Constructive Control by Adding Candidates

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY ADDING AN UNLIMITED NUMBER OF CANDIDATES ($\mathcal{E}$-CCAUC).

Given:

- Disjoint sets C and D of candidates,
- a list V of votes over $C \cup D$, and
- a distinguished candidate $p \in C$.

Question: Is there a subset D' of D such that p is the unique winner of the $\mathcal{E}$ election $(C \cup D', V)$?

Constructive Control by Adding Candidates

Definition (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY ADDING A LIMITED NUMBER OF CANDIDATES ($\mathcal{E}$-CCAC).

Given:
- Disjoint sets C and D of candidates,
- a list V of votes over $C \cup D$,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is there a subset D' of D such that $\|D'\| \leq k$ and p is the unique winner of the $\mathcal{E}$ election $(C \cup D', V)$?

Constructive Control by Deleting Candidates

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-Constructive Control by Deleting Candidates ($\mathcal{E}$-CCDC).

Given:
- A set C of candidates,
- a list V of votes over C,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is it possible to delete up to k candidates from C such that p is the unique winner of the resulting $\mathcal{E}$ election?

Constructive Control by Partition of Candidates

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY PARTITION OF CANDIDATES ($\mathcal{E}$-CCPC).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition C into C_1 and C_2 such that p is the unique winner (w.r.t. V) of the final stage of the two-stage election in which

- the winners of (C_1, V) surviving the tie-handling rule
- run against all candidates in C_2?

- "Ties eliminate" (TE): Only unique winners proceed to final stage.
- "Ties promote" (TP): All winners proceed to final stage.

Constructive Control by Runoff Partition of Candidates

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY RUNOFF PARTITION OF CANDIDATES ($\mathcal{E}$-CCRPC).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition C into C_1 and C_2 such that p is the unique winner (w.r.t. V) of the final stage of the two-stage election in which the runoff is between

- the winners of (C_1, V) surviving the tie-handling rule and
- the winners of (C_2, V) surviving the tie-handling rule?

- "Ties eliminate" (TE): Only unique winners proceed to final stage.
- "Ties promote" (TP): All winners proceed to final stage.

Constructive Control by Adding Voters

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY ADDING VOTERS ($\mathcal{E}$-CCAV).

Given:
- A set C of candidates,
- a list V of registered votes over C and an additional list W of as yet unregistered votes over C,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is there a subset W' of W such that $\|W'\| \leq k$ and p is the unique winner of the $\mathcal{E}$ election $(C, V \cup W')$?

Constructive Control by Deleting Voters

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathcal{E}$ be some voting system.

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY DELETING VOTERS ($\mathcal{E}$-CCDV).

Given:
- A set C of candidates,
- a list V of votes over C,
- a distinguished candidate $p \in C$, and
- a nonnegative integer k.

Question: Is it possible to delete up to k voters from V such that p is the unique winner of the resulting $\mathcal{E}$ election?

Constructive Control by Partition of Voters

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Name: $\mathcal{E}$-CONSTRUCTIVE CONTROL BY PARTITION OF VOTERS ($\mathcal{E}$-CCPV).

Given: An election (C, V) and a distinguished candidate $p \in C$.

Question: Is it possible to partition V into V_1 and V_2 such that p is the unique winner (with respect to the votes in V) of the final stage of the two-stage election in which the runoff is between

- the winners of (C, V_1) surviving the tie-handling rule and
- the winners of (C, V_2) surviving the tie-handling rule?

- "Ties eliminate" (TE): Only unique winners proceed to final stage.
- "Ties promote" (TP): All winners proceed to final stage.

Destructive Control

Remark:

- For each constructive control scenario, there is a corresponding destructive control type where the chair seeks to block the distinguished candidate's victory:

  $\mathcal{E}$-DCAUC, $\mathcal{E}$-DCAC, $\mathcal{E}$-DCDC, $\mathcal{E}$-DCPC-TE, $\mathcal{E}$-DCPC-TP, $\mathcal{E}$-DCRPC-TE, $\mathcal{E}$-DCRPC-TP, $\mathcal{E}$-DCAV, $\mathcal{E}$-DCDV, $\mathcal{E}$-DCPV-TE, and $\mathcal{E}$-DCPV-TP.

  In $\mathcal{E}$-DCDC it is not allowed to simply delete the distinguished candidate.

  $\Rightarrow$ This sums up to a total of 22 control types (and the corresponding control problems).

- The study of destructive control was initiated by Hemaspaandra, Hemaspaandra, and Rothe (2007).

Immunity and Susceptibility

Definition (Bartholdi, Tovey, and Trick (1992))

Let $\mathfrak{CT}$ be a control type.

1. We say a voting system is *immune to $\mathfrak{CT}$* if it is impossible for the chair to make the given candidate
   - the unique winner in the constructive case and
   - not a unique winner in the destructive case,

   respectively, via exerting control of type $\mathfrak{CT}$.
2. We say a voting system is *susceptible to $\mathfrak{CT}$* if it is not immune to $\mathfrak{CT}$.

Resistance and Vulnerability

Definition (Bartholdi, Tovey, and Trick (1992) & Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let $\mathfrak{CT}$ be a control type.

A voting system that is susceptible to $\mathfrak{CT}$ is said to be

1. *vulnerable to $\mathfrak{CT}$* if the control problem corresponding to $\mathfrak{CT}$ can be solved in polynomial time, and
2. *resistant to $\mathfrak{CT}$* if the control problem corresponding to $\mathfrak{CT}$ is NP-hard.

Links Between Susceptibility Cases

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

1. *A voting system is susceptible to constructive control by adding candidates if and only if it is susceptible to destructive control by deleting candidates.*
2. *A voting system is susceptible to constructive control by deleting candidates if and only if it is susceptible to destructive control by adding candidates.*
3. *A voting system is susceptible to constructive control by adding voters if and only if it is susceptible to destructive control by deleting voters.*
4. *A voting system is susceptible to constructive control by deleting voters if and only if it is susceptible to destructive control by adding voters.*

Links Between Susceptibility Cases

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

1. *If a voting system is susceptible to constructive control by partition of voters (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
2. *If a voting system is susceptible to constructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*
3. *If a voting system is susceptible to constructive control by partition of voters in model TE, then it is susceptible to constructive control by deleting voters.*
4. *If a voting system is susceptible to destructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to destructive control by deleting candidates.*

Links Between Susceptibility Cases

Definition

A voting system is *voiced* if in any election that has exactly one candidate, that candidate is always a (and thus, the unique) winner.

Theorem

1. *If a voiced voting system is susceptible to destructive control by partition of voters (in model TE or TP), then it is susceptible to destructive control by deleting voters.*
2. *Each voiced voting system is susceptible to constructive control by deleting candidates.*
3. *Each voiced voting system is susceptible to destructive control by adding candidates.*

Control Complexity of Plurality and Condorcet Voting

	Plurality		Condorcet	
Control by	Constructive	Destructive	Constructive	Destructive
Adding Candidates	R(esistant)	**R**	I(mmune)	**V(ulnerable)**
Deleting Candidates	R	**R**	V	**I**
Partition of Candidates	TE: R TP: R	**TE: R** **TP: R**	V	**I**
Run-off Partition of Candidates	TE: R TP: R	**TE: R** **TP: R**	V	**I**
Adding Voters	V	**V**	R	**V**
Deleting Voters	V	**V**	R	**V**
Partition of Voters	**TE: V** **TP: R**	**TE: V** **TP: R**	R	**V**

Boldface results are due to Hemaspaandra, Hemaspaandra, and Rothe (2007).
Nonboldface results are due to Bartholdi, Tovey, and Trick (1992).

Hitting Set

Definition

Name: HITTING SET.

Given:
- A set $B = \{b_1, b_2, \ldots, b_m\}$,
- a family $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets S_i of B, and
- a positive integer k.

Question: Does $\mathcal{S}$ have a hitting set of size at most k?
That is, is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i, $S_i \cap B' \neq \emptyset$?

Destructive Candidate Control in Plurality Voting

Construction: Given a HITTING SET instance $(B, \mathcal{S}, k)$, where $B = \{b_1, b_2, \ldots, b_m\}$, $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$, and $k \leq m$, construct the following election:

- The candidate set is $C = B \cup \{c, w\}$.
- The voter set V is defined as follows:
  1. $2(m-k) + 2n(k+1) + 4$ voters of the form $c\ w\ \cdots$, where "$\cdots$" means that the remaining candidates follow in an arbitrary order.
  2. $2n(k+1) + 5$ voters of the form $w\ c\ \cdots$.
  3. For each i, $1 \leq i \leq n$, there are $2(k+1)$ voters of the form $S_i\ c\ \cdots$, where "S_i" denotes the elements of S_i in some arbitrary order.
  4. For each j, $1 \leq j \leq m$, two voters of the form $b_j\ w\ \cdots$.

Destructive Candidate Control in Plurality Voting

Lemma (Hemaspaandra, Hemaspaandra, and Rothe (2007))

If B' is a hitting set of $\mathcal{S}$ of size k, then w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$.

Proof: See blackboard. ❑

Destructive Candidate Control in Plurality Voting

Lemma (Hemaspaandra, Hemaspaandra, and Rothe (2007))

Let $D \subseteq B \cup \{w\}$. If c is not a unique plurality winner of election $(D \cup \{c\}, V)$, then there exists a set $B' \subseteq B$ such that

1. $D = B' \cup \{w\}$,
2. *w is the unique plurality winner of the election $(B' \cup \{c, w\}, V)$, and*
3. *B' is a hitting set of $\mathcal{S}$ of size less than or equal to k.*

Proof: See blackboard. ❑

Destructive Candidate Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

$\mathcal{S}$ has a hitting set of size less than or equal to k if and only if destructive control by adding candidates can be executed for the election with qualified candidates $\{c, w\}$, spoiler candidates B, distinguished candidate c, and voter set V.

Proof: See blackboard. ❑

Corollary: Plurality voting is resistant to destructive control by adding candidates.

That is, Plurality-DCAUC (and also Plurality-DCAC) is NP-hard.

Destructive Candidate Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

$\mathcal{S}$ has a hitting set of size at most k if and only if the election with candidate set C, distinguished candidate c, and voter set V can be destructively controlled by deleting at most $m - k$ candidates.

Proof: See blackboard. ❑

Corollary: Plurality voting is resistant to destructive control by deleting candidates. That is, Plurality-DCDC is NP-hard.

Destructive Candidate Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

$\mathcal{S}$ has a hitting set of size at most k if and only if the election with candidate set C, distinguished candidate c, and voter set V can be destructively controlled by partition of candidates (with and without run-off, and for each both in model TE and TP).

Proof: See blackboard. ❑

Corollary: Plurality voting is resistant to destructive control by partition of candidates (with and without run-off, and for each both in model TE and TP). That is, Plurality-DCPC-TE, Plurality-DCPC-TP, Plurality-DCRPC-TE, and Plurality-DCRPC-TP are NP-hard.

Voter Control in Plurality Voting

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (2007))

1. *Plurality voting is vulnerable/certifiably-vulnerable to destructive control both by adding voters and by deleting voters.*

   "Certifiably-vulnerable" means the chair cannot only decide the problems Plurality-DCAV and Plurality-DCDV in polynomial time, but can even produce in polynomial time a "best possible" control action. Certifiable vulnerability implies vulnerability.
   (In particular, the "k" may be dropped from the problem instance.)

2. *In model TE, plurality voting is vulnerable/certifiably-vulnerable to constructive and destructive control by partition of voters.*
   That is, Plurality-CCPV-TE and Plurality-DCPV-TE is in P.

Destructive Control by Adding Voters in Plurality

Proof:

1. (a) Plurality voting is certifiably-vulnerable to destructive control by adding voters: "Smart Greedy"

   Given (C, c, V, W) as in DCAV (without k):

   - If c already is not a unique plurality winner in (C, V), adding no voters accomplishes our goal, and we are done.
   - Otherwise, sort all $d_i \in C - \{c\}$ by decreasing deficit, i.e., letting $\mathit{diff}(d_i)$ denote d_i's deficit of first-place votes needed to tie c, we have

   $$\mathit{diff}(d_1) \leq \mathit{diff}(d_2) \leq \cdots \leq \mathit{diff}(d_{\|C\|-1}).$$

   - For $i = 1, 2, \ldots, \|C\| - 1$, if

   $$\|\{w \in W \mid w\text{'s first choice is } d_i\}\| \geq \mathit{diff}(d_i),$$

     then add $\mathit{diff}(d_i)$ of these unregistered voters to ensure that d_i ties c (and c thus is not a unique winner) and halt.
   - If no iteration was successful, output "control impossible" and halt.

Destructive Control by Deleting Voters in Plurality Voting

(b) Plurality voting is certifiably-vulnerable to destructive control by deleting voters: "Dumb Greedy"

Given (C, c, V) as in DCDV (without k):

- If $C = \{c\}$, then output "control impossible" and halt;
- else if c already is not a unique plurality winner in (C, V), deleting no voters accomplishes our goal, and we are done.
- If every candidate other than c gets zero first-place votes, then output "control impossible" and halt.
- Otherwise, let d be the candidate closest to c in first-place votes, and let *diff*(d) denote d's deficit of first-place votes needed to tie c.

  Deleting *diff*(d) voters whose first choice is c assures that c is not a unique winner, and this is the fewest deletions that can achieve that.

Constructive Control by Partition of Voters (TE) in Plurality Voting

2. Plurality voting is certifiably-vulnerable to constructive control by partition of voters in model TE: Plurality-CCPV-TE is in P

   - Let (C, c, V) be given as in CCPV-TE.
   - For any partition (V_1, V_2) of V, let $Nominees(C, V_i)$, $i \in \{1, 2\}$, denote the set of candidates who are nominated by the subcommittee V_i (with candidates C) for the run-off in model TE.

Constructive Control by Partition of Voters (TE) in Plurality Voting

Consider the following cases (Cases 3 and 5 need not be disjoint):

Case 1: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \emptyset$ due to $V_2 = \emptyset$.

Case 2: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \{c\}$.

Case 3: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \emptyset$ due to c and d (and possibly additional other candidates) tying, where $c \neq d$.

Case 4: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \{d\}$, $c \neq d$.

Case 5: $Nominees(C, V_1) = \{c\}$ and $Nominees(C, V_2) = \emptyset$ due to d and e (and possibly additional other candidates) tying, where $c \neq d \neq e \neq c$.

Constructive Control by Partition of Voters (TE) in Plurality Voting

Given (C, c, V) as in CCPV-TE:

- If c is the unique plurality winner in (C, V) (thus catching Cases 1, 2, and 3), then output $(V, \emptyset)$ as a successful partition and halt;
- else if $\|C\| = 2$, then output "control impossible" (which in this context means that making c a unique winner is impossible) and halt.
- Otherwise, first try to make Case 4 hold in the Case 4 Loop;
- and then, if that fails, try to make Case 5 hold in the Case 5 Loop.
- Otherwise (i.e., if the Case 5 Loop was not successful either), c cannot win, so we output "control impossible" and halt.

Constructive Control by Partition of Voters (TE) in Plurality Voting

Case 4 Loop:

- For each $d \in C$, $d \neq c$, such that c beats d in a pairwise plurality election by the voters in V, do the following:
  If it holds that, for each $e \in C$ with $c \neq e \neq d$,

  $$score(e) \leq score(c) + score(d) - 2,$$

  then output (V_1, V_2) as a successful partition and halt, where
  - V_1 consists of
    - all $score(c)$ voters whose first choice is c and
    - exactly $\min(score(e), score(c) - 1)$ of the voters whose first choice is e, and
  - where $V_2 = V - V_1$.

Constructive Control by Partition of Voters (TE) in Plurality Voting

Case 5 Loop:

- For each $d \in C$ and for each $e \in C$ such that $\|\{c, d, e\}\| = 3$ and $score(d) \leq score(e)$, do the following:
  If it holds that, for each $f \in C - \{c\}$,

$$score(f) \leq score(c) + score(d) - 1,$$

  then output (V_1, V_2) as a successful partition and halt, where
  - V_1 consists of
    - all $score(c)$ voters whose first choice is c,
    - exactly $score(e) - score(d)$ of the voters whose first choice is e, and
    - for all $f \in C - \{c, d, e\}$, exactly $\min(score(f), score(c) - 1)$ of the voters whose first choice is f, and
  - where $V_2 = V - V_1$. ❑

Destructive Control by Partition of Voters (TE) in Plurality Voting

2. Plurality voting is certifiably-vulnerable to destructive control by partition of voters in model TE: Plurality-DCPV-TE is in P

- Let (C, c, V) be given as in DCPV-TE.
- If $C = \{c\}$, output "control impossible" and halt, as c must win;
- else if c already is not a unique plurality winner, output $(V, \emptyset)$ as a successful partition and halt.
- Otherwise, check if every voter's first choice is c or if $\|C\| = 2$, and if one of these two conditions is true, output "control impossible" and halt, since c cannot help but win.

Destructive Control by Partition of Voters (TE) in Plurality Voting

- Let d be a candidate who other than c got the most first-place votes, and let e be a candidate who other than c and d got the most first-place votes.

- We can certainly dethrone c if

$$score(c) \leq score(d) + score(e). \tag{1}$$

  Namely, if (1) holds, we output (V_1, V_2) as a successful partition and halt, where
  - V_1 consists of
    - all $score(d)$ voters whose first choice is d and
    - exactly $score(d)$ voters whose first choice is c (recall that in the current case we already know that $score(c) > score(d)$), and
  - where $V_2 = V - V_1$.

Destructive Control by Partition of Voters (TE) in Plurality Voting

- On the other hand, if Equation (1) is not satisfied, we have

  $$score(c) > score(d) + score(e),$$

  so in any partition (V_1, V_2), c wins in one of (C, V_1) or (C, V_2).

  Thus, it is impossible to make c lose in both subcommittees.

- If c is nominated by both subcommittees (in model TE), c trivially is the unique winner of the final run-off.

- So, we now check if it is possible for c
  - to win in *exactly* one subcommittee, and
  - yet can be made to *not be the unique winner* of the final run-off.

Destructive Control by Partition of Voters (TE) in Plurality Voting

For this to happen, it is (given the case we are in) a necessary and sufficient condition that there exists some candidate d such that:

- $d \neq c$,
- d ties or beats c in a pairwise plurality election, and
- for each candidate e, $c \neq e \neq d$, we have that $score(e) < score(c) + score(d) - 2$.

Destructive Control by Partition of Voters (TE) in Plurality Voting

We can in polynomial time brute-force check whether the above three conditions hold for some candidate d, and if they do, let d' be some such candidate d and output (V_1, V_2) as a successful partition and halt, where V_1 consists of

- all $score(c)$ voters whose first choice is c and,
- for each candidate e with $c \neq e \neq d'$, of exactly

$$\min(score(c) - 1, score(e))$$

  voters whose first choice is e,

and where $V_2 = V - V_1$.

Finally, if the above two conditions cannot be satisfied for any d, output "control impossible" and halt. ❑