Cake-cutting Algorithms

Folien zur Vorlesung

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Websites

- Vorlesungswebsite:

  http://ccc.cs.uni-duesseldorf.de/~rothe/cake

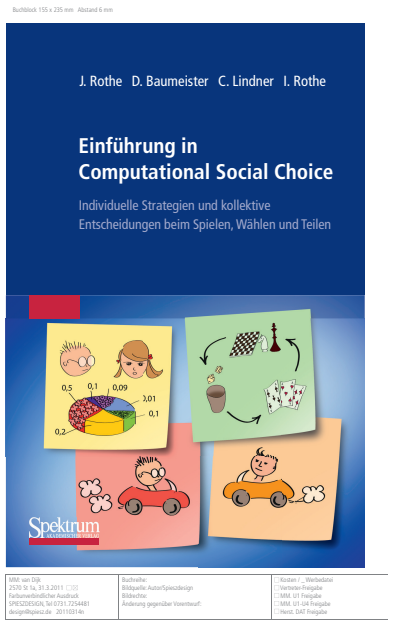
- Anmeldung nicht nur im LSF, sondern auch unter

  http://ccc.cs.uni-duesseldorf.de/verwaltung

  (CCC-System für alle meine Veranstaltungen)

Literature

- **J. Rothe (Herausgeber): Economics and Computation: An Introduction to Algorithmic Game Theory, Computational Social Choice, and Fair Division.** Springer-Verlag, 2015



Literature: Email from Prof. Michael Wooldridge, Oxford

Dear Joerg,

I just received a copy of "Economics and Computation". It looks FANTASTIC! I already started reading some of it, and I think we will use it on a course we are giving here next year.

It was tremendously kind of you to think about sending me a copy – I'm very grateful!

Congratulations, and thanks again!

Mike

–

Professor Michael Wooldridge mailto:mjw@cs.ox.ac.uk
Department of Computer Science, University of Oxford.

http://www.cs.ox.ac.uk/people/michael.wooldridge/

Literature (Recommended for Additional Reading)

- **Jack Robertson and William Webb: "Cake-Cutting Algorithms: Be Fair if You Can"**, A K Peters, 1998
- **Steven J. Brams and Alan D. Taylor: "Fair Division: From Cake-Cutting to Dispute Resolution"**, Cambridge University Press, 1996

The Players

Claudia

Doro

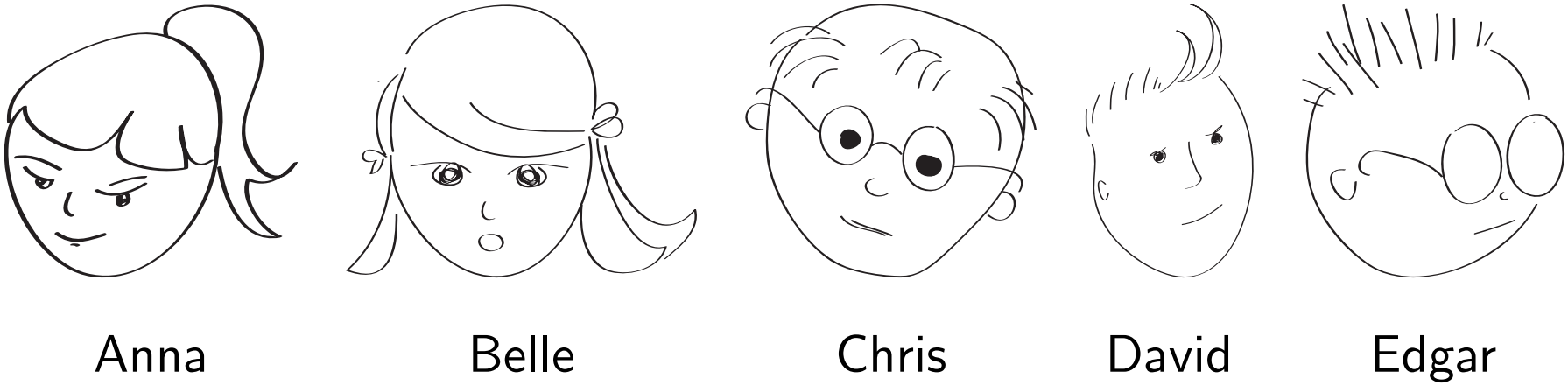
Edith

Felix

Gábor

Holger

More Players and the Cake



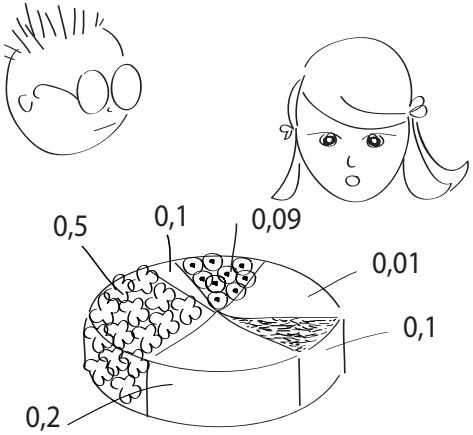


Figure: A (normalized) cake as a metaphor of a heterogeneous resource

What is Cake-cutting?

- Cake-cutting describes the problem of fairly dividing an infinitely divisible resource—the cake—among the players
  (as opposed to fair division of indivisible objects as in auctions).

- This in particular applies to areas such as
  - **economics & law** (e.g., the division of a property in a divorce or inheritance case),
  - **science & technology** (e.g., the division of computing time among several users sharing a single computer, or the division of bandwidth when sharing a network), and even
  - **politics** (historical example: the division of Germany among the allied powers—the USA, Great Britain, France, and Russia—as a result of the Second World War).

Historically: the Polish School

Hugo Steinhaus (1887–1972) | Stefan Banach (1892–1945) | Bronisław Knaster (1893–1990)

"It may be stated incidentally that if there are two (or more) partners with different estimations, there exists a division giving to everybody more than his due part; the fact disproves the common opinion that differences in estimations make fair division difficult."

– Hugo Steinhaus

Crucial Assumptions

- Every player p_i has an individual, private *valuation function* v_i measuring p_i's preferences/utilities over all pieces of cake. (Axiomatic formalization will be given later.)
- "Private" means: Every player knows only her own valuation function.
- The *cake* is a metaphor of a heterogeneous resource/good/object:
  - Distinct players may value one and the same piece of cake differently, i.e., their individual valuation functions will in general be distinct.
  - Pieces of cake having equal size can be valued differently by the same player: **It is not all about its size!**

Four Methods for Two Players

Mom

wants to fairly divide the cake among:

Claudia and **F**elix

Four Methods for Two Players

Method 1: **M**om cuts the cake into two pieces that she thinks are equal and presents **C**laudia with one piece and **F**elix with the other.

Method 2: **M**om cuts the cake into two pieces and **C**laudia and **F**elix toss a coin to decide who may choose first.

Method 3: **C**laudia cuts the cake into two pieces and **C**laudia may choose first.

Method 4: **C**laudia cuts the cake into two pieces and **F**elix may choose first.

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Cut-and-Choose Protocol for Two Players

Given: Cake X and players p_1 and p_2 with valuation functions v_1 and v_2.

Step 1: p_1 executes one cut to divide the cake into two pieces, S_1 and S_2, that he considers equal halves: $v_1(S_1) = v_1(S_2)$. It holds that $X = S_1 \cup S_2$.

Step 2: p_2 chooses her best piece.

Step 3: p_1 receives the remaining piece.

Figure: Cut-and-choose protocol for two players

Box Representations of Valuation Functions

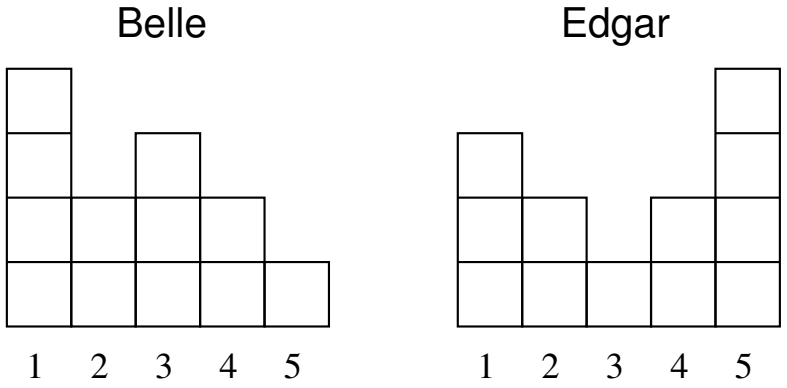


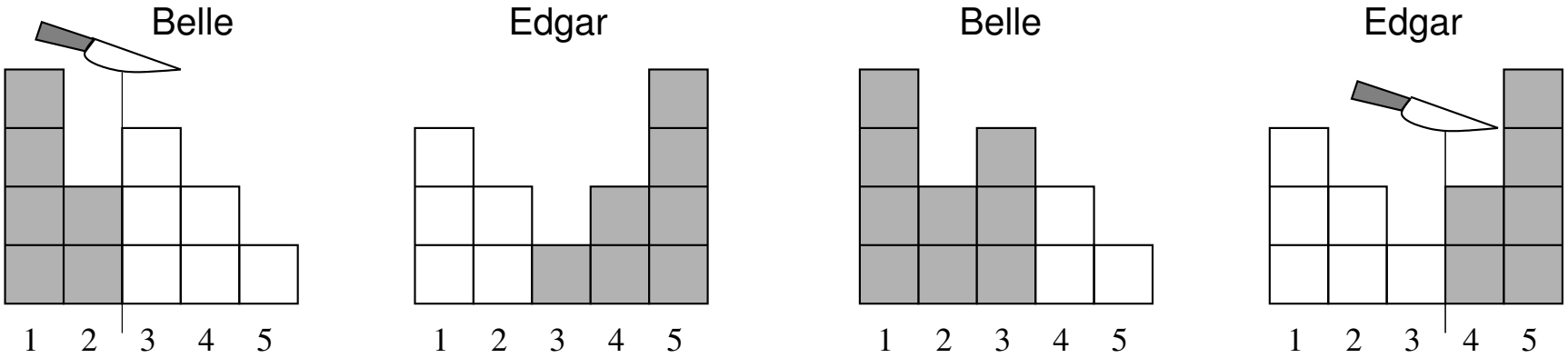
Figure: Example of a box representation of two valuation functions

- In this example, the cake consists of a total of 12 boxes.
- The number of boxes per column specifies the valuation of this part of the cake, i.e., every column represents a distinct portion of the cake.
- Here, Belle values the right edge of the cake significantly less than Edgar does, and Edgar shows only little interest in its middle part.

Box Representations of Valuation Functions

- More specifically, the left fifth of the cake is worth $4/12 = 1/3$ to Belle and $3/12 = 1/4$ to Edgar.
- The box representation is a very coarse and simplified representation of the players' valuation functions: There are valuation functions where the box representation fails to adequately describe them.
- However, the box representation is a very handy tool to visualize examples and to demonstrate step-by-step the procedure of a protocol for a given set of valuation functions.
- Markings and cuts (the latter of which will be labeled by a knife) can only be done in a vertical manner.
- For the sake of clarity, all examples using the box representation are designed such that cuts are always made where two columns join.

Cut-and-Choose Protocol: Using the Box Representation



(a) Belle cuts, Edgar chooses

(b) Edgar cuts, Belle chooses

Figure: Cut-and-choose protocol: Two examples

Some Questions

- Is the cut-and-choose protocol fair?
- How can it be generalized to more than two players?
- What does it take for a division to be "fair"?
- What other aspects need to be taken into account when evaluating the quality of a division?
- How can fairness be *guaranteed*, irrespective of the valuation functions of the players?
- *How* can we fairly divide a cake?

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Doesn't Cut It Method for Three Players

Claudia

Doro

Edith

Each player wants to get one third of the cake (in her valuation).

That is, they aim at a *proportional* division.

Doesn't Cut It Method for Three Players

Given: Cake X and players **C**laudia, **D**oro, and **E**dith with valuation functions $v_{\mathbf{C}}$, $v_{\mathbf{D}}$, and $v_{\mathbf{E}}$.

Step 1: **C**laudia cuts the cake X into two pieces, X_1 and X_2 with $X = X_1 \cup X_2$, such that $v_{\mathbf{C}}(X_1) = 1/3$ and $v_{\mathbf{C}}(X_2) = 2/3$.

Step 2: **D**oro cuts the piece X_2 into two pieces, X_{21} and X_{22} with $X_2 = X_{21} \cup X_{22}$, such that $v_{\mathbf{D}}(X_{21}) = (1/2) \cdot v_{\mathbf{D}}(X_2)$ and $v_{\mathbf{D}}(X_{22}) = (1/2) \cdot v_{\mathbf{D}}(X_2)$.

Step 3: The three players choose one piece each in the order:

1. **E**dith (chooses among X_1, X_{21}, and X_{22});
2. **C**laudia (chooses among the two remaining pieces);
3. **D**oro (takes the last piece).

Doesn't Cut It Method for Three Players: Who is Happy?

1. **E**dith? Certainly!
2. **C**laudia? As well!
3. **D**oro? **She is not guaranteed to be happy!**