Algorithmische Eigenschaften von Wahlsystemen I

Ausgewählte Folien zur Vorlesung

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Dozent: Prof. Dr. J. Rothe

HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

Websites

- Vorlesungswebsite:

  http://ccc.cs.uni-duesseldorf.de/~rothe/voting1

- Anmeldung nicht nur im LSF, sondern auch unter

  http://ccc.cs.uni-duesseldorf.de/verwaltung

  (CCC-System für alle meine Veranstaltungen)

Literature

- **A Richer Understanding of the Complexity of Election Systems**, P. Faliszewski, E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Chapter 14 in *Fundamental Problems in Computing: Essays in Honor of Professor Daniel J. Rosenkrantz*, pp. 375–406, S. Ravi and S. Shukla, Editors. Springer, Berlin, Heidelberg, New York, 2009.

- **Computational Aspects of Approval Voting**, D. Baumeister, G. Erdélyi, E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Chapter 10 in *Handbook on Approval Voting*, pp. 199–251, R. Sanver and J. Laslier, Editors. Springer-Verlag, Berlin, Heidelberg, 2010.

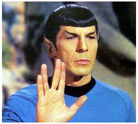
Literature

- **Voting Procedures**, S. Brams and P. Fishburn. Chapter 4 in Volume 1 of the *Handbook of Social Choice and Welfare*, pp. 173–236, K. Arrow, A. Sen, and K. Suzumura, Editors. North-Holland, 2002.
- **Chaotic Elections! A Mathematician Looks at Voting**, D. Saari. American Mathematical Society, 2001.
- **Original Papers** cited in this book and these book chapters.
- . . .

Elections

The Captain of Starship Enterprise is to be elected: 

Candidates:

Voters:

Elections

Definition

- An *election* (or *preference profile*) (C, V) is specified by a set

$$C = \{c_1, c_2, \dots, c_m\}$$

  of candidates and a list

$$V = (v_1, v_2, \dots, v_n)$$

  of votes over C.
- How the voters' preferences are represented depends on the voting system used, e.g., by
  - a *linear order (strict ranking)* or
  - an *approval vector*.

Elections

Definition

A *linear order* (or *strict ranking*) $>$ *on* C is a binary relation on C that is

- *total*: for any two distinct $c, d \in C$, either $c > d$ or $d > c$;
- *transitive*: for all $c, d, e \in C$, if $c > d$ and $d > e$ then $c > e$;
- *asymmetric*: for all $c, d \in C$, if $c > d$ then $d > c$ does not hold.

Remark:

1. Asymmetry of $>$ implies irreflexivity of $>$.
2. We often omit the symbol $>$ in the linear orders and write, e.g.,

   $$b \quad c \quad a \quad e \quad d \qquad \text{instead of} \qquad b > c > a > e > d$$

   to indicate that this voter (strictly) prefers b to c, c to a, a to e, and e to d. So the leftmost candidate is the most preferred one.

Elections

Remark:

3. Occasionally, by dropping asymmetry voters are allowed to be *indifferent* between candidates, as in:

$$b > c = a > e = d$$

If so, it will be mentioned explicitly.

4. One may distinguish between *weighted* and *unweighted* voters. Default case: unweighted voters (i.e., each voter has weight one).

5. Votes may be represented either *succinctly* or *nonsuccinctly*. Default case: nonsuccinct (i.e., one ballot per voter).

Elections

Example

Election (C, V) with $C = \{a, b, c, d, e\}$ and $V = (v_1, \ldots, v_7)$:

$$
\begin{array}{lccccc}
v_1: & c & b & a & e & d \\
v_2: & a & e & d & c & b \\
v_3: & b & a & c & e & d \\
v_4: & b & d & e & a & c \\
v_5: & c & b & a & e & d \\
v_6: & c & d & b & e & a \\
v_7: & e & d & a & b & c
\end{array}
$$

Who should win this election?

Election Systems

Definition

An *election system* is a rule determining the winner(s) of a given election (C, V). Formally, letting

- $\mathcal{P}(C)^n$ denote the set of all n-vote preference profiles (e.g., n linear orders or n approval vectors) over the set C of candidates and
- $\mathfrak{P}(S)$ the set of all subsets of a set S,

an election system defines a *social choice correspondence*

$$f : \mathcal{P}(C)^n \rightarrow \mathfrak{P}(C).$$

Given a preference profile $P \in \mathcal{P}(C)^n$, $f(P) \subseteq C$ is the *set of winners* (which may be empty and may have more than one winner).

Election Systems

Remark:

- A *social choice function* is a mapping

$$f : \mathcal{P}(C)^n \rightarrow C$$

  that assigns a single winner to each given preference profile.

- Letting $\mathcal{R}(C)$ denote the set of all transitive, total preference relations over C, a *social welfare function* is a mapping

$$f : \mathcal{P}(C)^n \rightarrow \mathcal{R}(C)$$

  that assigns a complete (possibly nonstrict) ranking to each given preference profile.

Election Systems: An Incomplete Taxonomy

- Preference-based Systems:
  - Positional scoring protocols (plurality, veto, k-approval, Borda, ...)
  - Majority-based voting (simple majority, Bucklin voting, ...)
  - Pairwise-comparison-based voting procedures (Condorcet, Black, Dodgson, Young, Kemeny, Copeland, Llull, ...)
  - Point distribution voting procedures (single transferable vote, ...)
- Nonranked Systems:
  - Approval voting
  - Negative voting
  - Plurality voting
  - Multistage voting procedures (plurality with runoff, ...)
- Hybrid Systems:
  - Sincere-strategy preference-based approval voting
  - Fallback voting

Election Systems: Plurality, Antiplurality, k-Approval

Definition

- *Plurality-rule elections*: The winners are precisely those candidates who are ranked first by the most voters.
- *Antiplurality-rule (a.k.a. veto) elections*: The winners are precisely those candidates who are ranked last by the fewest voters.
- *k-approval*: Each voter gives one point to each of the k most preferred candidates. Whoever scores the most points wins.

In our above example, c is the plurality winner, e is the antiplurality winner, and both a and b are 3-approval winners.

Election Systems: Borda Count

Definition

- *Borda Count*: With m candidates, each voter gives:
  - $m - 1$ points to the candidate ranked at first position,
  - $m - 2$ points to the candidate ranked at second position,

    $\vdots$
  - 0 points to the candidate ranked at last position.

  Whoever scores the most points wins.

In our above example, b is the Borda winner.

Election Systems: Borda Count

points :	4	3	2	1	0
v_1 :	c	b	a	e	d
v_2 :	a	e	d	c	b
v_3 :	b	a	c	e	d
v_4 :	b	d	e	a	c
v_5 :	c	b	a	e	d
v_6 :	c	d	b	e	a
v_7 :	e	d	a	b	c

Viewed as a social welfare function, the Borda system yields:

ranking	b	$>$	c	$>$	a	$>$	e	$>$	d
points	17	$>$	15	$>$	14	$>$	13	$>$	11

Election Systems: Scoring Protocols

Definition

A *scoring protocol* for m candidates is specified by a *scoring vector*, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, satisfying

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m.$$

Votes are linear orders. Each vote contributes

- α_1 points to that vote's most preferred candidate,
- α_2 points to that vote's second most preferred candidate,

  $\vdots$

- α_m points to that vote's least preferred candidate.

Whoever scores the most points wins.

Election Systems: Scoring Protocols for m Candidates

Voting System	Scoring Vector
Plurality	$\alpha = (1, \overbrace{0, \ldots, 0}^{m-1})$
Antiplurality (Veto)	$\alpha = (\overbrace{1, \ldots, 1}^{m-1}, 0)$
k-Approval ($(m-k)$-Veto)	$\alpha = (\overbrace{1, \ldots, 1}^{k}, \overbrace{0, \ldots, 0}^{m-k})$
Borda Count	$\alpha = (m-1, m-2, \ldots, 0)$
$\vdots$	$\vdots$

Simple Majority and Condorcet Voting

Definition

A candidate c wins by *(simple) majority* if c is ranked first by more than half of the voters.

In our above example, no candidate wins by simple majority. This obstacle is avoided by, e.g., Bucklin voting.

Definition

A candidate c is a *Condorcet winner* if c defeats every other candidate by a strict majority in pairwise comparisons.

In our above example, there is no Condorcet winner (as we have a top-3-cycle). This obstacle is avoided by, e.g., Black, Dodgson, Young, Copeland, and Kemeny voting.

Condorcet and Borda Paradox, Black Voting

- The *Condorcet Paradox* occurs whenever there exists no Condorcet winner.
- The *Borda Paradox* occurs whenever a plurality winner is defeated by every other candidate in pairwise contests by a majority of votes.

Definition

Black Voting:

1. Choose the Condorcet winner if there exists one.
2. Otherwise, choose all Borda winners.

Majority and Condorcet Criteria

Definition

A voting system satisfies the

1. *majority criterion* if it selects the majority winner whenever one exists;
2. *Condorcet criterion* if it selects the Condorcet winner whenever one exists.

Example

Black's system:

- satisfies the Condorcet criterion and
- monotonicity, but
- it is inconsistent.

Consistency and Monotonicity

Definition

1. A voting system is *consistent* if the following holds: When the electorate is divided arbitrarily into two (or more) parts and separate elections in each part result in the same winners, they also win an election of the entire electorate.
   The *multiple-districts paradox* shows inconsistency.

2. A voting system is *monotonic* if the following holds: If
   - some candidate w wins an election and
   - we then improve the position of w in some of the votes, leaving everything else the same,

   then w still wins in the changed election.
   The *winner-turns-loser paradox* shows failure of monotonicity.

Examples of (Non-)Monotonic Voting Systems

1. Examples of monotonic voting systems are:
   - plurality, Borda, and (more generally) all scoring protocols,
   - Condorcet,
   - Black, . . .

2. Examples of nonmonotonic voting systems are:
   - *Plurality with Runoff*:
     - Top two candidates wrt. plurality score proceed to runoff;
     - the winner is whoever is ranked higher by more voters than the other.
   - *Single Transferable Vote (STV)*, which proceeds in $m - 1$ rounds:
     - In each round, a candidate with lowest plurality score is eliminated (using some tie-breaking rule if needed) and all votes for this candidate transfer to the next remaining candidate in this vote's order.
     - The last remaining candidate wins.
   - Dodgson (two slides ahead).

A Stronger Notion of Monotonicity

Definition

A voting system is *strongly monotonic* if the following holds: If

- some candidate w wins an election and
- we then change the votes in such a way that every candidate originally ranked behind w is still ranked behind w after the change,

then w still wins in the changed election.

Which of the voting rules you know so far (if any) satisfies this strong monotonicity criterion?

Condorcet Systems: Dodgson, Young, and Copeland

Let (C, V) be a given election where votes are linear orders.

- *Dodgson*: The *Dodgson score of $c \in C$* (denoted by *$DScore(c)$*) is the smallest number of sequential swaps needed to make c a Condorcet winner. Whoever has the smallest Dodgson score wins.
- *Young*: The *Young score of $c \in C$* (denoted by *$YScore(c)$*) is the size of a largest sublist of V for which c is a Condorcet winner. Whoever has the maximum Young score wins.
- *Copeland*: For each $c, d \in C$, $c \neq d$, let $N(c, d)$ be the number of voters who prefer c to d. Let $C(c, d) = 1$ if $N(c, d) > N(d, c)$ and $C(c, d) = 1/2$ if $N(c, d) = N(d, c)$.
  The *Copeland score of c* is *$CScore(c)$* $= \sum_{d \neq c} C(c, d)$.
  Whoever has the maximum Copeland score wins.

Dodgson Voting Fails Monotonicity

Example (Fishburn (1977))

	Original Votes					Changed Votes			
15 votes :	c	a	d	b		c	a	d	b
9 votes :	b	d	c	a		b	d	c	a
9 votes :	a	b	d	c	$\Rightarrow$	a	b	d	c
5 votes :	a	c	b	d		a	c	b	d
5 votes :	b	a	c	d		a	b	c	d
	Dodgson winner a (3 swaps)					Dodgson winner c (2 swaps)			

Determining Young and Copeland Winners

Example

Consider the election (C, V) with $C = \{a, b, c, d\}$ and V:

$$
\begin{array}{lcccc}
v_1: & c & b & a & d \\
v_2: & a & d & c & b \\
v_3: & b & a & c & d \\
v_4: & d & b & a & c
\end{array}
$$

- b is the Young winner and
- a and b are the Copeland winners.

How Hard is it to Determine Copeland, Dodgson, and Young Winners?

Fact

Copeland winners can be determined in polynomial time.

Theorem (Hemaspaandra, Hemaspaandra, and Rothe (1997))

The problem of determining Dodgson winners is complete for "parallel access to NP." **without proof**

Theorem (Rothe, Spakowski, and Vogel (2003))

The problem of determining Young winners is complete for "parallel access to NP." **without proof**

An Incomplete Summary

	Majority	Condorcet	Consistent	Monotonic
a) Plurality	1	0		1
b) Borda	0			1
c) Veto				1
d) Condorcet		1		1
e) Copeland		1		
f) Dodgson		1		0
g) Young		1		
h) Black		1	0	1
i) Plurality w. Runoff				0
j) STV				0

Homogeneity

Definition

A voting system f is said to be *homogeneous* if for each preference profile (C, V) and for all positive integers q, it holds that

$$f((C, V)) = f((C, qV)),$$

where qV denotes V replicated q times.

Remark:

- Dodgson's system is not homogeneous.
- Fishburn (1977) proposed the following limit device to define a homogeneous variant of Dodgson Elections:

$$DScore^{*}_{(C,V)}(c) = \lim_{q \to \infty} \frac{DScore_{(C,qV)}(c)}{q}.$$

Dodgson Fails Homogeneity

Example (Fishburn (1977))

Original Profile						Changed Profile				
2 votes :	d	c	a	b		6 votes :	d	c	a	b
2 votes :	b	c	a	d		6 votes :	b	c	a	d
2 votes :	c	a	b	d		6 votes :	c	a	b	d
2 votes :	d	b	c	a	$\Rightarrow$	6 votes :	d	b	c	a
2 votes :	a	b	c	d		6 votes :	a	b	c	d
1 vote :	a	d	b	c		3 votes :	a	d	b	c
1 vote :	d	a	b	c		3 votes :	d	a	b	c
Dodgson winner a (3 swaps)						Dodgson winner d (6 swaps)				

“Dodgson” (“Weak Condorcet”) Fails Homogeneity

Example (Fishburn (1977))

Consider the election (C, V) with $C = \{a_1, a_2, \ldots, a_7, c\}$ and V:

$$
\begin{array}{cccccccc}
a_1 & a_2 & a_3 & a_4 & c & a_5 & a_6 & a_7 \\
a_7 & a_1 & a_2 & a_3 & c & a_4 & a_5 & a_6 \\
a_6 & a_7 & a_1 & a_2 & c & a_3 & a_4 & a_5 \\
a_5 & a_6 & a_7 & a_1 & c & a_2 & a_3 & a_4 \\
a_4 & a_5 & a_6 & a_7 & c & a_1 & a_2 & a_3 \\
a_3 & a_4 & a_5 & a_6 & c & a_7 & a_1 & a_2 \\
a_2 & a_3 & a_4 & a_5 & c & a_6 & a_7 & a_1
\end{array}
$$

- $DScore_{(C,V)}(c) = 7$ and $DScore_{(C,V)}(a_i) = 6$ for $1 \leq i \leq 7$.
- $DScore^*_{(C,V)}(c) = 3.5$ and $DScore^*_{(C,V)}(a_i) = 4.5$ for $1 \leq i \leq 7$, which implies that c wins in (C, qV) for large enough q.

Independence of Clones

Definition

- Two candidates are *clones of each other* if they are ranked next to each other in every individual ranking, i.e., both candidates perform identically in pairwise comparisons with any other alternative.

- A voting system is *independent of clones* if a losing candidate cannot be made a winning candidate by introducing clones.

Tideman's Example of Cloning

Example (Tideman (1987))

"When I was 12 years old I was nominated to be treasurer of my class at school. A girl named Michelle was also nominated. I relished the prospect of being treasurer, so I made a quick calculation and nominated Michelle's best friend, Charlotte. In the ensuing election

- *I received 13 votes,*
- *Michelle received 12, and*
- *Charlotte received 11,*

so I became treasurer."

In other words, Tideman cloned Michelle.

Cloning in Florida in 2000

In the 2000 US Presidential Election, Ralph Nader (Green Party) split votes away from Al Gore (Democrats), thus allowing George W. Bush (Republicans) to win the election. The final count in Florida was:

Republican	2,912,790	Workers World	1,804
Democratic	2,912,253	Constitution	1,371
Green Party	97,488	Socialist	622
Natural Law	2,281	Socialist Workers	562
Reform	17,484	Write-in	40
Libertarian	16,415		

Dodgson is Not Independent of Clones

Example (Brandt (2009))

	Original				Cloning c			
5 votes :	a	b	c		a	b	c	c'
4 votes :	b	c	a	$\Rightarrow$	b	c	c'	a
3 votes :	c	a	b		c	c'	a	b
	Dodgson winner a (2 swaps)				Dodgson winner b (3 swaps)			

Dodgson May Choose the Condorcet Loser and Fails the Reversal Symmetry Criterion

Definition

- A candidate c is a *Condorcet loser* if c is defeated by every other candidate by a strict majority in pairwise comparisons.
- A voting systems satisfies the *reversal symmetry criterion* if it holds that a unique winner becomes a loser whenever all individual rankings are reversed.

Dodgson May Choose the Condorcet Loser and Fails the Reversal Symmetry Criterion

Example (Brandt (2009))

Dodgson chooses the Condorcet loser					Dodgson fails the reversal symmetry criterion				
10 votes :	d	a	b	c	10 votes :	c	b	a	d
8 votes :	b	c	a	d	8 votes :	d	a	c	b
7 votes :	c	a	b	d	7 votes :	d	b	a	c
4 votes :	d	c	a	b	4 votes :	b	a	c	d
Dodgson winner d (3 swaps)					Dodgson winner d (no swaps)				

The No Show Paradox and the Twin Paradox

Definition

- The *no show paradox* occurs whenever a voter is better off not showing up (as this leads to the election of a candidate this voter prefers).
  A voting systems satisfies the *participation* criterion if the no show paradox never occurs.

- The *twin paradox* occurs if whenever a voter is joined by a “twin” (a voter with identical preferences), this gives less weight to their joint preferences.
  A voting systems satisfies the *twins welcome* criterion if the twin paradox never occurs.

The No Show Paradox and the Twin Paradox

Example (Moulin (1985))

Successive Elimination (*Regular Cup*):

Balanced binary tree whose leaves are labeled by the candidates. Each inner node is labeled by the winner of both children, where each vote is taken by majority.

The candidate at the root wins.

Here: a against b, next the winner against c.

Ties are broken lexicographically.

v_1 :	c	b	a
v_2 :	c	b	a
v_3 :	a	b	c
v_4 :	a	b	c
v_5 :	c	a	b
v_6 :	b	c	a
v_7 :	b	c	a

The No Show Paradox and the Twin Paradox

Remark:

- Voting systems immune to both paradoxes include:
  - plurality, Borda, and (more generally) all scoring protocols,
  - simple majority.
- Voting systems subject to the no show paradox include:
  - plurality with runoff,
  - successive elimination.

Fact

If a voting system is immune to the no show paradox, it is also immune to the twin paradox.

The No Show Paradox and the Twin Paradox

Theorem (Moulin (1988))

1. *For at most three candidates, there exist voting systems satisfying the Condorcet and participation criteria.*
2. *For at least four candidates (and at least 25 voters), no voting system satisfies the Condorcet and participation criteria.*

without proof

Further Properties of Voting Systems

Definition

A voting system is

- *anonymous* if it treats all voters equally: if any two voters trade their ballots, the outcome remains the same;
- *neutral* if it treats all candidates equally: if any two candidates are swapped in each vote, the outcome changes accordingly;
- *onto* (satisfies *citizens' sovereignty*) if for each candidate there are some votes that would make that candidate win;
- *nondictatorial* if there does not exist a dictator (i.e., a voter whose most preferred candidate always wins);
- *resolute* (*single-valued*) if it always selects a single candidate as the winner.

Further Properties of Voting Systems

Definition

- A voting system satisfies the *Pareto condition*: If c is ranked above d in all votes then the system ranks c above d;
- A voting system is *independent of irrelevant alternatives (Arrow's IIA)* if the social preferences between any two candidates c and d depend only on the individual preferences between c and d: If
  - the system ranks c above d and
  - we then change the votes but not who of c and d is ranked better,

  then the system should still rank c above d.

All our systems so far satisfy each of these conditions, except resoluteness and Arrow's IIA.

Arrow's Impossibility Theorem

Theorem (Arrow (1951))

Suppose there are at least three candidates.

There exists no voting system that simultaneously:

- *satisfies the Pareto condition,*
- *is independent of irrelevant alternatives, and*
- *nondictatorial.*

without proof

Muller–Satterthwaite Impossibility Theorem

Theorem (Muller and Satterthwaite (1977))

Suppose there are at least three candidates.

There exists no voting system that simultaneously is:

- *resolute,*
- *onto,*
- *strongly monotonic, and*
- *nondictatorial.*

without proof

Gibbard–Satterthwaite Impossibility Theorem

Theorem (Gibbard (1973) and Satterthwaite (1975))

Suppose there are at least three candidates.

There exists no voting system that simultaneously is:

- *resolute,*
- *onto,*
- *nondictatorial, and*
- *nonmanipulable.*

without proof

Remark: Intuitively, a voting system is *manipulable* if some voter can be better off revealing his or her vote insincerely.