there exists a set $S \in \mathscr{S}$ such that $\Delta(\beta_b) = \{\beta_{b'} \mid b' \in S\} \cup Q_S$. As a consequence, there exists an exact cover $\mathscr{S}' = \{S \in \mathscr{S} \mid \Delta(\zeta_{S,1}) = \{\beta_{b'} \mid b' \in S\} \cup Q_S\}$ of B in $\mathscr{S}$.

2. Consider the same reduction as above, except that Γ is not given. If, on the one hand, there is no exact cover of B in $\mathscr{S}$, a strictly popular coalition structure exists, namely, Γ as considered above. If, on the other hand, there is an exact cover of B in $\mathscr{S}$, note that Γ beats every other coalition structure in pairwise comparison, but is in a tie in comparison to Δ as defined above. Therefore, Γ as well as any other coalition structure cannot be strictly popular. ◻

5.4 Challenges and Future Work

We have studied three representations of hedonic games with respect to their axiomatic consistency. Next to enemy-oriented preference extensions, we have designed two new preference extension schemes, namely the generalized Bossong-Schweigert principle leaving a set of possible full extensions open, and preferences with three degrees of altruism. Both models are original and allow the expression of opinions that are not possible as yet in other compact representations. Particularly, the generalized Bossong–Schweigert extension principle is novel not only in the context of hedonic games. We have adapted several notions of stability to possibly and necessarily stable coalition structures and have presented characterizations of stability concepts when appropriate.

We have seen, for these representations, that coalition structures considered as stable might not be guaranteed to exist. Deciding whether they exist as well as identifying them might be possible only at great cost in terms of complexity.

For all encodings and preference extensions studied many questions and directions of future work have been left open and arise newly. In the context of enemy-oriented hedonic games, we provide a next step towards Θ_2^p-hardness of wonderful stability existence. Moreover, we will have a closer look at meta-theorems that help to determine the complexity of stability problems due to similar structures in hedonic games.

Bridging A Gap Chang and Kadin [CK95] define the following property: A problem *A has* AND_ω *functions*[6] if for all $n \in \mathbb{N} \setminus \{0\}$ there exists an $f \in \text{FP}$ such that for all possible input parameters $x_1, \ldots, x_n$, it holds that for all i, $1 \leq i \leq n$, $x_i \in A$ if and only if $f(x_1, x_2, \ldots, x_n) \in A$.

Lemma 5.59 ([CK95]). *1. If a problem is* NP*-complete, it has* AND_ω *functions.*

2. If a problem is DP*-complete, it has* AND_ω *functions.*

[6] Note that this is a different ω than the clique number, used here for consistency with the literature. Which ω is meant will always be clear from the context.

3. *If a problem is complete for any class of the Boolean hierarchy higher than the second level, it cannot have* AND_ω *functions, unless the Boolean hierarchy collapses to the second level.*

4. *If a problem is* Θ_2^p*-complete, it has* AND_ω *functions.*

Note that WSE has AND_ω functions by Property 5.6. By Lemma 5.59, we thus can conclude that WSE cannot be contained in any level of the Boolean hierarchy if it is not also contained in the second level: WSE is either complete for DP or Θ_2^p (or something completely different). Here, we discuss an approach for showing the conjecture that WSE is Θ_2^p-hard.

In order to apply Wagner's Lemma 2.3, the idea would be to generalize the construction for showing DP-hardness of WSE (see the proof of Theorem 5.10). From $2k$ given instances $x_1,\dots,x_{2k}$ of some NP-hard problem A, we construct a WSE instance as a graph G with $k+1$ independent components G_i, $1 \le i \le k+1$, in polynomial time. Then again, we can use Property 5.6 to deduce that G has a wonderfully stable partition if and only if each G_i, $1 \le i \le k+1$, has one. The single components G_i are constructed as illustrated in Figure 5.12: x_1 maps to G_i, x_{2k} maps to G_{k+1}, and the remaining $k-1$ components G_i, $2 \le i \le k$, are constructed from pairs (x_{2i-2}, x_{2i-1}) such that Property 5.60 holds.

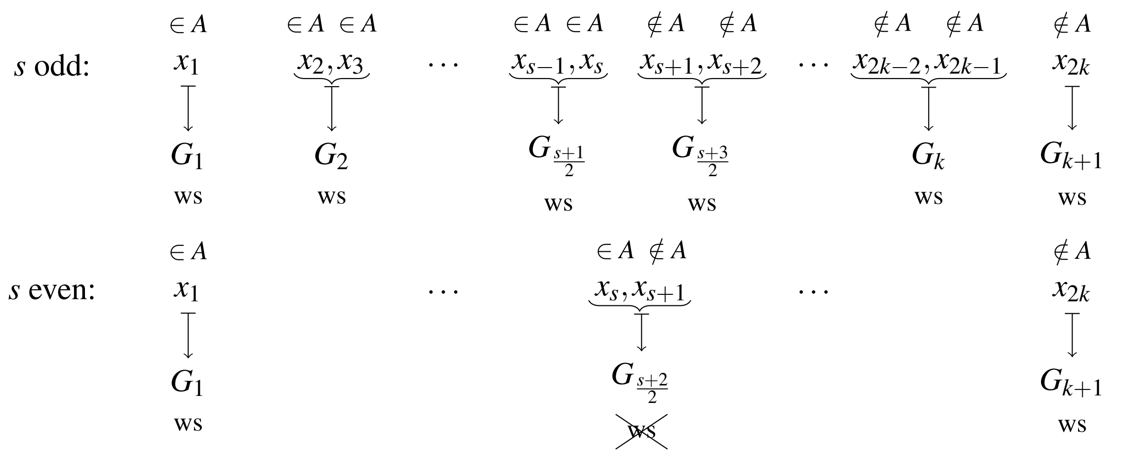


Figure 5.12: Illustration of a construction towards complexity of wonderful stability existence using Lemma 2.3. Here, ws is short for wonderfully stable.

Property 5.60. *Let* $x_1,\dots,x_{2k}$ *be given instances of an* NP*-hard problem* A*. Let* $G_1,\dots,G_{k+1}$ *be constructed graphs that satisfy:*

1. $x_1 \in A \iff G_1 \in \mathrm{WSE}$,

2. $x_{2k} \in A \iff G_{k+1} \notin \mathrm{WSE}$, *and*

3. *for each i,* $2 \le i \le k$, $(x_{2i-2}, x_{2i-1} \in A)$ *or* $(x_{2i-2}, x_{2i-1} \notin A) \iff G_i \in \mathrm{WSE}$.

Then, we can conclude a sufficient condition for Θ_2^p-hardness of WSE.

Proposition 5.61. *Let A be an* NP*-hard problem and let $x_1,\dots,x_{2k}$ be any $2k$ instances of A such that $x_j \in A$ implies $x_i \in A$ for $i < j$. If $G_1,\dots,G_{k+1}$ are graphs that can be constructed from $x_1,\dots,x_{2k}$ in polynomial time such that Property 5.60 is satisfied, then* WSE *is Θ_2^p-hard.*

Proof. Let f be a polynomial-time computable function such that $f(x_1,\dots,x_{2k}) = G$, where G is the graph consisting of $k+1$ independent components $G_1,\dots,G_{k+1}$ that satisfy Property 5.60. In order to apply Lemma 2.3, we have to show:

$$\|\{x_i \mid x_i \in A, 1 \leq i \leq 2k\}\| \text{ is odd } \iff G \in \text{WSE}.$$

Only if: Assume that $\|\{x_i \mid x_i \in A, 1 \leq i \leq 2k\}\|$ is odd. Since $x_j \in A$ implies that $x_i \in A$ for $i < j$, neither $x_1 \notin A$ nor $x_{2k} \in A$ can hold. By Property 5.60, both G_1 and G_{k+1} have a wonderfully stable partition. Furthermore, there exists an index s, $1 < s < 2k$, such that $x_i \in A$ for $i \leq s$, and $x_i \notin A$ for $i > s$. Again, due to the relation between the instances x_i only three cases can occur for each pair (x_{2i-2}, x_{2i-1}) of the remaining instances: (1) both x_{2i-2} and x_{2i-1} are in A; (2) neither x_{2i-2} nor x_{2i-1} are in A; or (3) x_{2i-2} is in A, yet x_{2i-1} is not. The latter case implies that s is of the form $s = 2i-2$ for some i which leads to a contradiction to s being odd. Therefore, all pairs have to be of the form stated in the first two cases. By Property 5.60, each component G_i, $2 \leq i \leq k$, has a wonderfully stable partition and so does G by Property 5.6.

If: Assume that there exists a wonderfully stable partition in G. This implies that every component G_i, $1 \leq i \leq k+1$, does as well. By Property 5.60, it holds that $x_1 \in A$, $x_{2k} \notin A$, and for all pairs (x_{2i-2}, x_{2i-1}), $2 \leq i \leq k$, either both x_{2i-2} and x_{2i-1} are in A, or neither x_{2i-2} nor x_{2i-1} are in A. In total, we have an odd number of instances in A among $x_1,\dots,x_{2k}$. ◻

With the reduction presented in the DP-hardness proof for WSE (see Theorem 5.10), the subgraphs G_1 and G_{k+1} can be constructed from given XC_3 instances such that the desired first two statements of Property 5.60 hold. In order to complete the Θ_2^p-hardness proof with the help of Proposition 5.61, we would have to construct the remaining subgraphs $G_2,\dots,G_k$ so as to satisfy the third property of Property 5.60. Looking closely at this property and letting the NP-complete set A be 3-SAT, we are searching for a polynomial-time reduction f such that for two given 3-SAT instances, φ_1 and φ_2, it holds that:

$$(\varphi_1, \varphi_2 \in \text{3-SAT}) \text{ or } (\varphi_1, \varphi_2 \notin \text{3-SAT}) \iff f(\varphi_1, \varphi_2) \in \text{WSE}. \tag{5.4}$$

Now, it seems reasonable to consider the DP-complete problem SAT-UNSAT, where we may assume that $\varphi_2 \in$ 3-SAT implies $\varphi_1 \in$ 3-SAT. By Lemma 2.1, this restriction of SAT-UNSAT is also DP-complete. Then Equivalence (5.4) simplifies to:

$$(\varphi_1, \varphi_2) \notin \text{SAT-UNSAT} \iff f(\varphi_1, \varphi_2) \in \text{WSE} \tag{5.5}$$

It follows that in order to prove Θ_2^p-hardness—and thus Θ_2^p-completeness—of WSE, it suffices to show coDP-hardness of WSE. By definition Θ_2^p-hardness implies coDP-hardness. To summarize, we have shown the following result.

Theorem 5.62. WSE *is* Θ_2^p*-complete if and only if it is* coDP*-hard.*

A similar statement to Proposition 5.61 applies to the existence problem of a strict core coalition structure. Hence, essentially the same argument works for SCSE as well: In order to prove a Θ_2^p-hardness lower bound, it would suffice to establish a coDP-hardness lower bound.

Corollary 5.63. *SCSE is* Θ_2^p*-hard if and only if it is* coDP*-hard.*

In this case this would, however, still leave an open gap between Θ_2^p-hardness and Σ_2^p-membership. Whether or not coDP-hardness holds for WSE or SCSE is left as an open problem.

Meta-Theorems Recently, Peters and Elkind [PE15] have presented a number of meta-theorems that imply NP-hardness results for existence problems for several representations of hedonic games and several stability concepts. We present approaches to adapt these theorems to the classes of hedonic games, we study. A class of hedonic games can be considered as a set of hedonic games with similar properties such as games that are induced by a certain representation. We, again, begin with single player deviations.

Theorem 5.64 ([PE15])**.** *For a class of hedonic games,* NASH STABILITY EXISTENCE *and* INDIVIDUAL STABILITY EXISTENCE *are* NP*-complete if this class satisfies the following properties:*

1. *The games in this class induce for each player $i \in N$, a weak preference order $\trianglerighteq_i$ over the set of players N which divides the set of players into friends $F_i = \{j \neq i \mid j \trianglerighteq_i i\}$ and enemies $E_i = \{j \mid i \triangleright_i j\}$ and allows each player to express an arbitrary order of coalitions of size two. We refer to this property as* arbitrary ordering of agents.
2. *For each player set N and each n-tuple of orderings $(\trianglerighteq_1, \dots, \trianglerighteq_n)$, the class contains a corresponding game such that:*
   (a) *the game is* consistent on pairs*, that is, for each $i \in N$ and for two players $j,k \in F_i \cup \{i\}$, $\{i,j\} \succeq_i \{i,k\}$ holds if and only if $j \trianglerighteq_i k$;*
   (b) *the game is* strictly 0-1-toxic*, that is, for each $i \in N$ and each coalition $S \in \mathscr{N}_i$, $\|S \cap F_i\| = 0$ and $\|S \cap E_i\| \geq 1$ implies $\{i\} \succ_i S$;*
   (c) *the game is* strictly 1-1-toxic*, that is, for each $i \in N$ and each coalition $S \in \mathscr{N}_i$, $\|S \cap F_i\| = 1$ and $\|S \cap E_i\| \geq 1$ implies $\{i\} \succ_i S$;*
   (d) *the game is* strictly 2-2-toxic*, that is, for each $i \in N$ and each coalition $S \in \mathscr{N}_i$, $\|S \cap F_i\| = 2$ and $\|S \cap E_i\| \geq 2$ implies $\{i\} \succ_i S$; and*
   (e) *the game is* not triangle-hating*, that is, for each $i \in N$ and for two friends $j \trianglerighteq_i k \in F_i$, it holds that $\{i,j,k\} \succeq_i \{i,k\}$.*

Remark 5.65. *Note that the first property is neither satisfied for the enemy-oriented preference extension, nor for the friend-oriented preference extension, nor for their altruistic modifications (irrespective of symmetric friendship relations); therefore, the problems studied in Sections 5.1 and 5.3 cannot be solved with this theorem.*

Nevertheless, it would be an interesting aspect for future work to formulate similar results for classes of games that do not satisfy this property. For hedonic games with ordinal preferences and thresholds (Section 5.2), we have to study the class of all possible extensions.

Lemma 5.66. *The class of all possible extensions to hedonic games with ordinal preferences and thresholds allows arbitrary ordering of agents as defined in Theorem 5.64.1.*

Proof. The well-defined ordering $\trianglerighteq_i$ is induced by $j \trianglerighteq_i k$ for two players $j,k \in N$ if

- $j,k \in N_i^+$ and $j \trianglerighteq_i^+ k$,
- $j \in N_i^+$ and $k \in N_i^0$,
- $j \in N_i^0$ and $k = i$ or $j = i$ and $k \in N_i^0$,
- $j,k \in N_i^0$,
- $j \in N_i^0$ and $k \in N_i^-$, or
- $j,k \in N_i^-$ and $j \trianglerighteq_i^- k$

and its transitive closure. That means we consider the set of neutral players as part of the set of friends F_i. The set of coalitions of size two can be ordered arbitrarily, as the set of players $N \setminus \{i\}$ can be ordered arbitrarily in a weak ranking with double threshold. ◻

Lemma 5.67. *The class of all possible extensions to hedonic games of hedonic games with ordinal preferences and thresholds satisfies Properties (a), (b), and (e) as defined in Theorem 5.64.2.*

Proof. For $\succeq^{+0-}$, and therefore, for each possible extension, (a) holds due to

$$\sigma : \{k\} \cap F_i \to \{j\} \qquad \text{with } k \mapsto j \text{ if } k \in F_i;$$

(b) holds via $\theta : \emptyset \to S \setminus (\{i\} \cup N_i^0)$; and (e) holds via $\sigma : \{k\} \cap F_i \to \{j,k\}$ with $k \mapsto k$. ◻

Lemma 5.68. *For each hedonic game with ordinal preferences and thresholds $\langle N, \trianglerighteq_1^{+0-}, \dots, \trianglerighteq_n^{+0-} \rangle$ there exists an extension $\langle \succeq_1, \dots, \succeq_n \rangle \in \times_{i=1}^{n} \mathrm{Ext}(\succeq_i^{+0-})$ that satisfies Properties (c) and (d) as defined in Theorem 5.64.2 and one that does not.*

Proof. For $S = \{i, j, e\}$ and $T = \{i, j, k, d, e\}$ with $j,k \in N_i^+$ and $d,e \in N_i^-$ the relation between S and $\{i\}$ as well as between T and $\{i\}$ is undecided in $\succeq_i^{+0-}$. Therefore, for (c) and (d), there exists an extension such that the properties holds, and one such that they do not hold. ◻

Theorem 5.69. *For the class of possible extensions that satisfy Properties (c) and (d) as defined in Theorem 5.64.2 (and for each hedonic game with ordinal preferences and thresholds there is one such extension),* Nash Stability *and* Individual Stability Existence *are* NP*-complete.*

Proof. By Lemma 5.66 the first point of Theorem 5.64 holds for all extensions of a hedonic game with ordinal preferences and thresholds. From the second point of Theorem 5.64 Properties (a), (b), and (e) are satisfied for all extensions of a hedonic game with ordinal preferences and thresholds by Lemma 5.67. Properties (c) and (d) hold for at least one extension by Lemma 5.68. By Theorem 5.64 for the class of all those extensions, NASH STABILITY EXISTENCE and INDIVIDUAL STABILITY EXISTENCE are NP-complete. ▭

Remark 5.70. *Note that this result does not imply* NP*-completeness of* POSSIBLE INDIVIDUAL STABILITY EXISTENCE*, yet.*

Next, we consider group deviations.

Theorem 5.71 ([PE15]). *For a class of hedonic games,* CORE STABILITY EXISTENCE *is* NP*-hard if this class satisfies the following properties:*

1. *The games in this class induce for each player $i \in N$, a weak preference order $\trianglerighteq_i$ over the set of players N as described in Theorem 5.64.1.*
2. *For each player set N and each n-tuple of orderings $(\trianglerighteq_1, \dots, \trianglerighteq_n)$, the class contains a corresponding game such that:*
   (a) *the game is consistent on pairs;*
   (b) *the game is strictly 0-1-toxic;*
   (f) *the game is* weakly 1-1-toxic*, that is, for each $i \in N$ and each coalition $S \in \mathscr{N}_i$, $\|S \cap F_i\| = 1$ and $\|S \cap E_i\| \geq 1$ implies $\{i, j\} \succ_i S$ for each $j \in F_i$;*
   (g) *the game is* weakly 2-2-toxic*, that is, for each $i \in N$ and each coalition $S \in \mathscr{N}_i$, $\|S \cap F_i\| = 2$ and $\|S \cap E_i\| \geq 2$ implies $\{i, j\} \succ_i S$ for each $j \in F_i$;*
   (h) *the game is* weakly 3-4-toxic*, that is, for each $i \in N$ and each coalition $S \in \mathscr{N}_i$, $\|S \cap F_i\| = 3$ and $\|S \cap E_i\| \geq 4$ implies $\{i, j\} \succ_i S$ for each $j \in F_i$;*

STRICT CORE STABILITY EXISTENCE *is* NP*-hard, if in 2. instead of (f) weakly 1-1-toxic, the game is (i)* 1-1-toxic*, that is, for each $i \in N$ and each coalition $S \in \mathscr{N}_i$, $\|S \cap F_i\| = 1$ and $\|S \cap E_i\| \geq 1$ implies $\{i\} \succeq_i S$.*

Theorem 5.72. *For the class of possible extensions that satisfy Properties (f), (g), (h) and (i) as defined in Theorem 5.71.2 (and for each hedonic game with ordinal preferences and thresholds there is one such extension),* CORE STABILITY EXISTENCE *and* STRICT CORE STABILITY EXISTENCE *is* NP*-hard.*

Proof. Again, by Lemma 5.66, the first point holds. Properties (a) and (b) hold for all extensions by Lemma 5.67. With arguments analogous to the proof of Lemma 5.68 it can be seen that the relation between the coalitions $\{i, j\}$ and $\{i, k, e\}$, $\{i, j\}$ and $\{i, j, k, d, e\}$, $\{i, j\}$ and $\{i, j, k, \ell, c, d, e\}$, as well as $\{i\}$ and $\{i, j, e\}$, for $j, k, \ell \in N_i^+$ with $k \triangleright_i j$ and $c, d, e \in N_x^-$, is undecided. Therefore there exists at least one extension such that (f), (g), (h) and (i) are satisfied. For the class of those extensions, CORE STABILITY EXISTENCE and STRICT CORE STABILITY EXISTENCE are NP-hard by Theorem 5.71. ▭

Remark 5.73. *Note that, again, this does not show* NP*-hardness of* POSSIBLE CORE STABILITY EXISTENCE *or* POSSIBLE STRICT CORE STABILITY EXISTENCE.

An adaption to this meta-theorem is also an interesting task for future work. Since the upper bounds for the core stability existence problems are higher classes in the polynomial hierarchy, it is also a challenging question to find meta-theorems for these classes.

The other two meta-theorems in [PE15] lead with an analogous study to the same results.

Open Questions and Future Directions This leaves the following problems open for hedonic games with ordinal preferences and thresholds, where the results and ideas from above are not immediately applicable (see Table 5.2): The complexity of possible verification problems for single player deviations and existence problems for individual and contractually individual stability remain open. Most problems for core stability (only the lower bound for possible verification for both, the core and the strict core, has been raised to coNP-hardness) are open regarding their complexity. There is a gap between coNP-hardness and Σ_2^p for possible verification of coalition structure comparison problems. The complexity of necessary Pareto optimality existence as well as of the existence problems for popularity are open as well. The exact complexity of the existence problems for strict popularity is also unknown up to now, although coNP-hardness has been shown as a lower bound. For altruistic hedonic games the perhaps most important open question is of how to characterize a perfect coalition structure and how hard it is to verify it for equally treated or altruistic influences. However, besides the necessary conditions presented, this question remains unsettled. Other questions, for instance, concerning group deviations are also open.

Apart from completing this analysis or considering other solution concepts in the new settings, we suggest introducing the notion of *partition correspondences* with the purpose to identify *good* coalition structures as an output for future work. As a first approach we propose defining partition functions using an adaption of a voting rule [BF02] or a similar mechanism that satisfies desirable properties [Tid06] reasonably redefined for this context. Stable outcomes (existential or universal) may be conditions for such correspondences. In contrast to the original idea of hedonic games where coalitions form in a decentralized manner, here a central correspondence is used, in order to decide which coalitions will work together. This might, for example, be the case in a setting where the head of a department has to divide a group of employees into teams. The teams should be stable, in the sense that the team members are as happy as possible with their group to create a good working atmosphere.

In the context of altruistic influences, one might think of redefining a player's happiness from not only taking friends' opinions into account, but also extending altruism to enemies' opinions, or, contrarily, acting selfishly against the enemies' will. However, this would contradict our view on a network, where communication is restricted to friends, and players further away are rather unknown than true enemies. Of course, we could ask for opinions, recursively, where friend's opinions also depend on their friends and so on, on more than one level. A question that occurs is of how much time it takes to determine a relation between two coalitions according to a player's preference; is this still possible in polynomial time in the number of players? Also, the model can be extended to edge-weighted graphs, or other encodings such as rankings of friends.

As one suggestion, we might extend the model and normalize by the size of the coalition, which can be compared to a friend-oriented restriction of a fractional hedonic game (see Section 2.3.2). For equally-treated influences, one could define for a player i in a network of friends and two coalitions A and B, i is contained in,

$$A \succeq_i^{eq} B \iff \sum_{a \in A \cap (N_i^+ \cup \{i\})} \frac{n\|A \cap N_a^+\| - \|A \cap N_a^-\|}{\|A\| \cdot \|A \cap (N_i^+ \cup \{i\})\|} \geq \sum_{b \in B \cap (N_i^+ \cup \{i\})} \frac{n\|B \cap N_b^+\| - \|B \cap N_b^-\|}{\|B\| \cdot \|B \cap (N_i^+ \cup \{i\})\|}.$$

It can be observed that in the selfish-first case only coalitions with the same cardinality are directly compared; therefore, the fractional variant results in the same weak order. The other two variants provide different preference extensions.

Example 5.74. *Consider the network of friends in Example 5.43*

friend-oriented, C:	$\{1,2,3\}$	$\{1,2,3,4\}$	$\{1,2,3,5\}$	N	$\{1,2\}$	$\{1,3\}$	$\{1,2,4\}$	$\{1,2,5\}$	$\{1,3,4\}$	$\{1,3,5\}$	$\{1,2,4,5\}$	$\{1,3,4,5\}$
$u_1(C)$	10	9	9	8	5	5	4	4	4	4	3	3
fractional	10/3	**9/4**			**5/2**							
$u_2(C)$	4	3	9	8	5	–	4	10	–	–	9	–
$u_3(C)$	4	9	3	8	–	5	–	–	10	4	–	9
equally treated	6	7	7	**8**	5	5	4	7	7	4	6	6
fractional	$\frac{120}{60}$	$\frac{105}{60}$	$\frac{105}{60}$	$\frac{96}{60}$	$\mathbf{\frac{150}{60}}$	$\mathbf{\frac{150}{60}}$	$\frac{80}{60}$	$\frac{140}{60}$	$\frac{140}{60}$	$\frac{80}{60}$	$\frac{90}{60}$	$\frac{90}{60}$
altruistic	**4**	**6**	6	8	5	5	4	**10**	**10**	4	9	9
fractional	$\mathbf{\frac{80}{60}}$	$\mathbf{\frac{80}{60}}$	$\frac{80}{60}$	$\frac{102}{60}$	$\frac{150}{60}$	$\frac{150}{60}$	$\frac{80}{60}$	$\mathbf{\frac{200}{60}}$	$\mathbf{\frac{200}{60}}$	$\frac{80}{60}$	$\frac{135}{60}$	$\frac{135}{60}$

Table 5.6: A player's utilities in a hedonic game with different altruistic influences based on a fractional friend-oriented preference extension

Combining friend-orientedness with the fractional approach, $\{1,2\}$ is preferred to $\{1,2,3,4\}$ as an example. For selfish-first preferences, it can be observed that only coalitions with the same cardinality are directly compared; therefore, the fractional variant results in the same weak order. Under the fractional approach related to equally treated preferences, the most preferred coalitions are $\{1,2\}$ and $\{1,3\}$. Differences to the fractional *variant of altruistic preferences occur, as both friends agree on* $\{1,2,3,4\}$ *being better than* $\{1,2,3\}$ *without the normalization; whereas they are indifferent between these coalitions when dividing by the coalition size, which means that player* 1 *dissolves this by her own valuation, preferring* $\{1,2,3\}$.

Note that some properties, such as a fractional-based variant of unanimity, would have to be redefined.

In addition, allowing different degrees of altruism for different players could be a natural and challenging extension. It may be interesting to study games with restricted inputs such as special graph classes that occur and for which, for example, verification of a strictly popular coalition structure would be tractable.

As a further interesting issue for future work, we finally suggest studying problems of strategic influence. A player might misreport his opinion to a friend in order to gain an advantage, might pretend to be a friend to achieve a goal if possible, or an external party might have a possibility to control the game from a bird's view.

6 Conclusions

All in all, this thesis deals with the computational complexity analysis of problems on the basis of cooperative game theory. These comprise known challenging problems from the computational social choice literature, such as false-name manipulation in weighted voting domains and wonderful stability existence in hedonic games with enemy-oriented preference extensions, as well as newly modelled related settings and their axiomatic evaluation. For instance, we propose a framework for beneficial merging and splitting in cooperative games in general, and structural control scenarios and bribery in particular settings. For hedonic games two new natural compact representations satisfying desirable properties are introduced, one an encoding open to a set of possible preference extensions and the other judging on coalitions with three degrees of altruistic influence.

Key results include the precise computational complexity, namely completeness for PP, likewise for the Shapley–Shubik and the probabilistic Penrose–Banzhaf index, of beneficial merging in weighted voting games, which solves previous conjectures in the affirmative. Next to that bribery in multiple-adversary path-disruption games with costs is Σ_2^p-complete. Other main results are settled in the context of hedonic games. We show that the challenging problem of wonderful stability existence in enemy-oriented games is DP-hard and tackle the question of its exact complexity by showing that coDP-hardness implies Θ_2^p-completeness. Relatedly, for hedonic games with ordinal preferences and thresholds, the possible Nash-stable existence problem is NP-complete. Altruistic hedonic games have the advantage that there always exist single player deviations and which are easy to detect. Strict popularity is, unfortunately, for hedonic games with preferences with selfish-first influences, NP-hard to verify. Interestingly enough, we see that some problems behave like their restricted variants when generalized to, e.g., uncertain targets in path-disruption games, two-player merging under desirable conditions, or when considering unanimity games in general.

Yet, a number of interesting questions of influence and stability remain open or emphasize anew as demanding for future work and will be summarized in the following.

On the one hand, our results are interesting in itself from a theoretical point of view. The studies complete some pictures in that we were able to show completeness results for NP, Σ_2^p, and PP as well as hardness results for NP, DP, and PP, while they bring up other interesting questions in challenging known or new settings. On the other hand, hardness results of natural problems from game theory for DP, Σ_2^p, and PP are by far rarer than for NP in this area.

Since merging, splitting, and annexation can be seen as manipulative behaviour, a high complexity can be interpreted as a protection shield against such strategic interference.

Equally, computational hardness may protect against undesired influences via bribery attacks and control scenarios in cooperative games. In often cases, even if NP-hardness of a problem is already known, it might be interesting to provide deeper insight into the computational complexity and therefore a better complexity shield. As pointed out by Woeginger, Σ_2^p-hardness indeed provides a much better security than mere NP-hardness. This holds due to the fact that while there are several common methods to circumvent NP-hardness such as approximation, fixed-parameter tractability, typical case analyses [Woe03], such methods are less applicable to circumvent hardness for higher complexity classes. For instance, there are good approximation schemes and dynamic methods known for computing the Shapley–Shubik index (see, e.g., [BMR$^+$10, FWJ08, KW05, BFJL00, MM00]). For parameterized complexity, see, e.g., [FN15, EHS15, EH15, BCF$^+$14a, Nie06, DF99]. For typical case studies applied to NP-hard voting problems, see, e.g., the survey [RS13b]. Other methods include a recent algebraic approach [BS14]. The distinction of slight differences in the definition of a decision problem is interesting and obligatory when studying their consequences for differences in complexity classes, see, e.g., [BF12].

In the same manner, it may be considered a positive result if a problem in some rather expressive setting, is *only* NP-complete instead of hard for a higher class that an intuitive algorithm would suggest.

Key ideas for future work cover, next to answering questions studied above that are not settled yet, such as conjectures of $\mathrm{NP}^{\mathrm{PP}}$- and Θ_2^p-completeness, the following issues. Details can be found in Sections 3.4, 4.2, and 5.4. New and existing models alike can be analysed with respect to further axioms; verification and existence problems studied for other stability concepts and for different representations of games. One may want to vary or refine models such as admitting external agents to bribe more selectively or distinguishing different player types, e.g., with different degrees of altruism within the same game. In the same way restrictions to, for example, special graph classes might give new insights into a problem's nature.

In any case finding interdisciplinary connections between topics provides promising research approaches for future work. On the subject of this, we are interested in axioms that may be transferred from one discipline to another, partition correspondences using aggregation methods known in other fields, and strategic influences in altruistic hedonic games. Perhaps impossibility results for certain scenarios not free from negative influences, like the famous Gibbard–Satterthwaite theorem [Gib73, Sat75] in the context of voting, can be found, in order to underline the advantage of a high complexity shield. If on the other hand cooperative games can be identified as immune against undesired side-effects, we may want to study their properties, both computational and axiomatic. Above all, challenging tasks are to translate concepts to abstract structures, especially to investigate manipulation more deeply, control scenarios in general and bribery in other representations, as well as to explore further meta-theorems that help to prove complexity results due to universal parallels in the composition of cooperative games.

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Contribution

The papers that have been published in the scope of this thesis, have been developed in equal parts together with my co-authors:

- The article [RR14a] elaborating and extending the conference contributions [RR10b], [RR10a], and [RR14b], contains my studies on false-name manipulation. The conference paper [RR16] is based on my findings on structural control in weighted voting games.

- The article *Path-Disruption Games: Bribery and a Probabilistic Model* [RRM16] as to appear in the journal *Theory of Computing Systems*, supersedes my studies on path-disruption games as published as the papers [RR11], [RR12], and [MRR14], the latter being joint work by A. Marple (Stanford University), J. Rothe, and myself in equal parts.

- The papers [RRSS14] and [RRSS16] about wonderful stability have been developed jointly in equal parts with the co-authors J. Rothe, H. Schadrack, and L. Schend.

  In the conference paper [LRR$^+$15] my contribution contains the joint development and characterization of the model in equal parts with my co-authors J. Lang, J. Rothe, H. Schadrack, and L. Schend. The section about possible and necessary stability is to be allocated to my contribution.

  The conference contribution entitled *Altruistic Hedonic Games* [NRR$^+$16] has been developed jointly with N. Nguyen, L. Rey, J. Rothe, and L. Schend in equal parts. The model and the axioms have been defined jointly. The complexity results of stability problems for single player deviations and coalition comparison are to be allocated to my contribution.