How Many Cuts Are Needed?

Questions:

- How many cuts are required to proportionally divide a cake among n players?
- How many cuts are needed to guarantee envy-freeness?

Finding answers to such questions means . . . :

- . . . to save some unneeded effort:
  It is more efficient if one can make do with as few cuts as possible.
- . . . to find a mathematically beautiful solution to a challenging problem.

What Does Count as a "Cut"?

- So far, we distinguished between **markings** and actual **cuts**.
- However, if we would maintain this distinction and would count real cuts only and not markings, we would be able to minimize the required number of cuts quite easily:
  - Instead of headily cutting pieces, the players would first make markings only and would make actual cuts just immediately prior to assigning the portions to the players.
  - The Last Diminisher protocol, for example, would then require no more than the optimal number of $n-1$ cuts for n players (only one cut per round), although there might be many markings in every round.

$\Rightarrow$ **We thus would have found merely the trivial solution to a trivial problem.**

What Does Count as a “Cut”?

- Certainly, this would not be the meaning Steinhaus (1948) had in mind when he wrote:
  “Interesting mathematical problems arise if we are to determine the minimal number of cuts necessary for fair division.”
- Therefore, we treat **markings** just like actual **cuts**—both count when we determine the minimal number of cuts required for fair division.
- Also, we will consider *finite* cake-cutting protocols only.
  - A bounded number of cuts required for fair division does not guarantee that the protocol is finite bounded, and not even that it is finite.
  - Moving-knife protocols require *infinitely (even uncountably) many* decisions, but usually require $n-1$ cuts only, and that is optimal.

The Minimal Number of Required Cuts

Definition (minimal number of cuts)

Let Π be a cake-cutting protocol for n players. The *minimal number of cuts required by* Π is the number $k(n)$ satisfying that:

1. Π always terminates with at most $k(n)$ cuts (including markings), and
2. in the worst case (with respect to the players' valuation functions), at least $k(n)$ cuts (including markings) are needed for Π to terminate.

Remark

- *That is, a cake-cutting protocol for n players requires exactly* $k(n)$ *cuts in the worst case, but may make do with fewer than that for suitably chosen valuation functions.*

The Minimal Number of Required Cuts

Remark (continued)

- *For example, if the valuation functions of the three players in the Selfridge–Conway protocol are chosen so that there remains no leftover, this protocol needs no more than merely* **two** *cuts ...*
- *... but it needs* **five** *in the worst case.*
- *No proportional cake-cutting protocol for $n \geq 1$ players can require a minimal number of fewer than $n-1$ cuts: All players must receive nonempty portions that they value to be worth at least $1/n > 0$ each.*

$\Rightarrow$ *$n-1$ is a lower bound on the minimal number of cuts required by proportional cake-cutting protocols.*

- *However, this is not the best lower bound for such protocols.*

The One-Cut-Suffices Principle

- First, however, let us investigate the minimal number of cuts required by concrete proportional cake-cutting protocols.
- Let us start with the Last Diminisher protocol with three players, Belle, Chris, and David.
- Suppose that
  - Belle cuts the piece S_1 with $v_{\text{Belle}}(S_1) = 1/3$
  - but $v_{\text{Chris}}(S_1) > 1/3$, so Chris then trims S_1 and passes the smaller piece S_2, with $v_{\text{Chris}}(S_2) = 1/3$, on to David.
  - Since $v_{\text{David}}(S_2) < 1/3$, S_2 goes to Chris who drops out with it.
  - Now, Belle and David play Cut & Choose for the reassembled remainings of the cake, $R = X - S_2$, which consists of *two* pieces, though: $A = X - S_1$ and $B = S_1 - S_2$.

The One-Cut-Suffices Principle

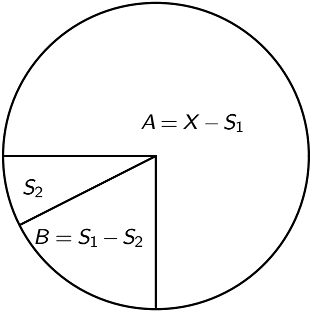

Figure: One cut suffices for the division of A and B with Cut & Choose

Do both pieces, A and B, have to be cut separately?

No, one cut suffices!

The One-Cut-Suffices Principle

- For concreteness, suppose that in our example

$$v_{\text{Belle}}(A) = \frac{2}{3} \quad \text{and} \quad v_{\text{Belle}}(B) = \frac{1}{6}.$$

- Then Belle, the cutter in Cut & Choose, ought to receive a piece of value $(1/2) \cdot (2/3 + 1/6) = 5/12$ according to her valuation function.
- So she cuts the more valuable of the two pieces, A, into two parts:
  - A_1 of value $v_{\text{Belle}}(A_1) = 5/12$ and
  - A_2 of value $v_{\text{Belle}}(A_2) = 2/3 - 5/12 = 3/12$.
- By additivity, $v_{\text{Belle}}(A_1) = v_{\text{Belle}}(A_2 \cup B) = 3/12 + 1/6 = 5/12$.
- David now has the choice between A_1 and $A_2 \cup B$.
- Thus, the Last Diminisher protocol requires no more than a total of three cuts for these valuations by the three players.

The One-Cut-Suffices Principle

- Note, however, that this is not a *worst-case* scenario.
- The worst case with respect to the minimal number of required cuts would have occurred when also David would have cut in the first round.
- Thus, the minimal number of cuts required by the Last Diminisher protocol for three players is four.
- The one-cut-suffices principle can be generalized to any number of pieces, as the following lemma shows.

The One-Cut-Suffices Principle

Lemma (one-cut-suffices principle, OCS)

Let $S_1, S_2, \ldots, S_m$ be m given pieces. A player who values S_i to be worth s_i, $1 \leq i \leq m$, can divide $S = \bigcup_{1 \leq i \leq m} S_i$ in the ratio of $x : y$ using just a single cut, where

$$x + y = s_1 + s_2 + \cdots + s_m.$$

Proof: See blackboard. ❑

Minimal Number of Cuts: Last Diminisher Protocol

Fact

The minimal number of cuts required by the Last Diminisher protocol is

$$\frac{n^2+n-4}{2}.$$

Proof: See blackboard. ❑

Minimal Number of Cuts: Modified Last Diminisher

Modify the Last Diminisher protocol as follows:

- In each execution of the second step, when one piece is being passed from one player to the next and is possibly being trimmed each time, the last player does *not* trim it if it is super-proportional, and drops out with this larger portion instead.
- The first three steps of the protocol are repeated $n-1$ times instead of $n-2$ times according to the same scheme, i.e., the last two players do not apply the Cut & Choose protocol.

Minimal Number of Cuts: Modified Last Diminisher

Fact

The minimal number of cuts required by the modified Last Diminisher protocol is

$$\frac{n(n-1)}{2}.$$

Proof: See blackboard. ❑

Example

- For $n = 50$ players, we have $1\,225$ cuts.
- For $n = 100$ players, we have $4\,950$ cuts.

Minimal Number of Cuts: Lone Chooser Protocol

- For the Lone Chooser protocol, the one-cut-suffices principle drastically decreases the minimal number of required cuts:
  Without the OCS we have
  $$n! - 1$$
  cuts, but with it we have:

Fact

The minimal number of cuts required by the Lone Chooser protocol is

$$\frac{(n-1)n(2n-1)}{6}.$$

Proof: See blackboard. ❑

Minimal Number of Cuts: Divide & Conquer Protocol

Case 1: $n = 1$. If there is only one player, she rakes in the whole cake: **no cut** is needed.

Case 2: $n = 2$. Two players apply the Cut & Choose protocol, making **one cut**.

Case 3: $n = 3$. With three players, the divide-and-conquer idea comes into play for the first time. This allows to recursively reduce this case to the simpler cases given above:

- Two of the three players divide the cake in the ratio of $1 : 2$.
- Two of the three players play Cut & Choose for a part of the cake they both value to be worth at least $2/3$.
- The remaining player receives at least $1/3$ in his valuation.

In total, $\mathbf{2 + 1 + 0 = 3}$ **cuts** are enough and needed in the worst case.

Minimal Number of Cuts: Divide & Conquer Protocol

Case 4: $n = 4$. With four players, we again reduce to simpler cases.

- Each of the first three players divides the cake into equal halves according to their valuation functions.
- The group of four players is then divided into two groups with two players each:
  - Two players play Cut & Choose for a part of the cake they both value to be worth at least $1/2$.
  - The other two players also play Cut & Choose for a part of the cake they both value to be worth at least $1/2$.

In total, $\mathbf{3+1+1=5}$ **cuts** are enough and needed in the worst case.

Case 5: $n = 5$. Analogously, the case can be reduced to two previous cases, leading to $\mathbf{4+3+1=8}$ **cuts**.

Minimal Number of Cuts: Divide & Conquer Protocol

- In general, let $D(n)$ denote the minimal number of cuts required by the Divide & Conquer protocol for n players.
- Being a recursive algorithm, the Divide & Conquer protocol for n players gives rise to the following recurrences for $k \geq 2$:

$$
\begin{aligned}
D(1) &= 0 \\
D(2) &= 1 \\
D(3) &= 3 \\
D(2k) &= 2k - 1 + 2D(k) \qquad (1) \\
D(2k+1) &= 2k + D(k) + D(k+1) \qquad (2)
\end{aligned}
$$

Minimal Number of Cuts: Divide & Conquer Protocol

- Equations (1) and (2) can be combined into just one recurrence for $n \geq 4$:

$$D(n) = n - 1 + D(\lfloor n/2 \rfloor) + D(\lceil n/2 \rceil). \qquad (3)$$

- Recurrence (3) can be solved by induction on n, and we obtain:

$$D(n) = n \cdot \lceil \log n \rceil - 2^{\lceil \log n \rceil} + 1.$$

Theorem (Even and Paz (1984))

The minimal number of cuts required by the Divide & Conquer protocol is $D(n) \in \mathscr{O}(n \log n)$.

Minimal Number of Cuts: Divide & Conquer Protocol

n	Method			$D(n)$
1	no cut needed			0
2	Cut & Choose			1
3	2 cuts reduce to the cases 1 & 2	$2+0+1$	$=$	3
4	3 cuts reduce to the cases 2 & 2	$3+1+1$	$=$	5
5	4 cuts reduce to the cases 2 & 3	$4+1+3$	$=$	8
6	5 cuts reduce to the cases 3 & 3	$5+3+3$	$=$	11
7	6 cuts reduce to the cases 3 & 4	$6+3+5$	$=$	14
8	7 cuts reduce to the cases 4 & 4	$7+5+5$	$=$	17
9	8 cuts reduce to the cases 4 & 5	$8+5+8$	$=$	21
10	9 cuts reduce to the cases 5 & 5	$9+8+8$	$=$	25
$\vdots$	$\vdots$		$\vdots$	
n	$n-1$ cuts reduce to the cases $\lfloor n/2 \rfloor$ & $\lceil n/2 \rceil$	$D(n)$	$=$	$n\lceil \log n \rceil - 2^{\lceil \log n \rceil} + 1$

Minimal Number of Cuts: Some Proportional Protocols

Table: Minimal numbers of cuts required by some proportional protocols

	Number of players						
Protocol	2	3	4	5	6	$\cdots$	n
Divide & Conquer	**1**	**3**	**5**	**8**	**11**	$\cdots$	$\mathbf{n \lceil \log n \rceil - 2^{\lceil \log n \rceil} + 1}$
Last Diminisher	1	4	8	13	19	$\cdots$	$(n^2+n-4)/2$
Last Diminisher (modified)	1	3	6	10	15	$\cdots$	$(n^2-n)/2$
Lone Chooser (without OCS)	1	5	23	119	719	$\cdots$	$n! - 1$
Lone Chooser (with OCS)	1	5	14	30	55	$\cdots$	$(n-1)n(2n-1)/6$

Upper and Lower Bounds in the Robertson–Webb Model

- While upper bounds on the minimal number of cuts commonly are presented more or less informally, the proof of a lower bound on that number requires a precise model that, in particular, specifies which operations are allowed.
- The model of Robertson and Webb allows
  - *evaluation requests* by which the protocol can gain information about how much a certain player values a certain piece of the cake, and
  - *cut requests* by which the protocol can suggest where a certain player ought to make a cut (or marking).
- In this model, Woeginger and Sgall (2007) prove a lower bound of $\Omega(n \log n)$ for the number of such operations in *any* finite, proportional cake-cutting protocol—under the condition that only *contiguous* portions are assigned.

Upper and Lower Bounds in the Robertson–Webb Model

- Without requiring this condition, Edmonds and Pruhs (2006) prove the same lower bound of $\Omega(n \log n)$ for *every* finite, proportional protocol—but by *"approximating"* both fairness and the positions of cuts in cut requests.
- It is easy to see that the upper bound of $\mathcal{O}(n \log n)$ remains valid when counting both cut and evaluation requests in the analysis of the Divide & Conquer protocol for n players.
- **How many cuts are required for envy-free protocols?**
  - The finite, envy-free protocol of Brams and Taylor (1995) is not finite bounded; its minimal number of required cuts is unbounded, too.
  - Aziz and Mackenzie's (2016) finite bounded, envy-free cake-cutting protocol for any number of players requires $\mathcal{O}(n^{n^{n^{n^{n^{n}}}}})$ Robertson–Webb operations.

Upper and Lower Bounds in the Robertson–Webb Model

- On the other hand, Procaccia (2009) proves, again in the model of Robertson and Webb, a lower bound of $\Omega(n^2)$ for the number of cut and evaluation requests in finite, envy-free cake-cutting protocols.
- His result highlights the difference between proportionality and envy-freeness:
  - An upper bound of $\mathcal{O}(n \log n)$ contrasting with
  - a lower bound of $\Omega(n^2)$

  indicates a *qualitative* discrepancy between these two concepts.
- Instead of considering asymptotic rates of growth, we now focus on the *exact* number of cuts required for proportionality to be guaranteed, especially for small values of n.

Minimal Number of Cuts Required for Proportionality

- Let $P(n)$ denote the minimal number of cuts required for a finite cake-cutting protocol to guarantee each of the n players a proportional share. This value $P(n)$ is not specific to a concrete protocol, but it is defined over *all* finite, proportional protocols.
- Recall that $D(n)$ is the minimal number of cuts required by the Divide & Conquer protocol.

Table: Comparison of $D(n)$ and the best known upper bound on $P(n)$

Number n of players	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$D(n)$ in Divide & Conquer	0	1	3	5	8	11	14	17	21	25	29	33	37	41	45	49
Upper bound on $P(n)$	**0**	**1**	**3**	**4**	**6**	**8**	13	15	18	21	24	27	33	36	40	44

Two Cuts Are not Enough for Three Proportional Shares

- How does one prove that a certain number of cuts (here two) are not enough to guarantee a certain property (here proportionality)?
- One would have to show that *every* finite cake-cutting protocol for three players requires at least three cuts in the worst case in order to assign a proportional share to each player.
- However, this might be difficult:
  - On the one hand, there are infinitely many of such protocols;
  - on the other hand, one would also have to consider those protocols that haven't even been found yet.
- Instead, we show that no finite protocol can guarantee all three players *more than a quarter—and thus, in particular, a third—of the cake with only two cuts*. **Why "one quarter"? Because . . .**

Two Cuts Are not Enough for Three Proportional Shares

Given: Cake $X = [0,1]$ and players p_1, p_2, and p_3 with valuation functions v_1, v_2, and v_3.

Step 1: p_1 divides the cake in the ratio of $1:2$ according to her valuation function, thus creating pieces S_1 and S_2 with:

$$v_1(S_1) = \frac{1}{3} \quad \text{and} \quad v_1(S_2) = \frac{2}{3}.$$

Figure: Quarter protocol for three players

Two Cuts Are not Enough for Three Proportional Shares

Step 2: Consider the following three cases.

Case 1: S_2 is worth at least one half of the cake for one of p_2 and p_3 (say, for p_2), and S_1 is worth at least one quarter of the cake for the other player (i.e., for p_3). In this case $(v_2(S_2) \geq 1/2$ and $v_3(S_1) \geq 1/4)$,

- S_1 goes to p_3, and
- p_1 and p_2 share S_2 using the Cut & Choose protocol.

The other case $(v_3(S_2) \geq 1/2$ and $v_2(S_1) \geq 1/4)$ is treated analogously.

Figure: Quarter protocol for three players

Two Cuts Are not Enough for Three Proportional Shares

Step 2 (continued):

Case 2: S_2 is worth at least one half of the cake for one of p_2 and p_3 (say, for p_2), and S_1 is worth less than one quarter of the cake for the other player (i.e., for p_3). In this case $(v_2(S_2) \geq 1/2$ and $v_3(S_1) < 1/4)$,

- S_1 goes to p_1, and
- p_2 and p_3 share S_2 using the Cut & Choose protocol.

The other case $(v_3(S_2) \geq 1/2$ and $v_2(S_1) < 1/4)$ is treated analogously.

Figure: Quarter protocol for three players

Two Cuts Are not Enough for Three Proportional Shares

Step 2 (continued):

Case 3: S_2 is worth less than one half of the cake for both p_2 and p_3 (i.e., we have $v_2(S_2) < 1/2$ and $v_3(S_2) < 1/2$). In this case,

- S_2 goes to p_1, and
- p_2 and p_3 share S_1 using the Cut & Choose protocol.

Figure: Quarter protocol for three players

Two Cuts Are not Enough for Three Proportional Shares

Fact

The Quarter protocol for three players guarantees each of the three players a quarter of the cake with only two cuts.

Proof: Two cuts are made in each case:

- One cut is made by p_1 in the first step, and
- another cut is added when executing the Cut & Choose protocol in each case of the second step.

It is easy to see that each of the three players indeed is guaranteed a quarter of the cake. ❑

Two Cuts Are not Enough for Three Proportional Shares

Theorem

No finite cake-cutting protocol for three players guarantees each of the players more than one quarter—and thus, in particular, one third—of the cake with only two cuts.

Proof: See blackboard. ❑

Four Cuts Guarantee Four Proportional Shares

- Three cuts are necessary to guarantee each of three players to receive one third of the cake.

$\Longrightarrow$ The upper bound "3" of $P(3)$ is **exact**, i.e., it coincides with the lower bound for $P(3)$.

- Also the upper bound for $P(4)$ is **exact**: A proportional division for four players can be guaranteed with four, yet not with three cuts.

  Even and Paz (1984) proposed a protocol that achieves a proportional division for four players by four cuts in the worst case.

- The Divide & Conquer protocol requires five cuts to guarantee a proportional division for four players.

Four Cuts Guarantee Four Proportional Shares

Idea behind the Even–Paz protocol:

- Suppose that some player divides cake X into four pieces, Y_1, Y_2, Z_1, and Z_2, such that
  - Y_2 is at least as valuable as Y_1 to him, and
  - Z_1 is at least as valuable as Z_2 to him.

$\Longrightarrow$ $Y_2 \cup Z_1$ is at least as valuable as one half of the cake to this player.

- Hence, he would be willing to share $Y_2 \cup Z_1$ with anyone else using the Cut & Choose protocol, as that would guarantee him one quarter of the cake.

Four Cuts Guarantee Four Proportional Shares

Definition

For any player p_i with valuation function v_i and for any two pieces of the cake, A and B, we say that:

1. p_i *prefers A to B* if $v_i(A) \geq v_i(B)$,
2. p_i *strictly prefers A to B* if $v_i(A) > v_i(B)$, and
3. A is *acceptable* for p_i if $v_i(A) \geq 1/n$, where n is the number of players.

Theorem (Even and Paz (1984))

The Quarter protocol for four players guarantees each of the four players a quarter of the cake with only four cuts.

Four Cuts Guarantee Four Proportional Shares

Proof: Look at the Quarter protocol for four players below.

Given: Cake $X = [0,1]$ and players p_1, p_2, p_3, and p_4 with valuation functions v_1, v_2, v_3, and v_4.

Step 1: p_1 divides cake X into two pieces, Y and Z, of equal value according to his valuation function. We have:

- $X = Y \cup Z$, $Y \cap Z = \emptyset$, and
- $v_1(Y) = v_1(Z) = 1/2$.

Figure: The Quarter protocol of Even and Paz (1984) for four players

Four Cuts Guarantee Four Proportional Shares

Step 2: Consider the following cases.

Case 1: Not all of p_2, p_3, and p_4 strictly prefer the same piece to the other. We thus may assume that

- $v_2(Y) \geq 1/2$,
- $v_3(Y) \geq 1/2$, and
- $v_4(Z) \geq 1/2$.

Then, p_2 and p_3 share Y and p_1 and p_4 share Z, both pairs using the Cut & Choose protocol.

Figure: The Quarter protocol of Even and Paz (1984) for four players

Four Cuts Guarantee Four Proportional Shares

Step 2 (continued):

Case 2: Each of p_2, p_3, and p_4 strictly prefer the same piece to the other. We thus may assume that $v_i(Z) > 1/2$ for $i \in \{2,3,4\}$. Then, p_1 divides Y into two pieces, Y_1 and Y_2, of equal value according to his valuation function. We have:

- $Y = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \emptyset$, and
- $v_1(Y_1) = v_1(Y_2) = 1/4$.

Figure: The Quarter protocol of Even and Paz (1984) for four players

Four Cuts Guarantee Four Proportional Shares

Step 2 (continued):

Subcase 2.1: For one of p_2, p_3, and p_4 (say, for p_2), Y_1 or Y_2 is acceptable. (This is possible, even if we have $v_i(Y) < 1/2$ for each $i \in \{2, 3, 4\}$.)

Hence, $v_2(Y_i) \geq 1/4$ for $i = 1$ or $i = 2$. Then

- p_2 receives Y_i,
- p_1 receives Y_j, $j \neq i$, and
- p_3 and p_4 share Z using Cut & Choose.

Figure: The Quarter protocol of Even and Paz (1984) for four players

Four Cuts Guarantee Four Proportional Shares

Step 2 (continued):

Subcase 2.2: For none of p_2, p_3, and p_4 is either of Y_1 or Y_2 acceptable.

However, since each of p_2, p_3, and p_4 prefers one of the pieces Y_1 and Y_2 to the other, two of these players (say, p_2 and p_3) must prefer the same (say, Y_2) to the other.

$\Longrightarrow$ $v_i(Y_2) \geq v_i(Y_1)$ for each $i \in \{2, 3\}$.

Figure: The Quarter protocol of Even and Paz (1984) for four players

Four Cuts Guarantee Four Proportional Shares

- Then p_1 receives Y_1 and drops out, while
- p_2, p_3, and p_4 share $X - Y_1 = Y_2 \cup Z$ among each other.

To this end, p_2 divides Z into two pieces, Z_1 and Z_2, of equal value according to her valuation function. We have:

- $Z = Z_1 \cup Z_2$, $Z_1 \cap Z_2 = \emptyset$, and
- $v_2(Z_1) = v_2(Z_2) > 1/4$.

Let us assume that p_3 prefers Z_1 to Z_2.

Figure: The Quarter protocol of Even and Paz (1984) for four players

Four Cuts Guarantee Four Proportional Shares

Step 2 (continued):

Case 2.2.1: Z_2 is acceptable for p_4. Then

- Z_2 goes to p_4, and
- p_2 and p_3 share $Y_2 \cup Z_1$ using Cut & Choose.

Case 2.2.2: Z_2 is not acceptable for p_4.

Then p_4 must prefer Z_1 to Z_2. In this case,

- Z_2 goes to p_2, and
- p_3 and p_4 share $Y_2 \cup Z_1$ using Cut & Choose.

Figure: The Quarter protocol of Even and Paz (1984) for four players

Four Cuts Guarantee Four Proportional Shares

- **Why is the Quarter protocol for four players correct?**
- Case 2.2 is the critical one:
  - p_1 first drops out with Y_1, which is acceptable for him.
  - Then, p_2, p_3, and p_4 share $X - Y_1 = Y_2 \cup Z$ among each other.
- **But wait!**

  Didn't we show earlier that two cuts (and we are not allowed to make more cuts at this point) are *not* enough to guarantee a proportional division among three players?
- We will exploit our (partial) knowledge of what p_2, p_3, and p_4 think about Y_1, Y_2, Z_1, and Z_2 in Case 2.2.

Four Cuts Guarantee Four Proportional Shares

Table: What do p_2, p_3, and p_4 think about Y_1, Y_2, Z_1, and Z_2 in Case 2.2?

	Y_1	Y_2	Z_1	Z_2
p_2	I	♡I	♡A	♡A
p_3	I	♡I	♡A	
p_4	I	I		

Key

A	:	acceptable (of value $\geq 1/4$)
I	:	inacceptable (of value $< 1/4$)
♡	:	prefers Y_i to Y_j or Z_i to Z_j

How might the still missing entries in this table look like?

To answer this question, we need to make a final case distinction.

Four Cuts Guarantee Four Proportional Shares

Table: What do p_2, p_3, and p_4 think about Y_1, Y_2, Z_1, and Z_2 in Case 2.2.1?

	Y_1	Y_2	Z_1	Z_2
p_2	I	♡I	♡A	♡A
p_3	I	♡I	♡A	
p_4	I	I		A

- All players receive a portion they consider to be acceptable.
- In total, they needed only four cuts for this division.

Four Cuts Guarantee Four Proportional Shares

Table: What do p_2, p_3, and p_4 think about Y_1, Y_2, Z_1, and Z_2 in Case 2.2.2?

	Y_1	Y_2	Z_1	Z_2
p_2	I	♡I	♡A	♡A
p_3	I	♡I	♡A	
p_4	I	I	♡A	I

- Also in this case all players receive a portion they each find acceptable.
- Again, only four cuts were needed. ❏

How Many Cuts Guarantee How Many Players Which Share of the Cake?

We have seen that in finite cake-cutting protocols:

- **with one cut**, two players can be guaranteed to receive at least one half of the cake by Cut & Choose, and that is optimal, as zero cuts obviously do not accomplish this;
- **with two cuts**, each of three players can be guaranteed to receive one quarter of the cake by Quarter protocol for three players, and that is optimal, as two cuts cannot guarantee the three players to receive a more valuable share of the cake;

How Many Cuts Guarantee How Many Players Which Share of the Cake?

- **with three cuts**, each of three players can be guaranteed to receive one third of the cake (e.g., by the modified Last Diminisher protocol or the Divide & Conquer protocol), and that again is optimal, as two cuts in particular cannot guarantee three players one third of the cake;
- **with four cuts**, each of four players can be guaranteed to receive one quarter of the cake by the Quarter protocol for four players, and that is optimal since three cuts can guarantee each of four players one sixth, but no more than one sixth of the cake.

How many cuts guarantee how many players which share of the cake?

How Many Cuts Guarantee How Many Players Which Share of the Cake?

Denoting by $M(n,k)$ the most valuable share of the cake that each of the n players can be guaranteed by k cuts in a finite cake-cutting protocol, the above results can be expressed as follows:

$$M(2,1) = 1/2,\ M(3,2) = 1/4,\ M(3,3) = 1/3,\ M(4,3) = 1/6,\ M(4,4) = 1/4.$$

Theorem (Robertson and Webb (1998))

1. $M(n, n-1) = 1/(2n-2)$ *for* $n \geq 2$.
2. $M(3,3) = 1/3$ *and* $M(n,n) = 1/(2n-4)$ *for* $n \geq 4$.
3. $M(n, n+1) \geq 1/(2n-5)$ *for* $n \geq 5$. **without proof**

How Many Cuts Guarantee How Many Players Which Share of the Cake?

Table: How many cuts guarantee how many players which share of the cake?

Number of cuts	Number of players 2	3	4	5	6	7	8	$\cdots$	n
$n-1$	**$1/2$**	**$1/4$**	**$1/6$**	**$1/8$**	**$1/10$**	**$1/12$**	**$1/14$**	$\cdots$	**$1/(2n-2)$**
n		**$1/3$**	**$1/4$**	**$1/6$**	**$1/8$**	**$1/10$**	**$1/12$**	$\cdots$	**$1/(2n-4)$**
$n+1$				**$1/5$**	$1/7$	$1/9$	$1/11$	$\cdots$	$1/(2n-5)$
$n+2$					$1/6$	$1/8$	$1/10$	$\cdots$	?