A Nonproportional Cake-Cutting Protocol for Three Players

Given: Cake $X = [0,1]$ and players p_1, p_2, and p_3 with valuation functions v_1, v_2, and v_3.

Step 1: p_1 cuts X into three pieces of equal value, S_1, S_2, and S_3, i.e.,

$$v_1(S_1) = v_1(S_2) = v_1(S_3) = \frac{1}{3}.$$

Step 2: p_3 chooses one of the three pieces that is most valuable to her.

Step 3: p_2 chooses one of the remaining two pieces.

Step 4: p_1 receives the remaining piece.

Figure: A nonproportional cake-cutting protocol for three players

Reminder: Cut-and-Choose Protocol

Given: Cake $X = [0,1]$ and players p_1 and p_2 with valuation functions v_1 and v_2.

Step 1: p_1 executes one cut to divide the cake into two pieces, S_1 and S_2, that he considers equal halves: $v_1(S_1) = v_1(S_2) = 1/2$. It holds that $X = S_1 \cup S_2$.

Step 2: p_2 chooses her best piece.

Step 3: p_1 receives the remaining piece.

Figure: Cut-and-choose protocol for two players

Remark

We know already that this protocol is proportional and even envy-free.

Austin's Moving-Knife Protocol

Definition (continuity)

The valuation function v_i of a player p_i is said to be *continuous* if we have the following: Given real numbers α and β where

$$v_i([0,a]) = \alpha \quad \text{and} \quad v_i([0,b]) = \beta$$

for a and b with $0 < a < b \leq 1$, for every real number $\gamma \in [\alpha, \beta]$ there exists some $c \in [a,b]$ with

$$v_i([0,c]) = \gamma.$$

Austin's Moving-Knife Protocol for Two Players

Given: Cake $X = [0,1]$ and players p_1 and p_2 with continuous valuation functions v_1 and v_2.

Step 1: A knife is being held above the cake and is moved gradually and continuously from the left to the right until any of the players, say p_1, calls: **"Stop!"**
When this happens, the knife position divides the cake into two pieces, S_1 and S_2 with $X = S_1 \cup S_2$, such that

$$v_1(S_1) = v_1(S_2) = \frac{1}{2}.$$

We will call this player the *step-1-caller*. If both players call **"Stop!"** at the same time, a random tie-breaker is applied (for simplicity, we assume p_1 to be chosen).

Austin's Moving-Knife Protocol for Two Players

Step 2:

- The step-1-caller, p_1, takes a second knife and holds it above the left edge of the cake.
- Now, p_1 moves both knives in parallel gradually and continuously from the left to the right of the cake such that p_1 values the part of the cake that is in-between the two knives to always be exactly half of the cake.
- We will refer to the (over time t constantly changing) piece in-between the knives as S_1^t.

Figure: Austin's moving-knife protocol for two players (continued)

Austin's Moving-Knife Protocol for Two Players

Step 3:

- p_2 is calling **"Stop!"** as soon as

$$v_2(S_1^{t_0}) = \frac{1}{2}$$

  (we will refer to this point in time as t_0) and both knives will cut the cake immediately at their positions.

- Now,
  - p_2 either chooses piece $S_1^{t_0}$ or the two adjacent pieces which give $X - S_1^{t_0}$, and
  - p_1 receives the remaining portion.

Figure: Austin's moving-knife protocol for two players (continued)

Austin's Moving-Knife Protocol for Two Players

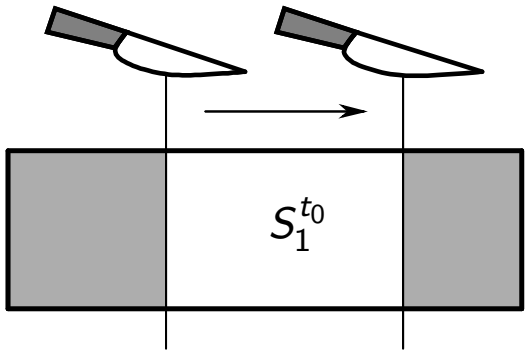


Figure: Austin's moving-knife protocol: p_2 calls **"Stop!"** at time t_0

Theorem

Austin's moving-knife protocol for two players is exact.

Proof: See blackboard. ❑

Austin's Moving-Knife Protocol for Two Players

Remark

- *The description of the protocol could be extended by adding that (in case p_2 does not call* **"Stop!"** *in Step 3) at the time when the right knife reaches the right edge of the cake, the left knife must be in the very same position as the right knife was at the beginning of Step 2 (i.e., before the knives started moving).*
- *However, this is just an additional note and is not required as long as the players follow the protocol's strategies.* **Exercises:**
  - *What would happen if Step 3 of the protocol would be modified such that the step-1-caller, here p_1, could choose which of the two portions $S_1^{t_0}$ or $X - S_1^{t_0}$ to receive?*
  - *How could any of the players deviate from the protocol's strategy in order to gain an advantage?*

Proportional Protocols for n Players

- **How can the cut-and-choose protocol be generalized to more than two players?**
- This question was first raised by Steinhaus (1947) and answered by Banach and Knaster (1948):

Hugo Steinhaus (1887–1972)

Stefan Banach (1892–1945)

Bronisław Knaster (1893–1990)

The Last Diminisher Protocol

Given: Cake $X = [0,1]$ and players $p_1, p_2, \ldots, p_n$, where p_i has valuation function v_i, $1 \leq i \leq n$. Set $N := n$.

Step 1: Player p_1 cuts piece S_1 such that $v_1(S_1) = 1/N$.

Step 2: The cut piece is passed successively to $p_2, p_3, \ldots, p_n$, each of whom trims it as appropriate. In more detail, let S_{i-1}, $2 \leq i \leq n$, be the piece player p_{i-1} passes on to p_i.

- If $v_i(S_{i-1}) > 1/N$, player p_i trims piece S_{i-1} and passes on the trimmed piece S_i where $v_i(S_i) = 1/N$.
- If $v_i(S_{i-1}) \leq 1/N$, player p_i passes on the untrimmed piece $S_i = S_{i-1}$.

The last player that either cut or trimmed this piece receives S_n and drops out.

The Last Diminisher Protocol

Step 3:
- Reassemble the remainings of the cake to $X := X - S_n$,
- rename the remaining players to $p_1, p_2, \ldots, p_{n-1}$, and
- set $n := n - 1$.

Step 4: Repeat Steps 1 to 3 until $n = 2$. The remaining two players, p_1 and p_2, apply the cut-and-choose protocol.

Figure: Last Diminisher protocol for n players

Example

- Belle and Edgar invited Chris and David for a pizza.
- The players will take their turn in alphabetical order.

The Last Diminisher Protocol

Example (continued)

- The players' valuation functions are represented using 20 boxes each:

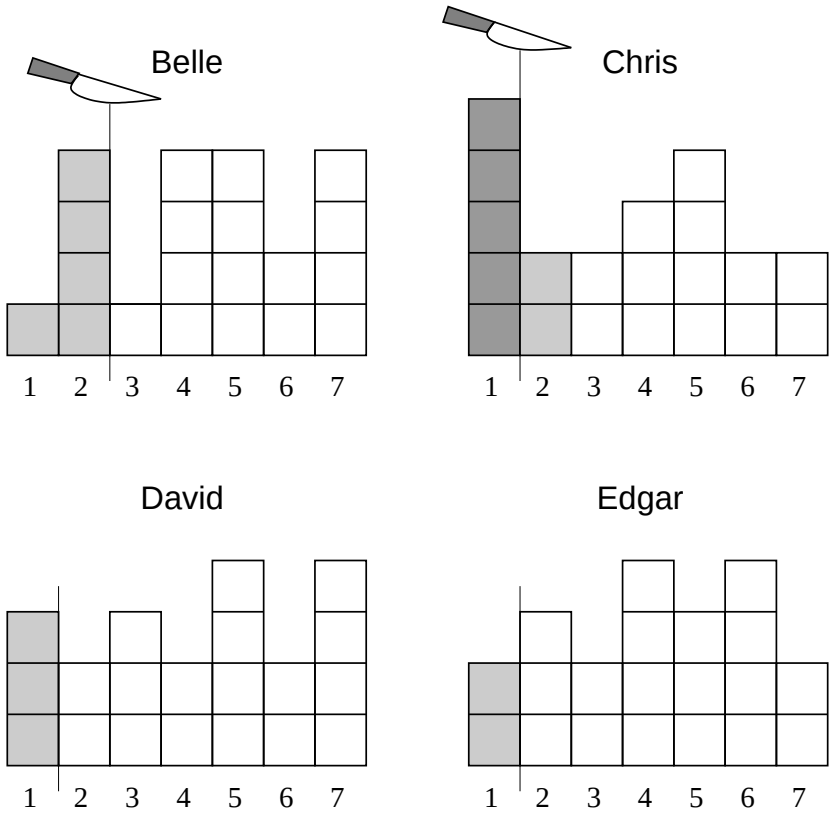


Figure: Last Diminisher protocol for four players: 1st round

The Last Diminisher Protocol

Example (continued)

- Second round:

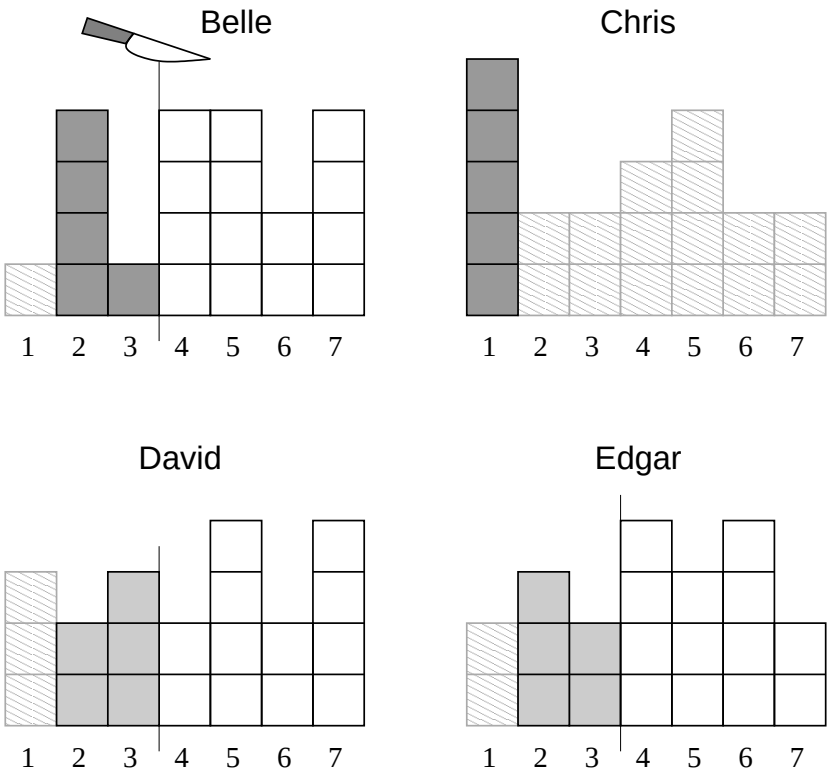


Figure: Last Diminisher protocol for four players: 2nd round

The Last Diminisher Protocol

Example (continued)

- Third and final round:

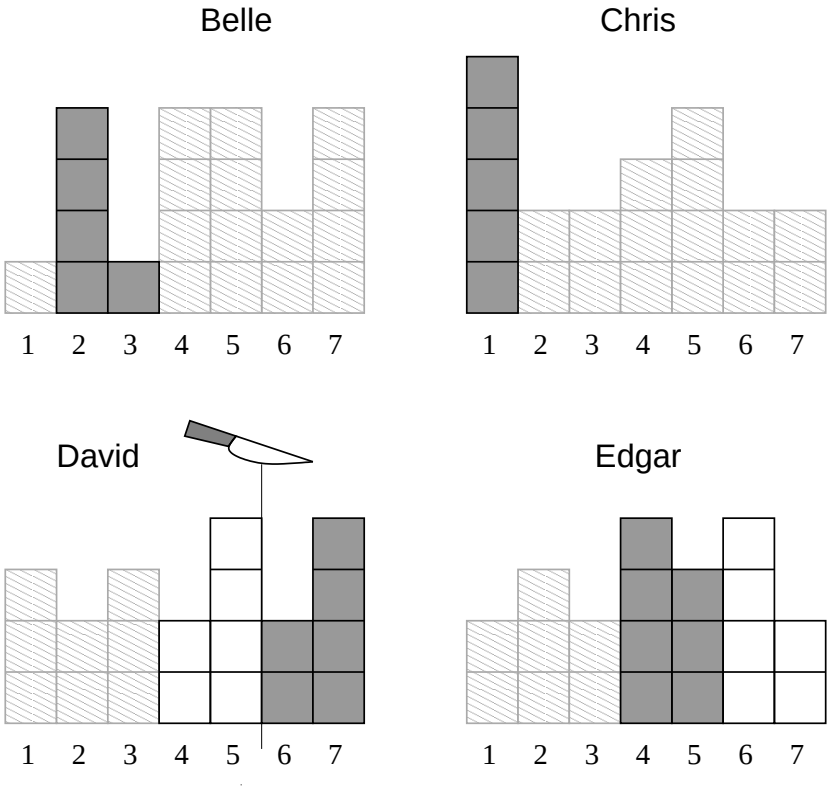


Figure: Last Diminisher protocol for four players: 3rd and final round

The Last Diminisher Protocol

Theorem

The Last Diminisher protocol is:

1. *proportional, but not envy-free,*
2. *finite bounded,*
3. *immune to manipulation for risk-averse players, and*
4. *not Pareto-optimal.*

Proof:

- For the proof of immunity to manipulation for risk-averse players, see the next slides.
- For the remaining claims, see blackboard.

The Last Diminisher Protocol: Immunity to Manipulation

- To show that the Last Diminisher protocol is immune to manipulation for risk-averse players, we need to show that

$$(\forall i)\,(\forall \sigma_i')\,(\exists v_{-i})\,[A_i' \neq A_i \implies v_i(X_i') < v_i(X_i)],$$

- Consider an arbitrary player p_i. We need to show that
  - in whatever way p_i may depart from the proposed strategies σ_i of the protocol using some σ_i' instead,
  - there exists some worst case (caused by the other players' valuation functions v_{-i}) such that
  - whenever p_i's actions $A_i' = \sigma_i'(v_{-i}, v_i)$ differ from the truthful actions $A_i = \sigma_i(v_{-i}, v_i)$, p_i is punished by receiving less than his best worst-case payoff: $v_i(X_i') < v_i(X_i)$, where X_i is the portion p_i receives when playing A_i and X_i' is p_i's portion when playing A_i'.

The Last Diminisher Protocol: Immunity to Manipulation

- How can p_i possibly deviate from the strategy proposed by the protocol?
- First, if p_i is the chooser when Cut & Choose is executed in Step 4, the only way for him to deviate from the protocol strategy is to choose a piece that is *at most* as valuable as the other for him.
- Clearly, this action would be punished in the worst case where the cutter in Cut & Choose cuts in a way that the piece chosen by p_i is less valuable than the other piece for p_i.

The Last Diminisher Protocol: Immunity to Manipulation

- Otherwise, p_i gets one of the following cut requests:
  - $\mathtt{cut}_i(X, 1/N)$ in Step 1, where $i = 1$ and X is the cake (possibly reassembled from the remainings of the previous round) at the start of this round, or
  - $\mathtt{cut}_i(S_{i-1}, 1/N)$ in Step 2, where p_i, $i > 1$, is to evaluate piece S_{i-1} and to trim it if it is super-proportional for him, and to pass it on otherwise, or
  - $\mathtt{cut}_i(X, 1/2)$ in Step 4, where $i = 1$, p_i is the cutter in Cut & Choose, and X is the cake (possibly reassembled from the remainings of the previous round) at the start of this final round.
- Suppose p_i deviates from the protocol strategy in response to $\mathtt{cut}_i(\cdot)$.
- We show that he is punished if his resulting action differs from the truthful action (i.e., only if $A'_i \neq A_i$).

The Last Diminisher Protocol: Immunity to Manipulation

Case 1: p_i cuts/trims the piece (call it X_i') to a smaller value in A_i' than in A_i, or he trims it in A_i' but passes it on in A_i.

Setting the other players' valuation functions v_{-i} so that

- p_i is the last player to cut/trim this piece X_i' in Step 1 or 2, or
- the chooser in Cut & Choose of Step 4 chooses the other piece,

p_i receives X_i' and drops out with it, thus being punished with a payoff of $v_i(X_i') < v_i(X_i)$.

Case 2: p_i cuts/trims the piece (call it T) to a greater value in A_i' than in A_i, or he passes it on in A_i' but trims it in A_i.

The Last Diminisher Protocol: Immunity to Manipulation

Case 2.1: If this happens in Cut & Choose of Step 4 in round $N-1$, set the other players' valuation functions v_{-i} so that the chooser chooses this piece T, which is worth more than $1/2$ of the remaining cake of value $2/N$ for p_i.

Hence, p_i receives a portion of value less than

$$\left(\frac{1}{2}\right)\left(\frac{2}{N}\right) = \frac{1}{N}$$

for himself.

The Last Diminisher Protocol: Immunity to Manipulation

Case 2.2: If this happens in Step 1 or Step 2 of round, say, $j < N-1$ (where N is the original number of players), set the other players' valuation functions v_{-j} so that

- at the start of this round, the (possibly reassembled, remaining) cake to be shared among the $N-j+1$ (remaining) players is of value

$$\frac{N-j+1}{N}$$

  for p_i, and

- p_i is not the last player to cut/trim this piece T in this round, but some other player drops out with a piece that p_i values to be worth more than $1/N$.

The Last Diminisher Protocol: Immunity to Manipulation

Thus, p_i does not receive T but remains in the game after round j, so the reassembled new cake to be shared among the $N-j$ remaining players in the following rounds is worth less than $(N-j)/N$ for p_i.

Hence, we can set the valuation functions v_{-i} of the other players still in the game so that p_i eventually must receive a portion of value less than $1/N$.

In all cases, p_i is punished with a payoff of

$$v_i(X_i') < v_i(X_i),$$

which completes the proof. ❑

The Dubins–Spanier Moving-Knife Protocol

- On the basis of the finite bounded Last Diminisher protocol, Dubins and Spanier (1961) presented a proportional moving-knife protocol.
- It is by and large a direct analogue of the Last Diminisher protocol, the only differences being that it
  - is a continuous rather than a finite bounded protocol and
  - it deals differently with the situation when there are only two players remaining:

    The Dubins–Spanier moving-knife protocol does not use the moving-knife analogue of the cut-and-choose protocol in this situation. Instead, the first $n-1$ portions are all assigned in the same way and the last remaining player receives all the remaining cake.
- It is easy to see that the Last Diminisher protocol could be adapted in a similar way while still being proportional.

The Dubins–Spanier Moving-Knife Protocol

Given: Cake $X = [0,1]$ and players $p_1, p_2, \ldots, p_n$ with continuous valuation functions.

Step 1:

- A knife is being held above the cake, and is moved gradually and continuously from the left to the right until any of the players calls **"Stop!"** because she considers the piece of cake to the left of the knife to be worth $1/n$.
- If several players call **"Stop!"** at the same time, a random tie-breaker is applied (for simplicity, we assume the outcome to be in favor of player p_i, where i is the smallest index amongst all players calling **"Stop!"**).

The Dubins–Spanier Moving-Knife Protocol

Step 1 (continued):
- The knife cuts the cake at the current position.
- The player who called **"Stop!"** receives the left piece of the cake and drops out.

Step 2, 3, . . . , n − 1: Repeat Step 1 for all remaining players and the remaining cake.

Step n: The last remaining player receives the remaining cake.

Figure: The Dubins–Spanier moving-knife protocol for n players (continued)

The Dubins–Spanier Moving-Knife Protocol

Theorem

The Dubins–Spanier Moving-Knife protocol is:

1. *proportional,*
2. *continuous,*
3. *immune to manipulation for risk-averse players,*
4. *not envy-free, and*
5. *not Pareto-optimal.*

Proof: Exercise. ❑

The Lone Chooser Protocol

- The Lone Chooser protocol due to Fink (1964) is a proportional and finite bounded cake-cutting protocol for any number of players.
- It also generalizes the idea of the cut-and-choose approach—though in a different way than the Last Diminisher protocol does.
- The name of the protocol results from the fact that following the rules of the protocol, in every round, all but one player cut the cake.
- The non-cutting player then chooses wisely from all the pieces available—she is the "*lone chooser*."
- Other than the Last Diminisher protocol, this protocol requires the number of players to *increase* with every round and it assigns all portions in the very last round; no player is dropping out before.
- **Advantage:** it can cope with an unknown number of players.

The Lone Chooser Protocol

Given: Cake $X = [0,1]$ and players $p_1, p_2, \ldots, p_n$, where p_i has valuation function v_i, $1 \leq i \leq n$.

Round 1:

- p_1 and p_2 apply the cut-and-choose protocol with p_1 being the one cutting the cake.
- Let S_1 be the piece player p_1 receives and S_2 be the piece p_2 chooses.
- It holds that $X = S_1 \cup S_2$ and

$$v_1(S_1) = \frac{1}{2} \quad \text{and} \quad v_2(S_2) \geq \frac{1}{2}.$$

Figure: Lone Chooser protocol for n players

The Lone Chooser Protocol

Round 2: p_3 shares S_1 with p_1 and S_2 with p_2 as follows:

- p_1 cuts S_1 into S_{11}, S_{12}, and S_{13} such that

$$v_1(S_{11}) = v_1(S_{12}) = v_1(S_{13}) = \frac{1}{6}.$$

- p_2 cuts S_2 into S_{21}, S_{22}, and S_{23} such that

$$v_2(S_{21}) = v_2(S_{22}) = v_2(S_{23}) \geq \frac{1}{6}.$$

- p_3 chooses a most valuable piece from $\{S_{11}, S_{12}, S_{13}\}$ and another most valuable piece from $\{S_{21}, S_{22}, S_{23}\}$.

$\vdots$

Figure: Lone Chooser protocol for n players (continued)

The Lone Chooser Protocol

$\vdots$

Round n – 1:

- For each i, $1 \leq i \leq n-1$, player p_i so far has a share T_i with $v_i(T_i) \geq 1/(n-1)$ and cuts T_i into n pieces $T_{i1}, T_{i2}, \ldots, T_{in}$ such that

$$v_i(T_{i1}) = v_i(T_{i2}) = \cdots = v_i(T_{in}) \geq \frac{1}{n(n-1)}.$$

- Player p_n then chooses for every i, $1 \leq i \leq n-1$, one out of $T_{i1}, T_{i2}, \ldots, T_{in}$ that is most valuable according to her valuation function v_n.

Figure: Lone Chooser protocol for n players (continued)

The Lone Chooser Protocol

Theorem

The Lone Chooser protocol is:

1. *proportional,*
2. *finite bounded,*
3. *immune to manipulation for risk-averse players,*
4. *not envy-free, and*
5. *not Pareto-optimal.*

Proof Sketch: See blackboard for proof of proportionality and finite boundedness. ❑

The Lone Chooser Protocol

Remark

- *Unlike the Last Diminisher protocol where the last two players will never envy any of the $n-2$ other players, the Lone Chooser protocol does not guarantee for any of the players that they will not envy any of the $n-1$ other players.*
- *The first player, p_1, is slightly disadvantaged, since he is only guaranteed to receive an exactly proportional share, whereas all other players might get more than that.*
- *In the Lone Chooser protocol, players are not guaranteed to receive a contiguous portion, unlike, for example, in Cut & Choose or the Dubins–Spanier moving-knife protocol.*

The Lone Chooser Protocol

Definition (contiguous portion)

Let a division of cake $X = \bigcup_{i=1}^{n} X_i$ be given with X_i being the portion assigned to player p_i. A portion X_i is called *contiguous* if there exist $x_1, x_2 \in \mathbb{R}$ with $0 \leq x_1 \leq x_2 \leq 1$ such that the following holds:

$$X_i = [x_1, x_2] = \{x \in \mathbb{R} \mid x_1 \leq x \leq x_2\}.$$

Remark

- *Stromquist (2008) has shown that there is no finite envy-free cake-cutting protocol for more than two players when only allowing contiguous portions.*
- *This implies that there is no finite envy-free cake-cutting protocol for $n \geq 3$ players that requires no more than $n-1$ cuts.*

The Lone Chooser Protocol

Remark

- *Austin (1982) points out that his moving-knife protocol can be combined with the Lone Chooser protocol such that the resulting division is not only guaranteed to be proportional but even exact. This combined version, too, is a moving-knife protocol.*

The Lone Divider Protocol

- Generalizing a method by Steinhaus for three players to any number of players, Kuhn (1967) presented the Lone Divider protocol.
- However, when increasing the number of players beyond three, the protocol relies on sophisticated mathematical theories:
  - Kuhn's original proof uses the Frobenius–König Theorem on the permanents of 0-1-matrices.
  - Other sources that discuss Kuhn's protocol use graph-theoretic and combinatorial ideas such as Hall's Marriage Theorem on the existence of a perfect matching in a graph.
- Therefore, we will not present the general Lone Divider protocol but will focus on the selected cases of three and four players only.
- Dawson (2001) introduced a recursive implementation of the Lone Divider protocol.

The Lone Divider Protocol for Three Players

- Belle, Chris, and David toss the dice to decide on which player is going to be the *lone divider*—a single player entitled to cut the cake. Let us assume Belle to be the lone divider.
- Belle cuts the cake into three equally valued (according to her valuation function) pieces, A, B, and C.
- Next, all other players to evaluate these pieces according to their valuation functions.
- All players indicate which of the pieces they find *acceptable* (i.e., to be a proportional share; here: *one third*).
- Clearly, all three pieces are acceptable for Belle, whereas for both the other players at least one piece must be acceptable.

The Lone Divider Protocol for Three Players

Case 1: For Chris or David (let us say, for Chris), at least two pieces are acceptable.

- David chooses any of the pieces he considers to be acceptable.
- Then, as Chris initially considered at least two pieces to be acceptable, there must be at least one acceptable piece left for him.
- Chris chooses any of these.
- Belle receives the remaining piece, which to her is as valuable as any of the other two pieces.

The Lone Divider Protocol for Three Players

Case 2: Both Chris and David consider only one of the pieces to be acceptable. (Note that the same argument applies if both Chris and David would find the very same piece acceptable.)

- There must be one piece (or even two pieces in the case that both players consider the very same piece to be acceptable), say C, that both Chris and David consider to be unacceptable.
- Belle gets C, as for her all three pieces are of equal value.
- Since both Chris and David consider the remainings of the cake, $A \cup B$, to be worth more than two thirds of the whole cake, they now share $A \cup B$ using the Cut & Choose protocol and, hence, are each guaranteed to receive more than $(1/2) \cdot (2/3) = 1/3$ of the whole cake.

The Lone Divider Protocol for Three and More Players

Remark

- *Every player is guaranteed to receive a proportional share.*
- *However, the Lone Divider protocol is not envy-free.*

  Who might envy whom in which case?

  Who is perhaps guaranteed to envy none of the other players?

- The procedure gets more complicated for more than three players. However, for **four** and **five** players, Custer (1994) found a clever way to only assign acceptable pieces to all the players.

The Lone Divider Protocol for Four Players

- Suppose there are four players:
  Belle (the lone divider), Chris, David, and Edgar.
- Again, Belle cuts the cake, this time into four pieces of equal value according to her valuation function: A, B, C, and D.
- All other players then evaluate all four pieces and decide which pieces they consider to be acceptable, i.e., to be worth at least one quarter of the cake.
- Again, Belle considers all four pieces to be acceptable, and for all other players at least one piece must be acceptable.

  Here, we need to distinguish the following three cases.

The Lone Divider Protocol for Four Players

Case 1: There is at least one piece (let us say A) that only Belle considers to be acceptable, i.e., all other players consider A to be unacceptable.

- If we now assign piece A to Belle, all remaining players will consider the remainings of the cake, $B \cup C \cup D$, to be worth more than three quarters of the whole cake.
- Hence, Chris, David, and Edgar can now apply the procedure for three players described above to fairly share the remainings of the cake.
- This guarantees each of them to receive more than $(1/3) \cdot (3/4) = 1/4$ of the whole cake.

The Lone Divider Protocol for Four Players

Case 2: There is at least one piece (let us say A) that only Belle and one other player (let us say Chris) consider to be acceptable.

- If we now assign piece A to Chris, all remaining players will consider the remainings of the cake $B \cup C \cup D$ to be worth at least three quarters of the whole cake:
  - for Belle exactly three quarters;
  - for David and Edgar even more than three quarters.
- Hence, Belle, David and Edgar can now apply the procedure for three players described above to fairly share the remainings of the cake.
- This guarantees each of them to receive at least $(1/3) \cdot (3/4) = 1/4$ of the whole cake.

The Lone Divider Protocol for Four Players

Case 3: Each of the pieces A, B, C, and D is considered acceptable by Belle and at least two other players.

A	B	C	D
Belle	Belle	Belle	Belle
?	?	?	?
?	?	?	?

Table: Case 3 of Custer's procedure

The Lone Divider Protocol for Four Players

- We conclude that
  - Belle considers all four pieces to be acceptable,
  - Chris considers at least three pieces to be acceptable,
  - David considers at least two pieces to be acceptable, and
  - Edgar considers at least one piece to be acceptable.

  Why? See blackboard.

- Hence, in this case, too, every player is guaranteed to receive a proportional share if the players are to choose a piece in reverse order:
  1. Edgar,
  2. David,
  3. Chris, and
  4. Belle.

How Can Disagreement Be Exploited?

Hugo Steinhaus (1887–1972)

Stefan Banach (1892–1945)

Bronisław Knaster (1893–1990)

"It may be stated incidentally that if there are two (or more) partners with different estimations, there exists a division giving to everybody **more than his due part***; the fact disproves the common opinion that differences in estimations make fair division difficult."*

– Hugo Steinhaus

Cut Your Own Piece Protocol

- This protocol, which is due to Steinhaus (1969), is very suitable to be used for dividing a property.
- Uncle **Ian** sadly passed away and left behind a lovely lakeside property to be divided amongst his three heirs, **Felix**, **George**, and **Helena**.

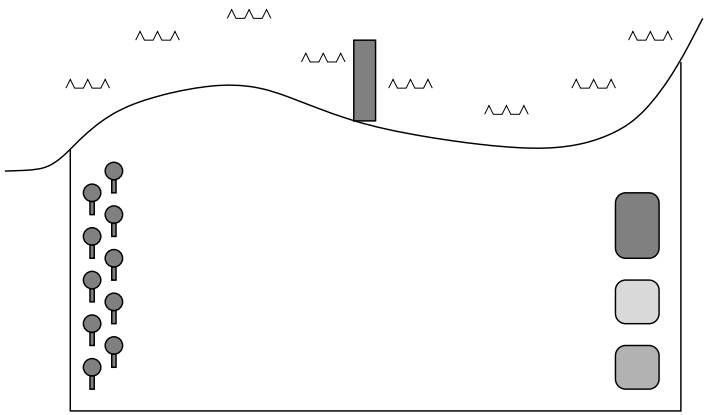
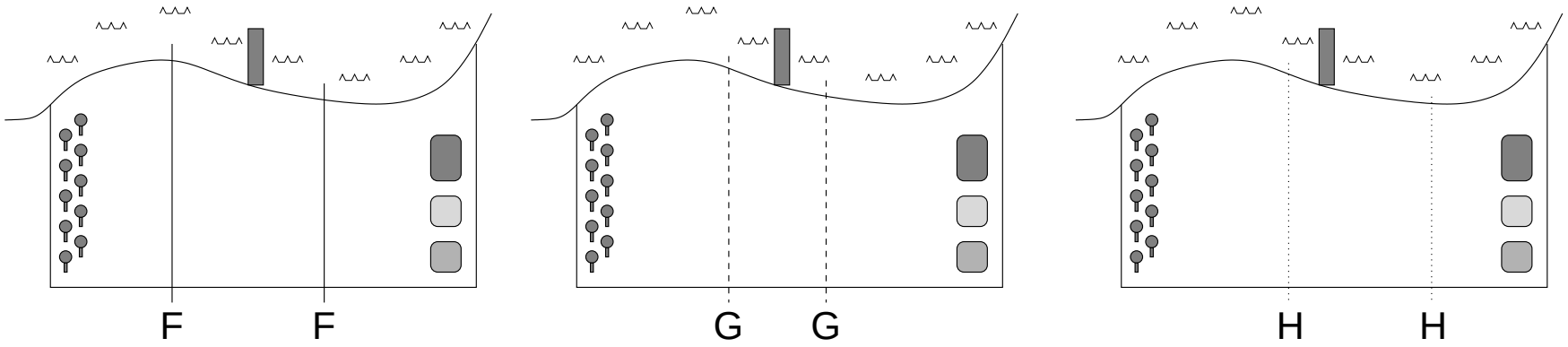


Figure: Cut Your Own Piece protocol: a lovely lakeside property

- They each mark three equally valued pieces of the property.

Cut Your Own Piece Protocol: Valuations and Markings

- **Felix** would love to get the forestry on the property's left-hand side.
- **George**, however, is very enthusiastic about sailing and hence is very keen on receiving the middle bit with the private jetty.
- **Helena** cares about neither the forest nor the boats, but would instead enjoy having a nice herb and flower garden as well as a bit of beach, preferably the sheltered bay on the right-hand side of the property.



(a) **Felix**'s markings (b) **George**'s markings (c) **Helena**'s markings

Figure: Cut Your Own Piece protocol: individual markings

Cut Your Own Piece Protocol: Valuations and Markings

- Each player makes the two markings on a separate transparent sheet of paper, a true-to-scale drawing of the property.
- Each marking needs to be in parallel to the left and right edge of the property and divides the property into three parts (see the figure).
- Markings are made such that the players consider each of their marked parts to be of the same value (w.r.t. their valuation functions).
- This implies that the more valuable a part of the property is to a player, the smaller the land area will be.
- None of the players knows or shall know the markings of the other players—which is why every single one of them is using a separate transparent sheet of paper.

Cut Your Own Piece Protocol: Valuations and Markings

- Now, Felix, George, and Helena place their three transparent sheets of paper on top of each other:

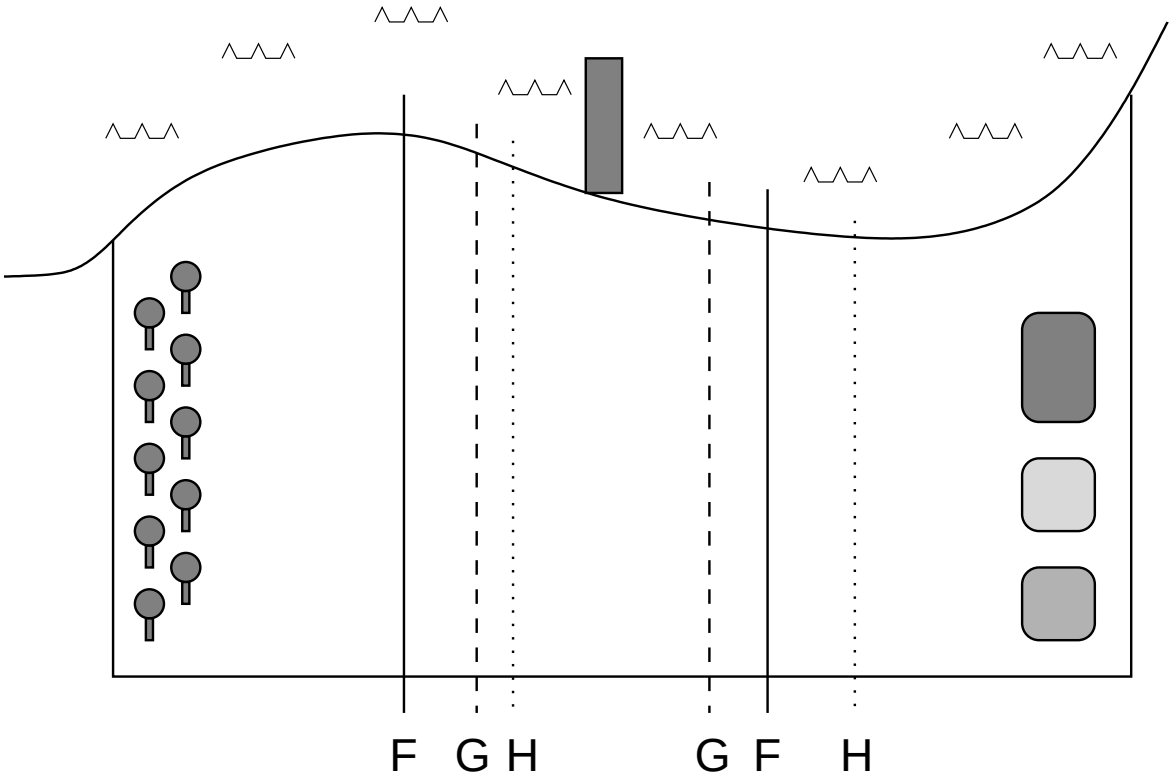
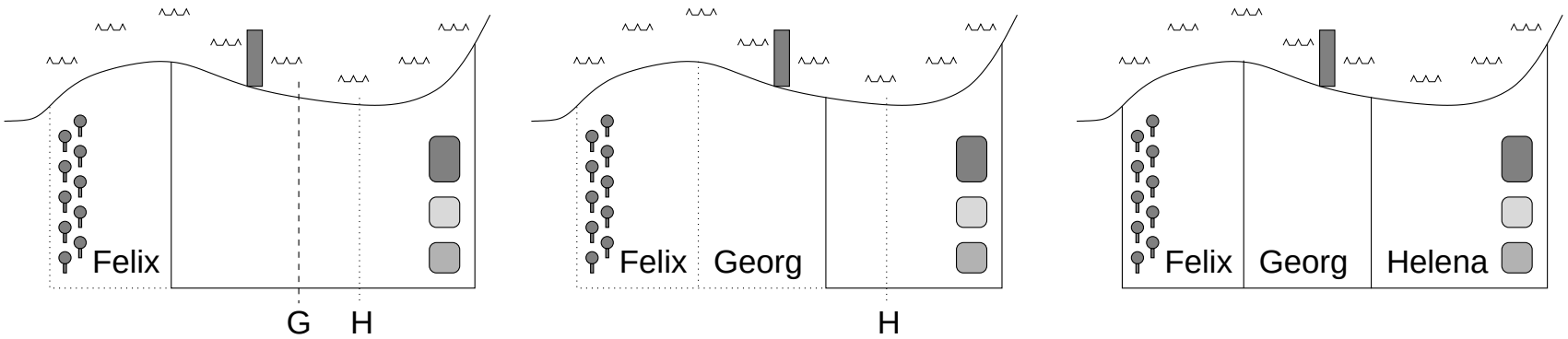


Figure: Cut Your Own Piece protocol: all markings

Cut Your Own Piece Protocol: Assignment

- **Goal:** Assign to all players a part of the property that they themselves have marked, where the three assigned parts need to be disjoint.
- **Method:** Start from the left edge of the property and keep moving to the right (ties can be broken arbitrarily):



(a) Felix gets his part (b) George gets his part (c) Helena gets her part

Figure: Cut Your Own Piece protocol: assignment of the shares

Cut Your Own Piece Protocol: Assignment

In more detail:

1. (a) The leftmost part of the property is assigned to **Felix**, as he made the leftmost marking. He receives a proportional share and drops out.
   (b) Remove all markings made by **Felix** as well as the leftmost markings of both the other players and repeat the procedure for the remaining part of the property and the remaining players.
2. **George** gets assigned the next part of the property, as his rightmost marking is to the left of **Helena**'s. Thus, he receives a proportional share as well and drops out.
3. The remaining part of the property goes to **Helena**, who also is guaranteed a proportional share and drops out.

Cut Your Own Piece Protocol for n Players

Given: Cake $X = [0,1]$ and players $p_1, p_2, \ldots, p_n$.

Step 1:

- Every player makes $n-1$ markings to divide the cake into n pieces of value $1/n$ (according to their valuation function).
- All $n(n-1)$ markings are required to be in parallel to the left and right edge of the cake, and no player knows the markings of any other player.

Figure: Cut Your Own Piece protocol for n players

Cut Your Own Piece Protocol for n Players

Step 2:
- Identify the leftmost marking.
- The piece of cake between the left edge of the cake and the leftmost marking is assigned to any of the players that placed a marking at this position, and this player drops out.
- Remove all markings of the player just dropped out and the leftmost markings of all remaining players.

Step 3: Repeat Step 2 for all remaining players and the remaining cake until all markings have been removed.

Step 4: The last remaining player receives the remaining cake.

Figure: Cut Your Own Piece protocol for n players

Cut Your Own Piece Protocol for n Players

Theorem

Suppose that at least two of the n players in the Cut Your Own Piece protocol have made distinct markings. Then the cake can be divided such that

1. *all players receive a piece that they themselves have marked,*
2. *no two of these pieces overlap, and*
3. *a nonempty piece remains.*

Proof: See blackboard. ❑

Divide & Conquer Protocol

- Even and Paz (1984) presented a cake-cutting protocol that makes use of a principle known from the Roman times: *"Divide et impera!"*
- The Divide & Conquer protocol is based on the idea that sharing the whole cake X among n players can be reduced to the problem of
  - sharing two disjoint parts partitioning the cake, A and B with $X = A \cup B$,
  - among two disjoint subgroups of players with roughly $n/2$ players each.
- Notice that a fair division of parts A and B leads to a fair division of the whole cake X if and only if parts A and B as well as the two subsets of players were determined in a fair way.

Divide & Conquer Protocol

- We distinguish between the number of players n being **even** or **odd**:
  - If the number of players is **even** then the group of players will be divided into **two subgroups with an equal number** of players.
  - If the number of players is **odd** then the players will be divided into two groups of roughly the same size, namely
    - **one subgroup with an even number** of players and
    - **another subgroup with an odd number** of players.
- Following this rule, the cake is recursively divided into smaller and smaller disjoint parts, and so are the groups of players.
- The procedure stops when every subgroup consists of one player only, who considers the part of the cake to be shared in this "subgroup" to be a proportional share of the whole cake.

Divide & Conquer Protocol for Two and Three Players

- For $n = 2$ players, Divide & Conquer is just Cut & Choose.
- For $n = 3$ players (e.g., Belle, Chris, and David), the Divide & Conquer protocol works as follows:
  - Belle and Chris each divide the cake in the ratio of **1 : 2** according to their measure using parallel markings.
  - Then it is David's turn to evaluate the piece between the **left edge** of the cake and the **leftmost marking**; call this piece A.
    - If David considers piece A to be worth at least one third of the whole cake, he accepts this piece and drops out.
      Belle and Chris now share $X - A$, which they both value to be worth at least two thirds of the cake, using the Cut & Choose protocol.
    - Otherwise, David shares $X - A$ by Cut & Choose with the player whose marking is the rightmost (where ties are broken arbitrarily).
      The player who marked A receives it and drops out.

Divide & Conquer Protocol for Four Players

- For $n = 4$ players (say Belle, Chris, David, and Edgar), the divide & conquer principle can also be applied so that proportionality will be "passed on" from one round to the next.
- Belle, Chris, and David each cut the cake in half, i.e., they divide the cake in the ratio of $\mathbf{1:1}$ w.r.t. their measure using parallel markings.
- This time, the **middle marking** is used to divide the cake into pieces A and $X - A$.
- Again, ties (i.e., markings at the very same position) are broken arbitrarily.
- We illustrate this by a specific example.

Divide & Conquer Protocol for Four Players

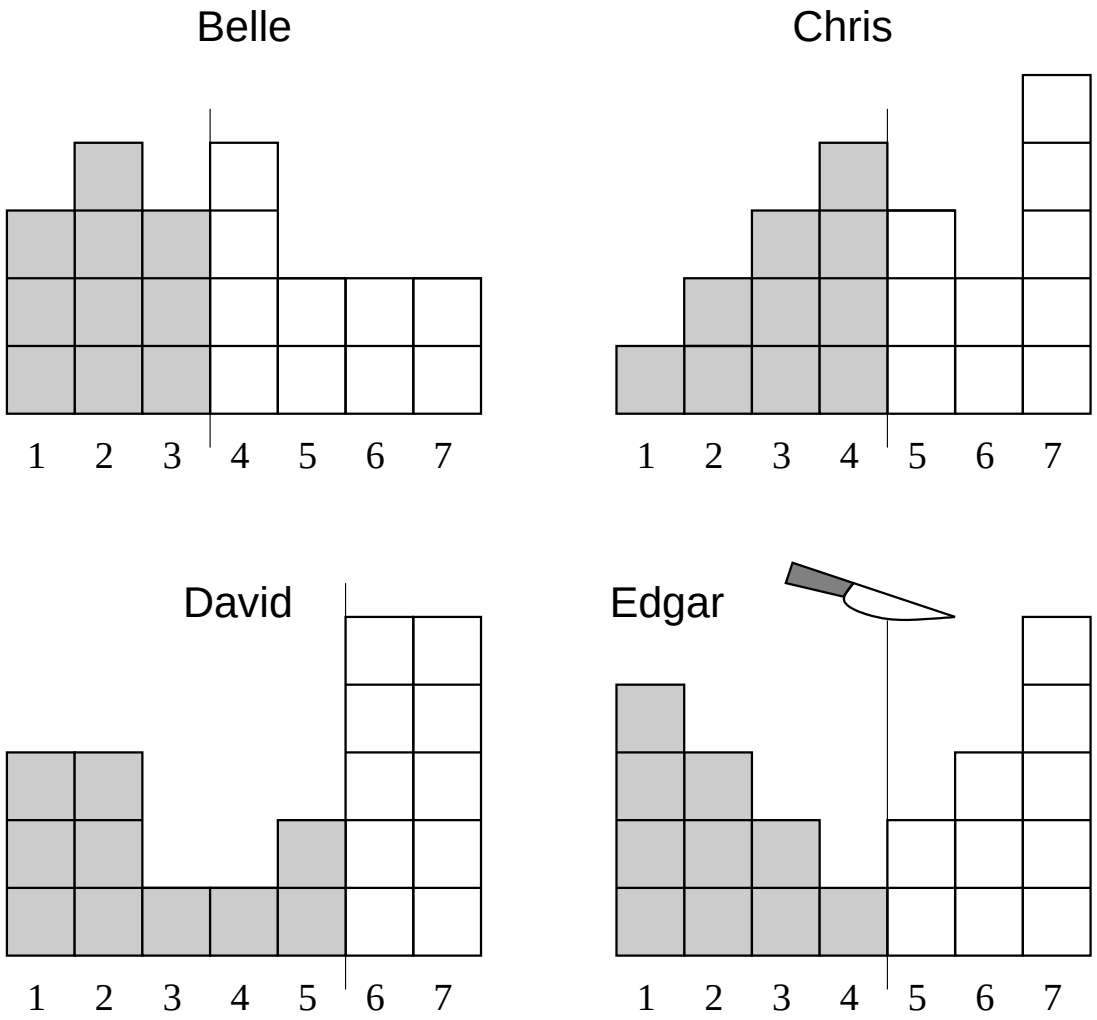
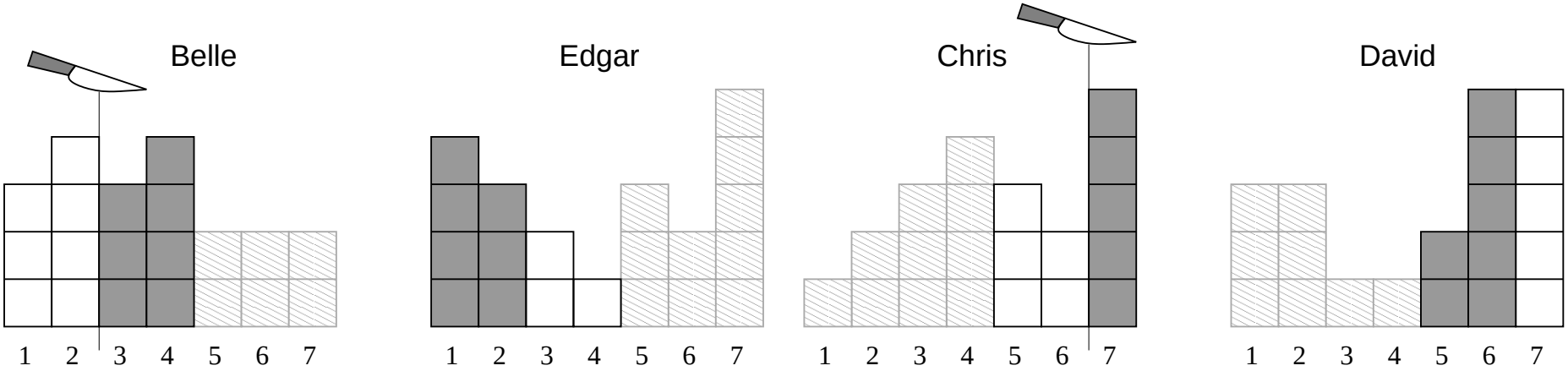


Figure: Divide & Conquer protocol for four players: 1st round

Divide & Conquer Protocol for Four Players



(a) Belle cuts, Edgar chooses

(b) Chris cuts, David chooses

Figure: Divide & Conquer protocol for four players: 2nd round

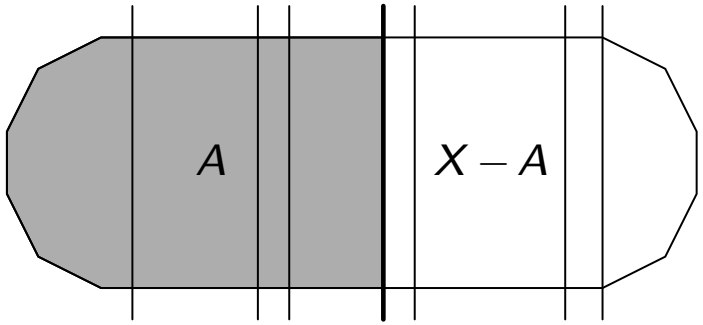
Divide & Conquer Protocol for n Players

Given: Cake $X = [0,1]$ and players $p_1, p_2, \ldots, p_n$, where p_i has valuation function v_i, $1 \leq i \leq n$.

Case 1: If $n = 1$ then player p_1 receives the whole cake X.

Case 2: If $n = 2k$ for $k \geq 1$ then apply the following steps.

Step 2.1: $p_1, p_2, \ldots, p_{n-1}$ divide the cake using parallel markings in the ratio of **k:k** according to their valuation function.
Let A be the piece of cake left of the kth marking.
For example, if $n = 8 = 2 \cdot 4$, this may result in:



Divide & Conquer Protocol for n Players

Step 2.2: If $v_n(A) \geq k/n = 1/2$ then p_n chooses piece A, and otherwise p_n chooses piece $X - A$.

Step 2.3: Recursively applying the Divide & Conquer protocol for k players,

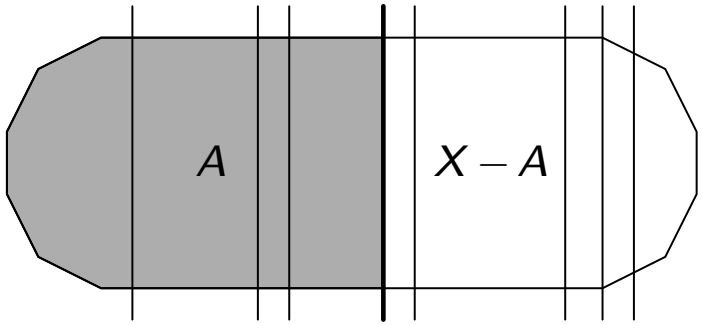
1. p_n shares the chosen piece with the $k-1$ players whose markings fall within this piece and are not the kth marking;
2. the other k players share the other piece.

Figure: Divide & Conquer protocol for n players

Divide & Conquer Protocol for n Players

Case 3: If $n = 2k+1$ for $k \geq 1$ then apply the following steps.

Step 3.1: $p_1, p_2, \ldots, p_{n-1}$ divide the cake using parallel markings in the ratio of $\mathbf{k:k+1}$ according to their valuation function. Let A be the piece of cake left of the kth marking. For example, if $n = 9 = 2 \cdot 4 + 1$, this may result in:



Step 3.2: If $v_n(A) \geq k/n = k/(2k+1)$, then p_n chooses piece A, and otherwise p_n chooses piece $X - A$.

Figure: Divide & Conquer protocol for n players

Divide & Conquer Protocol for n Players

Step 3.3:

1. If p_n chose piece A, then he shares this piece using the Divide & Conquer protocol for k players with the $k-1$ players whose markings fall within piece A and are not the kth marking;
2. if p_n chose piece $X-A$, then he shares this piece using the Divide & Conquer protocol for $k+1$ players with the k players whose markings fall within piece $X-A$ and are not the kth marking.

In both cases, the other $k+1$ (or k, respectively) players share the other piece using the Divide & Conquer protocol for $k+1$ (or k, respectively) players.

Divide & Conquer Protocol

Theorem

The Divide & Conquer protocol is:

1. *proportional,*
2. *finite bounded,*
3. *immune to manipulation for risk-averse players,*
4. *not envy-free, and*
5. *not Pareto-optimal.*

without proof

Divide & Conquer Protocol

Remark

1. *Brams, Jones, and Klamler (2011) presented an interesting version of the Divide & Conquer protocol that minimizes the number of players the most envious player envies.*

   Main differences:

   - *After all markings have been made to divide the cake in the ratio of*
     - **k:k** *if the number of players is even, or*
     - **k:k+1** *if the number of players is odd,*

     player p_n does not cut the cake at the kth marking but between two neighboring markings. This area is used to reduce envy.
   - *Furthermore, Brams, Jones, and Klamler (2011) add another step at the end of the protocol, which allows all players in an envy cycle to trade their portions to reduce the number of envy-relations.*

Divide & Conquer Protocol

Remark

2. *The number of (guaranteed) envy-relations of various cake-cutting protocols will later be discussed in more detail.*

3. *The* **main advantage** *of the Divide & Conquer protocol is that, among known proportional finite bounded protocols, it requires the least number of cuts in the worst case.*
   This will also be discussed later in more detail.